Keller-Segel 系の爆発解の挙動について

原田剛宇 (Go Harada)(大阪大、理、M)
鈴木貴 (Takashi Suzuki)(大阪大、理)

0 Introduction

We consider the behavior of blow-up solutions for (KS)

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot (\nabla u - \chi u \nabla v) \quad \text{in } \Omega, \; t > 0 \\
\tau \frac{\partial v}{\partial t} &= \Delta v - \gamma v + \alpha u \quad \text{in } \Omega, \; t > 0 \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega, \; t > 0 \\
u(\cdot, 0) &= u_0, \; v(\cdot, 0) = v_0 \quad \text{on } \Omega
\end{align*}
\]

Here \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) with smooth boundary \( \partial \Omega \), and \( \tau, \chi, \gamma, \) and \( \alpha \) are positive constants, and \( u_0, v_0 \) are nonnegative, nontrivial, smooth functions on \( \overline{\Omega} \).

In what follows we denote \( \| \cdot \|_{L^p(\Omega)} = \| \cdot \|_p, \; M = \| u_0 \|_1 \),

\[
\int_{\Omega} f dx = \frac{1}{|\Omega|} \int_{\Omega} f dx, \; D := \{ x \in \mathbb{R}^2 \mid |x| < 1 \}, \text{ and let } T \text{ be the maximal existence time of solution } (u, v).
\]

Theorem 1.1 in [1] states:
If \( M < \frac{4\pi}{\alpha \chi} \), then the solution \( (u, v) \) exists globally in time and globally bounded.
If \( \Omega = \{ x \in \mathbb{R}^2 \mid |x| < L \} \) and \( (u_0, v_0) \) is radial in \( x \), and \( M < \frac{8\pi}{\alpha \chi} \), then the solution \( (u, v) \) exists globally in time and globally bounded.

Then what happens if \( \frac{4\pi}{\alpha \chi} \leq M < \frac{8\pi}{\alpha \chi} \) and \( (u_0, v_0) \) is non radially symmetric? For simplicity, we put \( \alpha = \gamma = \chi = 1 \), and \( \Omega = D \).

Theorem 2 in [7] and Lemma 9 in [7] states:
Let \( \tau = 0, \; \Omega = D, \) and \( M < 8\pi \). If \( T < \infty \), then there exists \( x_0 \in \partial D \).
satisfying

\[ \lim \inf_{t \to T} \int_{D \cap B(x, \epsilon)} u(x, t) dx \geq 4\pi \text{ for any } \epsilon > 0. \]

In this paper, we consider to extend this result to \( \tau > 0 \). A main result is following.

**Theorem** Let \( \tau > 0 \), \( \Omega = D \), and \( M < 8\pi \). If \( T < \infty \), then there exists a continuous map \( p(t) : [0, T) \to \partial D \) satisfying

\[ \lim \sup_{t \to T} \int_{D \cap B(p(t), \epsilon)} u(x, t) dx \geq 2\pi \text{ for any } \epsilon > 0. \]

**1 Fundamental Lemmas for Theorem**

Following Lemmas are known.

**Lemma 1** The following holds:

\[ \| u(\cdot, t) \|_1 = \| u_0 \|_1, \]

and

\[ \| v(\cdot, t) \|_1 = e^{-\frac{t}{\tau}} \| v_0 \|_1 + \| u_0 \|_1 (1 - e^{-\frac{t}{\tau}}). \]

**Lemma 2** Put

\[ W(t) = \int_{\Omega} u \log u - uv + \frac{1}{2} (| \nabla v |^2 + v^2) dx. \]

Then we have

\[ \frac{dW}{dt}(t) + \tau \int_{\Omega} v^2 dx + \int_{\Omega} u | \nabla (\log u - v) |^2 dx = 0, \]

and it follows that

\[ \frac{dW}{dt}(t) \leq 0, \text{ and } W(t) \leq W(0). \]

**Lemma 3** Let \( M = \| u_0 \|_1 \). The following holds:

\[ a \int_{\Omega} uv dx \leq \int_{\Omega} u \log u dx + M \log \frac{1}{M} \int_{\Omega} e^{av} dx \text{ for any } a > 0. \]

**Lemma 4** (Corollary of Proposition[3]-2.3)

Let \( \Omega = D \). There exists \( C_\Omega \) such that

\[ \int_{\Omega} e^w dx \leq C_\Omega \exp \left( \frac{1}{8\pi} \| \nabla w \|_2^2 + \frac{1}{|\Omega|} \| w \|_1 \right) \text{ for any } 0 \leq w \in W^{1,2}(\Omega). \]
Proposition 4.8.1 Let $F$ be a set of $w(\cdot,t)(0 \leq t < T)$ such that $t \mapsto w(\cdot,t) \in H^1(D)$ is continuous and $\sup_{0 \leq t < T} \| w(\cdot,t) \|_{L^1(D)} < \infty$, then either one of the following holds:

1. There exists $\{t_k\} \nearrow T$ such that $w_k = w(\cdot,t_k) \in F$ satisfying the following.
   For any $\epsilon$, there exists $C_\epsilon$ such that
   $$\log \left( \int_D e^{w_k} \, dx \right) \leq \frac{1 + \epsilon}{16\pi} \int_D |\nabla w_k|^2 \, dx + C_\epsilon.$$  

2. There exists a continuous map $t \mapsto q(t) \in \partial D$ such that
   $$\lim_{t \to \tau} \frac{\int_{D \cap B(q(t),\epsilon)} \exp(w(x,t)) \, P_\ast(x) \, dx}{\int_D \exp(w(x,t)) \, P_\ast(x) \, dx} \geq \frac{1}{2}$$ for any $\epsilon > 0$,
   where $P_\ast(x) = \frac{8}{(1 + |x|^2)^2}$.

Brézis-Merle Type Inequality for Parabolic Equations of Second Order

We consider the following problem:

$$\begin{cases} 
\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{j=1}^2 b_j(x,t) \frac{\partial u}{\partial x_j} + c(x,t)u = f & \text{in } \Omega \times (0,T) \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \times (0,T) \\
u(x,0) = u_0(x) & \text{in } \Omega
\end{cases}$$

Let $b_j, c \in H^{\alpha,\frac{\alpha}{2}}(\bar{\Omega} \times [0,T])$ and $q \in \partial \Omega, \text{where } \alpha \text{ is a real number with } 0 < \alpha < 1$ and $h$ belongs to $H^{\alpha,\frac{\alpha}{2}}(\bar{\Omega} \times [0,T])$ if

$$|h(x,t) - h(y,s)| \leq \text{Const.} \left( |x - y|^{\alpha} + |t - s|^{\frac{\alpha}{2}} \right)$$

for any $(x,t), (y,s) \in \bar{\Omega} \times [0,T]$. Given $0 < \tau < T$ and $0 < \epsilon < 2\pi\nu$, there exist positive constants $\eta \in (0, 1/4)$ and $C > 0$ depending on $\tau, \epsilon, \eta \in (0, \eta_0), \| u_0 \|_{L^1(\Omega)}$, and $\| f \|_{L^1(\Omega \times (0,T))}$ such that $\eta \in (0, \eta_0)$ and $\sup_{0 < t < T} \| f(\cdot,t) \|_{L^1(\Omega \times (0,T))} \leq 2\pi\nu - \epsilon$ imply

$$\int_{B_q(x,\eta)} e^{u(x,t)} \, dx \leq C \text{ for } \tau \leq t \leq T,$$

where $u$ denote the solution of the above problem.

Proposition 1 The following holds:

1. $T < \infty$ implies $\lim_{t \to T} \int_{\Omega} uvdx = \infty$. 


\( (2) \) \( T < \infty \) implies \( \lim_{t \to T} \int_{\Omega} e^{av} dx = \infty \) for any \( a > \frac{M + \sqrt{M^2 - 4\pi M}}{2M} \).

\( (3) \) \( T < \infty \) implies \( \lim_{t \to T} \int_{\Omega} |\nabla v|^2 dx = \infty \).

2 Proof of Proposition 1

Before proving Proposition 1, we remark that \( T < \infty \) implies \( M \geq 4\pi \) by the contraposition of Theorem 1.1 in [1], so in the root sign \( M^2 - 4\pi M \) is not negative.

Proof of Proposition 1

Theorem 1 in [5] shows that \( T < \infty \) implies

\[
\lim_{t \to T} \| uv \|_1 = \lim_{t \to T} \| e^{av} \|_1 = \lim_{t \to T} \| \nabla v \|_2^2 = \lim_{t \to T} \| u \log u \|_1 = \infty \text{ for any } a > 1.
\]

So we prove only (2). From Lemma 3 and Lemma 4 with \( w = av \), we have

\[
ad \int_{\Omega} uv dx \leq \int_{\Omega} u \log u dx + \frac{Ma^2}{8\pi} \int_{\Omega} |\nabla v|^2 dx + C \text{ for any } a > 0 . \tag{2.1}
\]

From Lemma 2,

\[
\int_{\Omega} u \log u - uv + \frac{1}{2}(|\nabla v|^2 + v^2) dx \leq W(0). \tag{2.2}
\]

By (2.1) + \( \frac{Ma^2}{4\pi} \) (2.2),

\[
\left( a - \frac{Ma^2}{4\pi} \right) \int_{\Omega} uv dx \leq \left( 1 - \frac{Ma^2}{4\pi} \right) \int_{\Omega} u \log u dx + C \text{ for any } a > 0.
\]

Put \( a = \frac{M + \sqrt{M^2 - 4\pi M}}{M} \) in the above inequality, then

\[
\int_{\Omega} u \log u dx \leq \frac{M + \sqrt{M^2 - 4\pi M}}{2M} \int_{\Omega} uv dx + C.
\]

Using this and Lemma 3, we have

\[
\left( a - \frac{M + \sqrt{M^2 - 4\pi M}}{2M} \right) \int_{\Omega} uv dx \leq M \log \frac{1}{M} \int_{\Omega} e^{av} dx + C \text{ for any } a > 0.
\]

Since \( \lim_{t \to T} \int_{\Omega} uv dx = \infty \),

\[
\lim_{t \to T} \int_{\Omega} e^{av} dx = \infty \text{ for any } a > \frac{M + \sqrt{M^2 - 4\pi M}}{2M}.
\]
Remark
1. Proposition 3.1 in [6] shows that $\|v(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C$ for any $q \in (1, 2)$. By using this and Hölder's inequality and Sobolev's imbedding theorem, we have
\[
\int_{\Omega} uv dx \leq \|u\|_p \|v\|_{p'} \leq C \|u\|_p \text{ for any } p > 1.
\]
So, it follows from Proposition 1(1) that $T < \infty$ implies
\[
\lim_{t \to T} \|u(\cdot, t)\|_p = \infty \text{ for any } p > 1.
\]

3 Proof of Theorem

Proof of Theorem
Suppose the first alternative (1) of Proposition 4-8.1 holds, then there exists $\{t_k\} \nearrow T$ such that $v_k = v(\cdot, t_k)$ satisfy the following:
\[
\log \left( \frac{1}{\pi} \int_{D} e^{v(x,t_k)} dx \right) \leq \frac{1 + \epsilon}{16\pi} \int_{D} |\nabla v(x, t_k)|^2 dx + C_{\epsilon} \text{ for any } \epsilon > 0. \quad (3.1)
\]
From Lemma 2 and Lemma 3 with $a = 1$, we have
\[
\frac{1}{2} \int_{D} |\nabla v|^2 dx \leq W(0) + M \log \frac{1}{M} \int_{D} e^{v} dx \quad (3.2)
\]
By $(3.1) + (3.2)$,
\[
\left( \frac{1}{2} - \frac{1 + \epsilon}{16\pi} M \right) \int_{D} |\nabla v(x, t_k)|^2 dx \leq W(0) - M \log M + M \log \pi + C_{\epsilon} M.
\]
Since $M < 8\pi$, We can take $\epsilon$ such that
\[
\frac{1}{2} - \frac{1 + \epsilon}{16\pi} M > 0.
\]
Then
\[
\int_{D} |\nabla v(x, t_k)|^2 dx < \infty.
\]
This contradicts to Proposition 1.
Therefore the second alternative (2) of Proposition 4-8.1 holds. Then there exists a continuous map $t \in [0, T) \mapsto q(t) \in \partial D$ such that
\[
\liminf_{t \to T} \frac{\int_{D \cap B(x, \epsilon)} e^{v} P_*(x) dx}{\int_{D} e^{v} P_*(x) dx} \geq \frac{1}{2} \text{ for any } \epsilon > 0. \quad (3.3)
\]
Since $P_*(x) = \frac{8}{(1 + |x|^2)^2}$, $x \in D$ implies $2 \leq P_*(x) \leq 8$.
From Proposition 1
\[
\lim_{t \to T} \int_{D} e^{v} dx = \infty,
\]
it follows from (3.3) that

$$\lim_{t \to T} \int_{D \cap B(q(t), \epsilon)} e^u \, dx = \infty \text{ for any } \epsilon > 0. \quad (3.4)$$

(a) In case that there exists \( q \in \partial D \) such that \( q(t) \to q \) \((t \to T)\).

We suppose for this \( q(t) \) there exists \( \eta_1 \) such that

$$\limsup_{t \to T} \int_{D \cap B(q(t), \eta_1)} u \, dx < 2\pi.$$  

Then there exists \( \epsilon > 0 \) such that

$$\limsup_{t \to T} \int_{D \cap B(q(t), \eta_1)} u \, dx \leq 2\pi - \epsilon,$$

and there exists \( t_0 \) such that \( t_0 < t < T \) implies

$$\int_{D \cap B(q(t), \eta_1)} u \, dx \leq 2\pi - \frac{\epsilon}{2}.$$

Because of the continuity of \( q(t) \), for this \( \eta_1 \) there exists \( T_1 \) such that \( t > T_1 \)

implies \(| q(t) - q | < \frac{\eta_1}{2} \). Since \( B(q, \frac{\eta_1}{2}) \subset B(q(t), \eta_1) \),

\( t > \max\{T_0, T_1\} =: T_2 \) implies

$$\int_{D \cap B(q, \frac{\eta_1}{2})} u \, dx \leq 2\pi - \frac{\epsilon}{2}.$$

That is

$$\int_{D \cap B(q, \frac{\eta_1}{2})} u \, dx \leq 2\pi - \frac{\epsilon}{2} \text{ for any } \eta \in (0, \eta_1).$$

By using Brézis-Merle's inequality, given \( t_0 \in (T_2, T) \) there exists \( \eta_0 \in (0, \min\{\eta_1, \frac{1}{4}\}) \) and \( C = C(t_0, \epsilon, \eta) > 0(\eta \in (0, \eta_0)) \) such that \( \eta \in (0, \eta_0) \) implies

$$\int_{D \cap B(q, \frac{\eta_0}{2})} e^u \, dx \leq C \text{ for any } t \in [t_0, T].$$

This contradicts to (3.4). Therefore

$$\limsup_{t \to T} \int_{D \cap B(q(t), \eta)} u \, dx \geq 2\pi \text{ for any } \eta > 0.$$  

Put \( p(t) = q(t) \).

(b) In case that there doesn't exist \( q \in \partial D \) such that \( q(t) \to q \) \((t \to T)\).

Put

\[ A := \{ \gamma \in \partial D \mid \text{for any } T_0 < T \text{ there exists } t \in (T_0, T) \text{ such that } q(t) = \gamma \}. \]

For any \( \gamma \in A \), by the definition of \( A \) and (3.4), we have

$$\limsup_{t \to T} \int_{D \cap B(\gamma, \epsilon)} e^u \, dx = \infty \text{ for any } \epsilon > 0. \quad (3.5)$$
We suppose for this $\gamma$ there exists $\eta_1$ such that

$$\limsup_{t \to T} \int_{D \cap B(\gamma, \eta_1)} udx < 2\pi.$$  

Then there exists $\epsilon > 0$ such that

$$\limsup_{t \to T} \int_{D \cap B(\gamma, \eta_1)} udx \leq 2\pi - \epsilon,$$

and there exists $T_0$ such that $T_0 < t < T$ implies

$$\int_{D \cap B(\gamma, \eta_1)} udx \leq 2\pi - \frac{\epsilon}{2}.$$  

That is

$$\int_{D \cap B(\gamma, \eta)} udx \leq 2\pi - \frac{\epsilon}{2} \text{ for any } \eta \in (0, \eta_1).$$  

By using Brézis-Merle's inequality, given $t_0 \in (T_0, T)$ there exists $\eta_0 \in (0, \min\{\eta_1, \frac{\epsilon}{2}\})$ and $C = C(t_0, \epsilon, \eta) > 0(\eta \in (0, \eta_0))$ such that $\eta \in (0, \eta_0)$ implies

$$\int_{D \cap B(\gamma, \frac{\eta}{2})} e^v dx \leq C \text{ for any } t \in [t_0, T].$$  

This contradicts to (3.5). Therefore

$$\limsup_{t \to T} \int_{D \cap B(\gamma, \eta)} udx \geq 2\pi \text{ for any } \eta > 0.$$  

Put $p(t) = \gamma$.

**Remark**

1. We use Proposition1(2) with $a = 1$ to prove Theorem. But using $a > \frac{M + \sqrt{M^2 - 4\pi M}}{2M}$, we can improve the constant $2\pi$ to a larger one in Theorem, which is now studying.

2. If $M = 4\pi$, then $W(t)$ is bounded from below by putting $a = 1$, $M = 4\pi$ in (2.1). So when this, it follows from [6] that limsup can be changed to liminf in Theorem.

3. Theorem is correct even if $\Omega$ is a simply connected bounded domain in $\mathbb{R}^2$ with smooth boundary.

**References**


