Robustness and Global Bifurcation of Three-species Ecological Model

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§0. INTRODUCTION

We shall consider three-dimensional Lotka-Volterra equations defined by the following vector fields:

\[(LV) \quad \dot{x}_i = \gamma_i x_i (1 + \sum_{j \in \mathbb{Z}_3} a_{ij} x_j), \quad i \in \mathbb{Z}_3,\]

on the closed positive cone \( \mathbb{R}_+^3 := \{ x \in \mathbb{R}^3 : x \geq 0 \} \), where \( x = (x_1, x_2, x_3) \). We denote by \( \frac{d}{dt} \) a differential \( \frac{d}{dt} \) and \( \mathbb{Z}_3 := \{1, 2, 3\} \) is considered cyclic.

In the case of the competitive systems (see Hirsch [H]), the global behaviors of the three-dimensional Lotka-Volterra systems (with \( a_{ij} < 0 \)) have been studied by Driessche and Zeeman [DZ], Chi, Hsu and Wu [CHW].

The analysis of the global behaviors of orbits in this paper is featured that the system is not always competitive. For example in the case when \( \gamma_1 = \gamma_2 = \gamma_3 (> 0) \), we have Theorem 1.1. below.

Consider the following systems:

\[(LV_1) \quad \dot{x}_i = x_i (1 + \sum_{j \in \mathbb{Z}_3} a_{ij} x_j), \quad i \in \mathbb{Z}_3.\]

**THEOREM 1.1.**[UO] *In the vector field of (LV_1) with \( \gamma_i > 0 \), \( a_{ii} < 0 \) and \( a_{i,i-1} < a_{ii} < a_{i,i+1} \) (\( i \in \mathbb{Z}_3 \)), almost every orbit \( \psi \) in \( (\mathbb{R}_+^3)\) satisfies one of the following:

1. If \( \prod_{i \in \mathbb{Z}_3} (a_{i+1,i} - a_{ii}) > \prod_{i \in \mathbb{Z}_3} (a_{ii} - a_{i,i-1}) \) and \( \det A < 0 \), then every orbit \( \psi \) in \( (\mathbb{R}_+^3)\) tends toward the equilibrium point \( z^* \).
2. If \( \prod_{i \in \mathbb{Z}_3} (a_{i+1,i} - a_{ii}) < \prod_{i \in \mathbb{Z}_3} (a_{ii} - a_{i,i-1}) \), then every orbit \( \psi \) in \( (\mathbb{R}_+^3)\) \( \backslash \Gamma_1 \) winds toward the heteroclinic cycle \( T \).

Here we denote by \( (\cdot)^\circ \) the interior of \( \cdot \), by \( A = (a_{ij}) \) called the interactive matrix of the system, by \( x^* \) the equilibrium point which is in \( (\mathbb{R}_+^3)\) and by \( \Gamma_1 \) the half line from 0 passing through the equilibrium \( x^* \). Three singularities \( e_1 := (\frac{-1}{a_{ii}}, 0, 0), e_2 := (0, \frac{-1}{a_{ii}}, 0), e_3 := (0, 0, \frac{-1}{a_{ii}}) \) are connected one another by three orbits called the heteroclinic orbits. The union of these singularities and orbits forms a curved triangle \( T \), called the heteroclinic cycle.

On the other hand, in the case when \( \gamma_i \)'s are not necessarily the same, there is an impressive result Theorem 1.3. below.

**DEFINITION 1.2.** The system (LV) is said to be *permanent*, if there exists a compact set \( K \subset (\mathbb{R}_+^3)\) such that for any \( \psi(0) \in (\mathbb{R}_+^3)\), we have \( \psi(t) \in K \) for \( t \) sufficiently large.

Namely the system (LV) is permanent, if \( \partial \mathbb{R}_+^3 \) is a repeller on \( \mathbb{R}_+^3 \), where \( \partial \mathbb{R}_+^3 \) denotes the boundary of \( \mathbb{R}_+^3 \) including points at infinity.

**THEOREM 1.3.**[HS] *Consider the vector field of (LV) with \( \gamma_i > 0 \), \( a_{ii} < 0 \) and \( a_{i,i-1} < a_{ii} < a_{i,i+1} \) (\( i \in \mathbb{Z}_3 \)).*
(1) If $\prod_{i \in \mathbb{Z}_3} (a_{i,i+1} - a_{ii}) > \prod_{i \in \mathbb{Z}_3} (a_{ii} - a_{i,i-1})$ and $\det A < 0$, then the system (LV) is permanent.

(2) If $\prod_{i \in \mathbb{Z}_3} (a_{i,i+1} - a_{ii}) < \prod_{i \in \mathbb{Z}_3} (a_{ii} - a_{i,i-1})$, then the heteroclinic cycle $T$ is an attractor.

The above results determine the behaviors of the orbits in the neighborhood of $\partial \mathbb{R}_+^3$ in $\mathbb{R}_+^3$.

Now we shall define the structural stable-like idea on a stability of the system (LV).

Consider the system (LV) which is obtained by perturbing the system (LV) as follows:

$$(LV_{\epsilon}) \quad \dot{x}_i = \gamma_i x_i (1 + \sum_{j \in \mathbb{Z}_3} a_{ij} x_j + \epsilon_i \phi_i(x)), \quad i \in \mathbb{Z}_3,$$

where the $\phi_i$ are affine linear, the $|\epsilon_i|$ are sufficiently small and $\epsilon := (\epsilon_1, \epsilon_2, \epsilon_3)$.

**Definition 1.4.** We say a property of the system (LV) is robust, if it holds in the system (LV$_{\epsilon}$).

The following theorem is our first result.

**Theorem 1.5.** Consider a vector field of (LV$_1$) as in Theorem 1.1.

1. The global property that $x^*$ is a global attractor on $(\mathbb{R}_+^3)^o$ are robust.
2. The global property that $T$ is a global attractor on $(\mathbb{R}_+^3)^o \Gamma_\epsilon$ are robust.

Here $\Gamma_\epsilon$ is the one-dimensional stable manifold of the $x^*$. This result has the following corollary.

**Corollary 1.6.** Consider the system (LV) as in Theorem 1.3.[HS]. Then there exists sufficient small $\epsilon > 0$, such that for $|\gamma_i - \gamma_j| < \epsilon$ (i, j $\in \mathbb{Z}_3$), every orbit $\psi$ in $(\mathbb{R}_+^3)^o \Gamma_\epsilon$ satisfies one of the following:

1. If $\prod_{i \in \mathbb{Z}_3} (a_{i,i+1} - a_{ii}) > \prod_{i \in \mathbb{Z}_3} (a_{ii} - a_{i,i-1})$ and $\det A < 0$, then $\psi$ tends toward the equilibrium point $x^*$.
2. If $\prod_{i \in \mathbb{Z}_3} (a_{i,i+1} - a_{ii}) < \prod_{i \in \mathbb{Z}_3} (a_{ii} - a_{i,i-1})$ then $\psi$ winds toward the heteroclinic cycle $T$.

In the case (1) of Theorem 1.3.[HS], the system is permanent but the $x^*$ need not be locally stable. Infact it may be locally unstable with a certain condition. And with some additional conditions, the system may have a limit cycle which is robust in competitive systems. We shall show the above case in Proposition 2.2. and Corollary 2.3. as our second result.

**Proposition 2.2.** Consider the system (LV) with $a_{ii} = -1$, $\det A < 0$. Suppose that $\gamma_i x_i^*$ are constant $k$, $a_{i,i-1}$ (i $\in \mathbb{Z}_3$) have negative values (and not all the same) and satisfy the following conditions (C1) and (C2):

(C1) $\quad 1 + \prod_{i \in \mathbb{Z}_3} (1 + a_{i,i-1}) > 0.$

(C2) $\quad 8 + \prod_{i \in \mathbb{Z}_3} a_{i,i-1} < 0.$

If $a_{i,i+1}$ (i $\in \mathbb{Z}_3$) close enough to 0, then there exists a non-trivial $\omega$-limit set in $(\mathbb{R}_+^3)^o$. In particular, when $a_{i,i+1}$ (i $\in \mathbb{Z}_3$) close enough to 0 from below, the $\omega$-limit set is a limit cycle.
For our interests where the limit sets exist, we conclude by showing the existence of a positively invariant set which includes them in Theorem 3.1 and Corollary 3.2.

**Theorem 3.1.** Given system (LV) with \( a_{ij} + a_{ji} < 0 \) \((i, j \in \mathbb{Z}_3)\). If the set \( I \) satisfies (G1), (G2) and (G3), then the set \( I \) is positively invariant and the every orbit from \((R^3_+)^o\) has an \( \omega \)-limit in the set \( I \).

For details see §3.

§1. Proof of Theorem 1.5.

We denote by \((LV_{1\epsilon})\) the system \((LV_{1})\) with \( \gamma_1 = \gamma_2 = \gamma_3 > 0 \). The assertion of Theorem 1.5. means that, in the system \((LV_{1\epsilon})\), every orbit \( \psi \) from \((R^3_+)^o\) tends toward \( x^* \) or every orbit \( \psi \) from \((R^3_+)^o \setminus \Gamma_{\epsilon} \) winds toward \( T \) if \( \det A < 0 \) and \( \prod_{i \in \mathbb{Z}_3} (a_{ii+1} - a_{ii}) \neq 0 \).

**Lemma 1.7.** There exists an open set \( K_{\epsilon} \subset (R^3_+)^o \), a neighborhood \( n(\Gamma_1) \) of \( \Gamma_1 \) and a smooth scalar function \( G(x) \) on \((R^3_+)^o \setminus \Gamma_1\) such that \((LV_{1\epsilon})\) is transverse to \( C_{\theta} := \{ x \in (R^3_+)^o : G(x) = \theta \} \) for any \( \theta \) in \((0, \pi)\), and \( \dot{G} \) has a constant sign on \( K_{\epsilon} \setminus n(\Gamma_1) \).

Proof of Lemma 1.7. We consider the projected vector field \((\hat{x})\) of \((LV_{1})\) on \( S^2_\theta := \{ x \in R^3_+ : |x| = 1 \} \) as follows:

\[
\hat{x} = F - |x|^{-2}(x \cdot F)x,
\]

where \( F = (F_1, F_2, F_3) \) is the vector field of the system \((LV_{1})\). We denote by \( \gamma_x(t) \) the orbit of \((\hat{x})\) through the point \( x \) at \( t = 0 \).

We define the map \( \psi : R \times S^2_+ \to S^2_+ \) by \( \psi(t, x) = \gamma_x(t) \).

Then we have

\[
\psi(0, x) = x \quad x \in S^2_+
\]

and

\[
\psi(t_1 + t_0, x) = \psi(t_1, \psi(t_0, x)) \quad t_1, t_0 \in R.
\]

For each \( t \in R \) we have a map

\[
\overline{\psi_t} : S^2_+ \to S^2_+
\]

defined by

\[
\overline{\psi_t}(x) = \psi(t, x) \quad (t, x) \in R \times S^2_+.
\]

We shall consider the system \((LV_{1})\) in the case (1). Because \( x^* \) is locally asymptotically stable, in the vector field \((\hat{x})\) there exists a positive real number \( d \) such that \((\hat{x})\) is transverse to the closed curve \( \overline{C_d} \) inward, where \( \overline{C_d} := \{ x \in (S^2_+)\setminus: |x - x^*| = d \} \).

For each \( t < 0 \) the closed curve \( \overline{\psi_t}(\overline{C_d}) = \{ \overline{\psi_t}(x) \in (S^2_+)\setminus: x \in \overline{C_d} \} \) is smooth because \( \overline{\psi_t} \) is a diffeomorphism. For each \( t < 0 \), let

\[
C_\theta := \bigcup_{s > 0} s \overline{\psi_t}(\overline{C_d}) \subset (R^3_+)^o,
\]

where \( \theta = \cot^{-1} t \).

Now we consider the system \((LV_{1\epsilon})\) in the case (1). We define the function \( G : (R^3_+)^o \to R \) as follows:

\[
G(x) = \begin{cases} 
\theta & (x \in (R^3_+)^o \setminus \Gamma_1), \\
0 & (x \in \Gamma_1).
\end{cases}
\]
We consider
\[ \dot{G} := \nabla G \cdot f = \sum_{i \in \mathbb{Z}^3} \frac{\partial G}{\partial x_i} \dot{x}_i, \]
where \( \nabla := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}) \) and \( f := (f_1, f_2, f_3) \) is the vector field of the system \((LV_1)\).

Hence for the sufficiently small \( \epsilon \), there exists a open set \( K_{\epsilon} \subset (\mathbb{R}_+^3)^{\circ} \) and the neighborhood \( n(\Gamma_1) \) of \( \Gamma_1 \) such that
\[ \dot{G} < 0 \quad \text{on} \quad K_{\epsilon} \setminus n(\Gamma_1). \]

Similarly in the case \((2)\), we have
\[ \dot{G} > 0 \quad \text{on} \quad K_{\epsilon} \setminus n(\Gamma_1). \]

\[ \square \]

**Proof of Theorem 1.5.** First we consider sail-like surfaces \( D_r := \{ x \in (\mathbb{R}_+^3)^{\circ} : D(x) = r \} \) and a family of cone-like annuluses \( C_{\theta} \), where \( D(x) := |x|^2 \) on \((\mathbb{R}_+^3)^{\circ}\) and \( G(x) \) are smooth scalar functions defined in Lemma 1.7.

For \( r_1, r_2 (0 < r_1 < r_2 < \infty) \) and \( \theta_1, \theta_2 (0 < \theta_1 < \theta_2 < \pi) \) we suppose the domain:
\[ D_{(r_1, r_2)} := \{ x : r_1 < D(x) < r_2 \}, \quad C_{(\theta_1, \theta_2)} := \{ x : \theta_1 < G(x) < \theta_2 \}. \]

Clearly we have
\[ D_{(0,p)} \subseteq D_{(0,q)}, \quad C_{(0,p)} \subseteq C_{(0,q)}, \]
for \( p < q \). And we have
\[ \lim_{\theta \to 0} C_\theta = \Gamma_1, \quad \lim_{\theta \to \pi} C_\theta = \partial \mathbb{R}_+^3 \setminus \{ \infty \} \quad \text{and} \quad \lim_{r \to 0} D_r = \{ 0 \}. \]

We shall consider the system \((LV_1)\). If \( \det A < 0 \) and \( \prod_{i \in \mathbb{Z}^3} (a_{i,i+1} - a_{ii}) + \prod_{i \in \mathbb{Z}^3} (a_{i,i-1} - a_{ii}) > 0 \), then the system \((LV_1)\) is permanent by Theorem 1.3. Therefore there is some bounded domain \( I(q) \subset (\mathbb{R}_+^3)^{\circ} \) such that for any orbit \( \psi \) from any point in \( I(q) \) and sufficiently large \( t \), \( \psi(t) \in (\mathbb{R}_+^3)^{\circ} \setminus I(q) \).

Hence there is some \( \delta, \delta' (0 < \delta < \delta' < \infty) \) such that for any orbit \( \psi \) from any point in \((\mathbb{R}_+^3)^{\circ}\), \( \liminf_{t \to -\infty} \psi^i(t) > \delta \) and \( \limsup_{t \to -\infty} \psi^i(t) < \delta' \) \((i \in \mathbb{Z}^3)\), where the \( \psi^i(t) \) are the components of \( \psi(t) \).

When we consider the system \((LV_\epsilon)\), there exists some domain \( I(q)_\epsilon \), the construction of which is similar to that of \( I(q) \) in the \((LV_1)\), because the permanence of the system is robust in this case (see [HS]). In addition to existence of \( I(q)_\epsilon \), from the Appendix 1 if \( \epsilon \) is enough close to 0, then \( I(q)_\epsilon \) is close to \( I(q) \) enough.

Now we consider the system \((LV_{1\epsilon})\) in the case \((1)\). For sufficiently large \( l \) we have
\[ \dot{D} \geq D(1 - l(x_1 + x_2 + x_3)), \]
where \( \dot{D} := \nabla D \cdot f \).

Therefore for sufficiently small \( m \) we have
\[ \dot{D}|_{D_{(0,m)}} > 0. \]

On the other hand for sufficiently large \( L \) and \( T \) we have
\[ \psi(T) \in D_{(0,L)} \quad \text{if} \quad \psi(0) \in D_{(L,\infty)}, \]
where \( \psi(t) \) is the orbit of the system \((LV_{1\epsilon})\).
In the system (LV), if necessary we shall exchange once, there exists a family \( \{C_{\theta}\} \), such that \( \hat{\dot{C}} < 0 \) on \( \{x \in C_{\theta} : 0 < \theta < \infty\} \). Therefore in the system \((LV_{1e})\), for any \( \tau, \bar{\tau} (0 < \tau < \bar{\tau} < \infty) \) and any \( \theta, \bar{\theta} (0 < \theta < \bar{\theta} < \pi) \), if necessary by resetting \( \epsilon_{i} \) enough small, then there is \( r_{1}, r_{2} \) \( (0 < r_{1} < \tau < \bar{\tau} < r_{2} < \infty) \) and \( \theta_{1}, \theta_{2} \) \( (0 < \theta_{1} < \theta < \bar{\theta} < \theta_{2} < \pi) \) such that

\[
\dot{\hat{C}} < 0 \quad \text{on} \quad D_{(r_{1}, r_{2})} \cap C_{(\theta_{1}, \theta_{2})},
\]

and

\[
D_{(0, r_{1})} \cup D_{(r_{2}, \infty)} \cap C_{(\theta_{2}, \pi)} \subset I(g).
\]

On the other hand, there exists a tubular neighborhood \( n(\Gamma_{\epsilon}) \) of one dimensional manifold \( \Gamma_{\epsilon} \) which is tangent to the eigenspace spanned by an eigenvector with real eigen value of the Jacobian matrix \( D_{x_{\epsilon}^{*}}f \) of \( f \) at \( x_{\epsilon}^{*} \) such that for any orbit \( \psi \) from any point in \( n(\Gamma_{\epsilon}) \) we have \( \omega(\psi) = x_{\epsilon}^{*} \), where \( \omega(\cdot) \) is \( \omega \)-limit set of \( \cdot \) and \( x_{\epsilon}^{*} \) is an interior equilibrium point of the system \((LV_{\epsilon})\).

Now we define a domain \( J_{\epsilon} \):

\[
J_{\epsilon} := \max_{r_{i}^{\epsilon}, \theta} \min_{r_{i}^{\epsilon}} \{ x \in (R_{+}^{3})^{\circ} : x \in D_{(r_{1}, r_{2})} \cap C_{(0, \theta)} \subset n(\Gamma_{\epsilon}) \}.
\]

If necessary by resetting \( \epsilon_{i} \) more close to 0, \( r_{1}^{\epsilon} \) can be close to 0 enough, \( r_{2}^{\epsilon} \) can be large enough and \( \theta' \) can be close to \( \pi \) enough, namely for any \( r_{1}, r_{2} (> 0) \) and \( \theta_{1} \in (0, \pi) \), there are \( \epsilon_{i} \) \( (0 < |\epsilon_{i}| \ll 1) \), such that \( 0 < r_{1}^{\epsilon} < r_{1} < r_{2} < r_{2}^{\epsilon} < \infty \) and \( 0 < \theta_{1} < \theta' < \pi \). Hence,

\[
C_{(0, \theta_{1})} \subset \{I(g)_{\epsilon} \cup J_{\epsilon} \} \subset \{I(g)_{\epsilon} \cup n(\Gamma_{\epsilon})\}.
\]

Therefore by (1), (2) and (3), for any orbit \( \psi \) from any point in \((R_{+}^{3})^{\circ}\), we have \( \omega(\psi) = x_{\epsilon}^{*} \).

In the other case that \( \prod_{i \in Z_{3}} (a_{i,i+1} - a_{ii}) + \prod_{i \in Z_{3}} (a_{ii} - a_{i,i-1}) < 0 \), the proof is done similarly.

This theorem has the following corollary.

**Corollary 1.6.** Consider the system \((LV)\) as in Theorem 1.3.[HS]. Then there exists sufficient small \( \epsilon > 0 \), such that for \( |\gamma_{i} - \gamma_{j}| < \epsilon \) \( (i,j \in Z_{3}) \), every orbit \( \psi \) in \((R_{+}^{3})^{\circ} \Gamma_{\epsilon} \) satisfies one of the following:

1. If \( \prod_{i \in Z_{3}} (a_{i,i+1} - a_{ii}) > \prod_{i \in Z_{3}} (a_{ii} - a_{i,i-1}) \) and \( \det A < 0 \), then \( \psi \) tends toward the equilibrium point \( x^{*} \).
2. If \( \prod_{i \in Z_{3}} (a_{i,i+1} - a_{ii}) < \prod_{i \in Z_{3}} (a_{ii} - a_{i,i-1}) \) then \( \psi \) winds toward the heteroclinic cycle \( T \).

§2. EXISTENCE OF A LIMIT CYCLE

In another case that the system is not so near the system \((LV_{1})\), an \( \omega \)-limit set which is another type one in Theorem 1.1. is happened. In some case, it is a limit cycle. We shall show these cases in Proposition 2.2.

**Proposition 2.2.** Consider the system \((LV)\) with \( a_{ii} = -1 \), \( \det A < 0 \). Suppose that \( \gamma_{i}x_{i}^{*} \) are constant \( k \), \( a_{i,i-1} \) \( (i \in Z_{3}) \) have negative values \( (\text{and not all the same}) \) and satisfy the following conditions (C1) and (C2):

\[
(C1) \quad 1 + \prod_{i \in Z_{3}} (1 + a_{i,i-1}) > 0.
\]
\[ 8 + \prod_{i \in \mathbb{Z}_3} a_{i,i-1} < 0. \]

If \( a_{i,i+1} (i \in \mathbb{Z}_3) \) close enough to 0, then there exists a non-trivial \( \omega \)-limit set in \((\mathbb{R}^3_+)\). In particular, when \( a_{i,i+1} (i \in \mathbb{Z}_3) \) close enough to 0 from below, the \( \omega \)-limit set is a limit cycle.

The above proposition has the following corollary.

**Corollary 2.3.** Consider the system \((LV)\) with \( a_{ii} = -1, \) \( \det A < 0, \) \( \gamma_i > 0 \) and \( x^* \in (\mathbb{R}^3_+) \) \((i \in \mathbb{Z}_3)\). Suppose that \( a_{i,i-1} \) are negative and not all the same, and the parameters of the system holds the following condition (D1) and (D2):

(D1) \[ \prod_{i \in \mathbb{Z}_3} (1 + a_{i,i+1}) + \prod_{i} (1 + a_{i,i-1}) > 0. \]

(D2) \[ (\sum_{i \in \mathbb{Z}_3} \gamma_i x^*_i) \left( \sum_{i \neq j \in \mathbb{Z}_3} \gamma_i \gamma_j x^*_i x^*_j (1 - a_{ij} a_{ji}) \right) + (\prod_{i \in \mathbb{Z}_3} \gamma_i x^*_i) \det A < 0. \]

Then there exists a non-trivial \( \omega \)-limit set in \((\mathbb{R}^3_+)\). In particular, if \( a_{i,i+1} < 0 \) \((i \in \mathbb{Z}_3)\), then the \( \omega \)-limit set is a limit cycle.

**§3. Statement and Proof of Theorem 3.1.**

For our interests where the limit set exists, we shall consider the following set \( I \) and conditions (G1), (G2) and (G3).

\[ I := \{ x \in \mathbb{R}^3_+ : C_{\min} \leq \frac{x_1}{\gamma_1} + \frac{x_2}{\gamma_2} + \frac{x_3}{\gamma_3} \leq C_{\max} \} \]

where

\[ C_{\min} := \min(C'_{\min}, C_{\parallel}) \quad \text{and} \quad C_{\max} := \max(C'_{\max}, C_{\parallel}). \]

Here

\[ C'_{\min} := \inf\{ c > 0 : (G1) \cap (G2) \}, \quad C'_{\max} := \sup\{ c > 0 : (G1) \cap (G2) \} \]

and \( C_{\parallel} \) is defined in (G3).

(G1): For some \( i,j \in \mathbb{Z}_3 \) \((i \neq j)\),

\[ (1 - c\gamma_i)(1 - c\gamma_j) < 0. \]

(G2): For some \( i,j \in \mathbb{Z}_3 \) \((i \neq j)\) if

\[ \gamma_i^2 + (a_{ij} + a_{ji}) \gamma_i \gamma_j + \gamma_j^2 > 0, \]

\[ c(2\gamma_i^2 + (a_{ij} + a_{ji}) \gamma_i \gamma_j) > \gamma_i - \gamma_j \quad \text{and} \quad c(2\gamma_j^2 + (a_{ij} + a_{ji}) \gamma_i \gamma_j) > \gamma_j - \gamma_i \]

holds, then

\[ h_c(1 - c\gamma_i) < 0 \quad \text{or} \quad h_c(1 - c\gamma_j) < 0, \]

where

\[ h_c := \frac{\gamma_i^2 \gamma_j^2 (a_{ij} + a_{ji} + 2)(a_{ij} + a_{ji} - 2)c^2 + 2\gamma_i \gamma_j (a_{ij} + a_{ji} + 2)(\gamma_i \gamma_j)c + (\gamma_i - \gamma_j)^2}{\gamma_i^2 + (a_{ij} + a_{ji}) \gamma_i \gamma_j + \gamma_j^2}. \]
(G3): We put $\gamma a_1, \gamma a_\gamma, s$ and $\bar{x}$ as follows.

$$1a_1 := (1,1,1) (A + tA)^{-1} t(1,1,1),$$

$$\gamma a_\gamma := \left( \frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3} \right) (A + tA)^{-1} t(1,1,1).$$

Thus

$$1a_\gamma := (1,1,1) (A + tA)^{-1} t \left( \frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3} \right) = \gamma a_1.$$

And

$$s := -(1a_1/\gamma a_\gamma)^{1/2},$$

$$\bar{x} := \{s(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}) -(1,1,1)\} (A + tA)^{-1}.$$

If $\bar{x} \in (R_+^3)^\circ$, then

$$C_{\bar{x}} = -1a_\gamma - (1a_1 \cdot \gamma a_\gamma)^{1/2}, \quad (8)$$

where $\cdot$ is the ordinary multiplication of numbers.

**Theorem 3.1.** Given system (LV) with $a_{ij} + a_{ji} < 0$ ($i,j \in Z_3$). If the set $I$ satisfies (G1), (G2) and (G3), then the set $I$ is positively invariant and the every orbit from $(R_+^3)^\circ$ has an $\omega$-limit in the set $I$.

**Proof** We consider the function $S$ as follows:

$$S(x) := \frac{x_1}{\gamma_1} + \frac{x_2}{\gamma_2} + \frac{x_3}{\gamma_3},$$

on $R_+^3$. Thus the differential of $S$ is

$$\dot{S} = x_1 + x_2 + x_3 + x A^t x. \quad (9)$$

Define a plane $S_c$ for $c > 0$ and a quadratic surface $Q$ as follows:

$$S_c := \{x \in R_+^3 : S(x) = c\},$$

$$Q := \{x \in R_+^3 : \dot{S} = 0\}.$$

Remark that the surface $Q$ includes all equilibrium points of the system (LV). For enough large $l \in R$ and the matrix $E_1$ whose all elements are equal to 1,

$$\dot{S} \geq x_1 + x_2 + x_3 - l x E_1 t x = (x_1 + x_1 + x_2)(1 - l(x_1 + x_2 + x_3)).$$

Thus for an enough small $c' > 0$,

$$\dot{S}|_{x \in S_{c'}} > 0 \text{ on } (R_+^3)^\circ.$$

And for the identity matrix $E_0$,

$$\dot{S} \leq x_1 + x_2 + x_3 - x E_0 t x = x_1(1 - x_1) + x_2(1 - x_2) + x_3(1 - x_3).$$

Thus for an enough large $C' > 0$, 


\[ \dot{S}|_{x \in S_{c}}' < 0 \text{ on } (R^{3}_{+})^{\circ}. \]

If for some \( c > 0 \)

\[ Q \cap S_{c} \cap R^{3}_{+} = \emptyset, \]

then for an arbitrary initial value \( \psi(0) \in R^{3}_{+} \cap S_{c} \) and arbitrary \( t > 0 \),

\[ \psi(t) \in \{ x : S(x) > c \} \text{ or } \psi(t) \in \{ x : S(x) < c \}. \]

Here we denote the minimum value and maximum value of \( c \) such that \( Q \cap S_{c} \cap R^{3}_{+} \neq \emptyset \) by \( C_{\min} \) and \( C_{\max} \) respectively.

Now \( \gamma_{i} \) and \( a_{ij} \ (i,j \in Z_{3}) \) are fixed. The quadratic curve \( Q \cap S_{c} \) does not change the type of it (but may degenerate).

Hence we shall consider the value of \( c \) such that this quadratic curve \( Q \cap S_{c} \) and the compact set \( S_{c} \cap R^{3}_{+} \) have a common point.

In the case of (G1) and (G2), we shall consider the common set of \( S_{c} \cap \partial R^{3} \) and \( Q \cap S_{c} \), where \( \partial R^{3} := \{ x \in R^{3} : \prod_{i}x_{i} = 0 \} \). We may assume \( x_{3} = 0 \) without loss of generality. So if

\[ \{ x \in R^{3}_{+} : x \in Q \cap S_{c} \text{ and } x_{3} = 0 \} \subset \{ x \in R^{3}_{+} : x \in S_{c} \cap \partial R^{3} \}, \]

then \( Q \cap S_{c} \cap R^{3}_{+} \neq \emptyset \).

It becomes a simple problem where the solution of the quadratic equation exist.

In the case of (G3), we can not determine the value of \( C_{\min} \) and \( C_{\max} \) in this way. But just on \( C_{\min} \) or \( C_{\max} \), the following equality holds.

\[ s \nabla S_{c} = \nabla Q|_{\{ x : s_{c} = C_{\min} \text{ or } C_{\max} \}}, \quad (10) \]

where \( s \neq 0 \in R \).

Since \( \nabla S_{c} = \left( \frac{1}{\gamma_{1}}, \frac{1}{\gamma_{2}}, \frac{1}{\gamma_{3}} \right) \), and \( Q \) is quadratic, the point \( \tilde{x} \in R^{3} \) which holds (10) are less than two or equal two. For those points, the just value of \( C \) such that \( \tilde{x} \in S_{c} \) is \( C_{\min} \) or \( C_{\max} \).

We shall calculate in the state as follows:

\[ \nabla Q = g + xD_{x}g = (1,1,1) + (x_{1},x_{2},x_{3})(A + {}^{t}A), \]

where we put the system (LV), \( x_{i} = \gamma_{i}x_{i}g_{i} \ (i \in Z_{3}) \) and \( g := (g_{1},g_{2},g_{3}) \).

From the equation (10),

\[ \tilde{x} = \{ s(\frac{1}{\gamma_{1}}, \frac{1}{\gamma_{2}}, \frac{1}{\gamma_{3}}) - (1,1,1) \}(A + {}^{t}A)^{-1}. \quad (11) \]

Since \( \tilde{x} \in Q \),

\[ (1,1,1)\tilde{x} + \frac{1}{2}\tilde{x}(A + {}^{t}A)^{t}x = 0. \quad (12) \]

By (11) and (12),

\[ s^{2} = a_{1}a_{\gamma}. \quad (13) \]

When \( \tilde{x} \in R^{3}_{+} \), we may put \( \tilde{c} := \{ c : \tilde{x} \in S_{c} \} \).

Thus

\[ \tilde{c} = (\frac{1}{\gamma_{1}}, \frac{1}{\gamma_{2}}, \frac{1}{\gamma_{3}})^{t}\tilde{x} = -(\frac{1}{\gamma_{1}}, \frac{1}{\gamma_{2}}, \frac{1}{\gamma_{3}})(A + {}^{t}A)^{-1} {}^{t}(1,1,1) - (a_{1} \cdot a_{\gamma})^{\frac{1}{2}}, \quad (14) \]
where we put \( s = -(1a_1/\gamma a_\gamma)^{1/2} \) by (13), because if \( s = (1a_1/\gamma a_\gamma)^{1/2} \) then \( \tilde{c} \leq 0 \). The value of \( \tilde{c} \) is \( C_{\min} \) or \( C_{\max} \) in the case. □

For arbitrary \( \gamma'_i > 0 \ (i \in \mathbb{Z}_3) \), we shall consider the set:

\[
I_{S_\gamma} := \{ x \in \mathbb{R}^3_+ : C_{\min} \leq \frac{x_1}{\gamma'_1} + \frac{x_2}{\gamma'_2} + \frac{x_3}{\gamma'_3} \leq C_{\max} \}.
\]

Here the value of \( C_{\min} \) and \( C_{\max} \) are the minimum and the maximum value respectively such that

\[
Q_\gamma \cap S_\gamma^c \cap \mathbb{R}^3_+ \neq \emptyset,
\]

where

\[
S_\gamma^c := \{ x \in \mathbb{R}^3 : S_\gamma := \frac{x_1}{\gamma'_1} + \frac{x_2}{\gamma'_2} + \frac{x_3}{\gamma'_3} = c \},
\]

\[
\dot{S}_\gamma = \frac{x_1}{\gamma'_1}x_1 + \frac{x_2}{\gamma'_2}x_2 + \frac{x_3}{\gamma'_3}x_3 + x \text{ diag}\left(\frac{\gamma'_1}{\gamma'_1}, \frac{\gamma'_2}{\gamma'_2}, \frac{\gamma'_3}{\gamma'_3}\right) A^t x,
\]

\[
Q_\gamma := \{ x \in \mathbb{R}^3_+ : \dot{S}_\gamma = 0 \}.
\]

Here \( \text{diag}\{\cdot\} \) is a diagonal matrix with components \( \cdot \). Remark that all equilibrium points of the system (LV) are included in \( Q_\gamma \), too.

**Corollary 3.2.** Consider the system as in Theorem 3.1. For an arbitrary vector \( \gamma' := (\gamma'_1, \gamma'_2, \gamma'_3) \ (\gamma'_i > 0 \ i \in \mathbb{Z}_3) \), we put

\[
I := \bigcap_{\gamma} I_{S_\gamma}.
\]

Then the set \( I \) is positively invariant, and every orbit from \((\mathbb{R}^3_+)^o\) has an \( \omega \)-limit in the set \( I \).

Remark that in the case of (G3), it is well known that if the \( Q_\gamma \) is elliptic, then the \( x^* \) is exactly a global attractor on \((\mathbb{R}^3_+)^o\).

**REFERENCES**


