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Difference Equation for A Population Model

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1. Introduction.

Recently many kinds of population models are formulated with difference euations [**1**,3,4]. Here we consider the following difference equation :

$$u(t+2) = \alpha u(t+1) + \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)}, \tag{1.1}$$

seems to be a general statements of a relative socio-spatial dynamics, where $\alpha = 1 + r$, in which r is the net (births minus death) endogenous population (stock) growth rate, and the second term is a function depicting net immigration at $t+1$, in turn a function of a "momentum" to grow from t to $t+1$. So that we assume that $\alpha > 0$ $(r > -1)$ and $\beta > 0$ in (1.1).

The model (1.1) was proposed by Prof. Dimitrios Dendrinos [**2**]. Let

$$u(t+2) = u_1(t+2) + u_2(t+2),$$

where $u_1(t+2) = \alpha\{u_1(t+1) + u_2(t+1)\}$, $u_2(t+2) = \beta\frac{u_2(t+1)}{u_1(t+1)}$. In this model we assume that α and β are constants. $u_1(t+2)$ is a term for endogenous population growth rate from $t+1$ to $t+2$, and $u_2(t+2)$ due to net in-migration rate.

Here we will study it, at first, from the viewpoint of (real) state-space analysis, especially of stability properties as $t \to \infty$, for positive solutions. Next we study it from the complex analytic view point.

2. Stability properties of positive solutions

In this section, we suppose that $u(t)$ is a solution of (1.1) which is positive for $t \geqq t_0$, with the initial values $u(t_0)$ and $u(t_0+1)$.

We say $u(t)$ is *stable* if there are positive constants L_1, L_2 such that

$$0 < L_1 \leqq u(t_0+n) \leqq L_2 \quad \text{for } n \in \mathbb{N}.$$

$u(t)$ is *asymptotically stable* if there is b_0 such that, for any $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$|u(t_0+n) - b_0| < \epsilon \quad \text{for } n \geqq n_0, \quad \text{i.e.} \quad u(t_0+n) \to b_0 \ (n \to \infty).$$

We write (1.1) as

$$(2.1) \qquad u(t+2) - \alpha u(t+1) = \frac{c}{u(t)}(u(t+1) - \alpha u(t)), \quad c = \frac{\beta}{\alpha}.$$

2.1. When $\alpha > 1$

Suppose $u(t_0+1) - \alpha u(t_0) \geqq 0$ for some t_0. Then we have $u(t_0+n) \to \infty$ $(n \to \infty)$ and the solution $u(t)$ is not stable. So if the solution $u(t)$ is stable, we assume that $u(t+1) - \alpha u(t) < 0$ for any $t \geqq 0$.

(i) Suppose $u(t_0) \geqq c$. If $u(t_0+1) > u(t_0)$, then we have $u(t) \to +\infty$. Therefore we must have that $u(t+1) \leqq u(t)$ whenever $u(t) \geqq c$. If $u(t)$ is stable and $u(t) \geqq c$ for any t, then c is an asymptotically stable equilib. point : $u(t) \downarrow c$.

(ii) Suppose $u(t_0) < c$. If $u(t_0+1) < u(t_0)$, then $u(t_0+2) < u(t_0+1) < u(t_0) \leqq c$, and $u(t_0+n) \to -\infty$. So that $u(t)$ is not stable. Therefore, if $u(t)$ is stable, we must have

(2.2)

$$u(t+1) \leqq u(t) \text{ whenever } u(t) > c, \qquad u(t+1) \geqq u(t) \text{ whenever } u(t) < c.$$

Conversely, if (2.2) holds, then $u(t) \to c$ as $t \to \infty$. Therefore we see that $u(t)$ is stable if and only if $u(t)$ is asymptotically stable, and c is the only stable equilib. point.

Indeed we can show the existence of a positive asymptotically stable solution from the section 4.

2.2. When $\alpha < 1$

Suppose $u(t_0+1) - \alpha u(t_0) \leqq 0$, for some t_0. Then we have $u(t_0+n) \to 0$ and the population tends to death. So we may assume that $u(t+1) - \alpha u(t) > 0$ for any t.

(i)When $u(t_0) \leq c$ for some t_0. We have

$$\alpha(u(t_0+1) - u(t_0)) \leqq u(t_0+2) - u(t_0+1).$$

(i-a) Suppose $u(t_0+1) - u(t_0) \geqq 0$ and there is k_0 such that $u(t_0+k) \leqq c \, (0 \leqq k < k_0)$ and $u(t_0+k_0) > c$. Furthermore if we assume $u(t_0+k_0+n+1) - u(t_0+k_0+n) \geqq 0$ for any $n \geqq 1$, we would have

$$u(t_0+k_0+n) \uparrow u_0 < \infty (u_0 > c).$$

This is contradiction. Hence there is n_0 such that $u(t_0+k_0+n) \leqq u(t_0+k_0+n+1)$ $(n < n_0)$ and $u(t_0+k_0+n_0) > u(t_0+k_0+n_0+1)$.

(i-b) Suppose $u(t_0+1) < u(t_0) \leqq c$. If $u(t_0+n) \leqq u(t_0+n-1)$ $(n \geqq 1)$, then $u(t_0+n) \downarrow 0$.

If there is k_0 such that $u(t_0+k) \leqq u(t_0+k-1)$ $(k < k_0)$ and $u(t_0+k_0) > u(t_0+k_0-1)$, we suppose there is h_0 such that $u(t_0+k_0+h) \leqq c$ $(h < h_0)$ and $u(t_0+k_0+h_0) > c$. Then we have $u(t_0+k_0+h_0+1) > u(t_0+k_0+h_0) > c$. Now we proceed to the following case (ii).

(ii) When $u(t_0) > c$. Then we have

$$\alpha\{u(t_0+1) - u(t_0)\} > u(t_0+2) - u(t_0+1). \tag{2.3}$$

(ii-a) Suppose $u(t_0+1) > u(t_0) > 0$. If $u(t_0+n) > u(t_0+n-1)\,(n \geq 1)$, we have

$$c < u(t_0+n) \uparrow u_0 < \frac{u(t_0+1)}{1-\alpha},$$

which is a contradiction. Thus there is n_0 such that

$$u(t_0+n) > u(t_0+n-1)\,(n < n_0) \quad \text{and} \quad u(t_0+n+1) < u(t_0+n).$$

Thus we proceed to the following case (ii-b).

(ii-b) Suppose $u(t_0+1) < u(t_0)$. Then $u(t_0+2) < u(t_0+1)$ by (2.3). If $u(t_0+n) \geqq c$, then we get $u(t_0+n) \downarrow c$. If $u(t_0+n) \geqq c\,(n < n_0)$ and $u(t_0+n_0) < c$, then we come back to the case (i).

Thus we see that, when $0 < \alpha < 1$, we have either $u(t+n) \to 0$ or $u(t)$ is stable.

2.3. When $\alpha = 1$

We have in this case

$$u(t+2) - u(t+1) = \frac{c}{u(t)}(u(t+1) - u(t))$$

Note that,

$$\text{if } \lim_{n\to\infty} u(t_0+n) = u_0 \neq \pm\infty, \quad \text{then } \lim_{n\to\infty} \frac{u(t_0+n+1)}{u(t_0+n)} = 1.$$

(i) When $u(t_0+1) > u(t_0)$. If $u(t_0) \geqq c$. Then we have

$$u(t_0+n) \uparrow u_0 \leqq \frac{(2-b)u(t_0+1) - u(t_0)}{1-b}, \quad (u_0 > c).$$

If $u(t_0) < c$. Hereafter $u(t_0+n)$ behaves as above and we get $u(t_0+n) \uparrow u_0 < \infty$.

(ii) When $u(t_0+1) < u(t_0)$.
If $u(t_0) \leqq c$, and $u(t_0+n) > 0$. $u(t_0+n)$ oscillate.
If $u(t_0) > c$ and $u(t_0+n_1) \leqq c$ for some n_1, then $u(t_0+n)$ oscillate.
If $u(t_0) > c$ and $u(t_0+n) > c$ for all n, then there is a u_0 such that $u(t_0+n) \downarrow u_0 \geqq c$. Therefore, when $\alpha = 1$, we get that

$$\text{either} \quad u(t_0+n_0) \leqq 0 \ (\exists n_0), \quad \text{or} \quad u(t_0+n) \downarrow u_0 \geqq c, \quad \text{or} \quad u(t_0+n) \uparrow u_0 > c.$$

3. Analytic Solution

3.1. When $\alpha > 1$

We have showed that, if $u(t_0+n)$ is stable, then $u(t_0+n) \to c$ as $n \to \infty$. Then $v(t) = u(t) - c$ satisfies $v(t_0+n) \to 0$ and

$$v(t+2) = (\alpha+1)v(t+1) - v(t) + F(v(t), v(t+1)), \tag{3.1}$$

where $F(v_1, v_2)$ is the sum of terms of higher degree with respect to v_1, v_2. Let λ_1, λ_2, be roots of the characterristic equation

$$\lambda^2 - (\alpha+1)\lambda + 1 = 0, \quad (0 < \lambda_2 < 1 < \lambda_1). \tag{3.2}$$

If we suppose $u(t+n)$ is stable when $n \to \infty$. Therefore we consider a paticular solution of (1.1) in the form

$$u(t_0+t) = c + \sum_{k=1}^{\infty} a_k \lambda_2^{kt}. \tag{3.3}$$

If we suppose $u(t+n)$ is stable when $n \to -\infty$, then we also consider solutions in the form

$$u(t_0+t) = c + \sum_{k=1}^{\infty} a_{-k} \lambda_2^{-kt}. \tag{3.4}$$

Furthermore, if $u(t_0-n) \to 0$ $(n \to \infty)$, then we have

$$\lim_{n\to-\infty} \frac{u(t+n+2)}{u(t+n+1)} = \alpha.$$

And we have a solution $u(t) = a_1\alpha^t$ (a_1 : arbitrary). Suppose $x(t) = \phi(t)\alpha^t$ be a solution of (1.1) such that $x(t-n) \to 0$ as $n \to +\infty$, and ϕ satisfies $\lim_{n\to\infty} |\phi(t-n)-a_1| < M$ uniformly on any compact set for some constant M. Then we have $\lim_{n\to\infty}(\phi(t-n+1) - \phi(t-n)) = 0$.

3.2. When $\alpha < 1$
There is no asymptotically stable solution.

3.3. When $\alpha = 1$
Suppose $u(t_0 + n) \uparrow u_0$ or $u(t_0 + n) \downarrow u_0$ with $u_0 > c$, then we have

$$u(t_0 + t + 1) - u(t_0 + t) = \sum_{k=1}^{\infty} d_k \left(\frac{c}{u_0}\right)^{kt}, \quad \frac{c}{u_0} < 1$$

and $u(t_0 + t)$ will be obtained.

4 Existence of analytic stable solutions in the case $\alpha > 1$

In this section, we write $u(t_0 + t)$ simply as $u(t)$. Time t is of course a real variable. But in this section, we consider t to be a complex variable, and we will prove existance of analytic solutions.

When $\alpha > 1$, we can determined a formal solution to (3.1) in the form (4.1),

$$v(t) = \sum_{n=1}^{\infty} a_n \lambda^{nt}, \; 0 < \lambda = \lambda_2 < 1, \tag{4.1}$$

where

$$a_n\beta\{\lambda^{2n} - (1+\alpha)\lambda^n + 1\} = \alpha \sum_{k=1}^{n-1} a_k a_{n-k} \lambda^{n-k}(\alpha - \lambda^{n-k}).$$

4.1. Make a Map T and existence of a fixed poit of T.
We rewrite (3.1) as

$$v(t) = \beta \frac{(1+\alpha)v(t+1) - v(t+2)}{\alpha v(t+2) - \alpha^2 v(t+1) + \beta} = f(v(t+1), v(t+2)). \tag{4.2}$$

Let N be a positive integer. Put $P_N(t) = \sum_{n=1}^{N} a_n \lambda^{nt}$ and $y(t) = v(t) - P_N(t)$. We rewrite (4.2) as

$$y(t) = g(t, y(t+1), y(t+2)) \tag{4.3}$$

Put

$$\begin{cases} S(\rho) = \{t \in \mathbb{C} : |\lambda^t| \leq \rho\} \\ H(A,\rho) = \{y : y(t) \text{ is holomorphic and } |y(t)| \leqq A|\lambda^t|^{N+1} \text{ for } t \in S(\rho)\}. \end{cases}$$

Take $A > 0$ and $0 < \rho < 1$, which will be determined later. For $y(t) \in H(A,\rho)$, put

$$T[y](t) = g(t, y(t+1), y(t+2)). \tag{4.4}$$

Then we can show that T has a fixed point $y(t) = y_N(t) \in H(A,\rho)$ by schauder's fixed point theorem in [**5**].

Furthermore we can show the uniqueness of the fixed poit and arbitrariness of N (indepence of N). Thus we have proved that a solution $v(t)$ is defined and holormorphic in $S(\rho)$ for a $\rho > 0$, which has the expansion (4.1).

4.2. Analytic General Solutions.

Analytic solutions of some difference equations are investigated in [**6**]-[**9**]. In this section, we shall investigate analytic general solutions of (1.1).

From in [**6**], we have following theorem.

THEOREM 1. *Let $v(s)$ be the solution of* (4.2) *obtained in section* **3**. *Suppose $x(t)$ be an analytic solution of* (4.2) *such that $x(t-n) \to 0$ as $n \to +\infty$, uniformly on any compact set. Then there is a periodic entire function $\pi(t)$, ($\pi(t+1) = \pi(t)$), such that*

$$x(t) = v\left(t + \frac{\log \pi(t)}{\log \lambda}\right).$$

Conversely if we put

$$x(t) = v\left(t + \frac{\log \pi(t)}{\log \lambda}\right),$$

where π is a periodic function whose period is one, then $x(t)$ is a solution of (3.1).

Now we have sought general solutions of the population model which is given by the equation (1.1) such that

$$u(t) = \frac{\beta}{\alpha} + v\left(t + \frac{\log \pi(t)}{\log \lambda}\right)$$

where $\pi(t)$ is an arbitrarily peroidic function whose period is one.

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