Difference Equation for A Population Model

Mami Suzuki (鈴木麻美)  Teikyo Heisei University

1. Introduction.

Recently many kinds of population models are formulated with difference equations [1,3,4]. Here we consider the following difference equation:

\[(1.1) \quad u(t+2) = \alpha u(t+1) + \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)},\]

seems to be a general statements of a relative socio-spatial dynamics, where \(\alpha = 1 + r\), in which \(r\) is the net (births minus death) endogenous population (stock) growth rate, and the second term is a function depicting net immigration at \(t+1\), in turn a function of a "momentum" to grow from \(t\) to \(t+1\). So that we assume that \(\alpha > 0\) (\(r > -1\)) and \(\beta > 0\) in (1.1).

The model (1.1) was proposed by Prof. Dimitrios Dendrinos [2]. Let

\[u(t+2) = u_1(t+2) + u_2(t+2),\]

where \(u_1(t+2) = \alpha \{u_1(t+1) + u_2(t+1)\},\) \(u_2(t+2) = \beta \frac{u_2(t+1)}{u_1(t+1)}\). In this model we assume that \(\alpha\) and \(\beta\) are constants. \(u_1(t+2)\) is a term for endogenous population growth rate from \(t+1\) to \(t+2\), and \(u_2(t+2)\) due to net in-migration rate.

Here we will study it, at first, from the viewpoint of (real) state-space analysis, especially of stability properties as \(t \to \infty\), for positive solutions. Next we study it from the complex analytic view point.

2. Stability properties of positive solutions

In this section, we suppose that \(u(t)\) is a solution of (1.1) which is positive for \(t \geq t_0\), with the initial values \(u(t_0)\) and \(u(t_0+1)\).

We say \(u(t)\) is stable if there are positive constants \(L_1, L_2\) such that

\[0 < L_1 \leq u(t_0 + n) \leq L_2 \quad \text{for} \ n \in \mathbb{N}.\]
$u(t)$ is asymptotically stable if there is $b_0$ such that, for any $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$|u(t_0 + n) - b_0| < \epsilon \quad \text{for} \quad n \geq n_0,$$

i.e. $u(t_0 + n) \to b_0 \quad (n \to \infty)$.

We write (1.1) as

$$(2.1) \quad u(t + 2) - \alpha u(t + 1) = \frac{c}{u(t)}(u(t + 1) - \alpha u(t)), \quad c = \frac{\beta}{\alpha}.$$  

2.1. When $\alpha > 1$

Suppose $u(t_0 + 1) - \alpha u(t_0) \geq 0$ for some $t_0$. Then we have $u(t_0 + n) \to \infty \quad (n \to \infty)$ and the solution $u(t)$ is not stable. So if the solution $u(t)$ is stable, we assume that $u(t + 1) - \alpha u(t) < 0$ for any $t \geq 0$.

(i) Suppose $u(t_0) \geq c$. If $u(t_0 + 1) > u(t_0)$, then we have $u(t) \to +\infty$. Therefore we must have that $u(t + 1) \leq u(t)$ whenever $u(t) \geq c$. If $u(t)$ is stable and $u(t) \geq c$ for any $t$, then $c$ is an asymptotically stable equilib. point : $u(t) \downarrow c$.

(ii) Suppose $u(t_0) < c$. If $u(t_0 + 1) < u(t_0)$, then $u(t_0 + 2) < u(t_0 + 1) < u(t_0) \leq c$, and $u(t_0 + n) \to -\infty$. So that $u(t)$ is not stable. Therefore, if $u(t)$ is stable, we must have

$$(2.2) \quad u(t + 1) \leq u(t) \quad \text{whenever} \quad u(t) > c, \quad u(t + 1) \geq u(t) \quad \text{whenever} \quad u(t) < c.$$  

Conversely, if (2.2) holds, then $u(t) \to c$ as $t \to \infty$. Therefore we see that $u(t)$ is stable if and only if $u(t)$ is asymptotically stable, and $c$ is the only stable equilib. point.

Indeed we can show the existence of a positive asymptotically stable solution from the section 4.

2.2. When $\alpha < 1$

Suppose $u(t_0 + 1) - \alpha u(t_0) \leq 0$, for some $t_0$. Then we have $u(t_0 + n) \to 0$ and the population tends to death. So we may assume that $u(t+1) - \alpha u(t) > 0 \quad \text{for any} \quad t$.

(i) When $u(t_0) \leq c$ for some $t_0$. We have

$$\alpha(u(t_0 + 1) - u(t_0)) \leq u(t_0 + 2) - u(t_0 + 1).$$

(i-a) Suppose $u(t_0 + 1) - u(t_0) \geq 0$ and there is $k_0$ such that $u(t_0 + k) \leq c (0 \leq k < k_0)$ and $u(t_0 + k_0) > c$. Furthermore if we assume $u(t_0 + k_0 + n + 1) - u(t_0 + k_0 + n) \geq 0$ for any $n \geq 1$, we would have

$$u(t_0 + k_0 + n) \uparrow u_0 < \infty (u_0 > c).$$
This is contradiction. Hence there is \( n_0 \) such that \( u(t_0 + k_0 + n) \leq u(t_0 + k_0 + n + 1) \) \((n < n_0)\) and \( u(t_0 + k_0 + n_0) > u(t_0 + k_0 + n_0 + 1)\).

(i-b) Suppose \( u(t_0 + 1) < u(t_0) \leq c \). If \( u(t_0 + n) \leq u(t_0 + n - 1) \) \((n \geq 1)\), then \( u(t_0 + n) \downarrow 0 \).

If there is \( k_0 \) such that \( u(t_0 + k) \leq u(t_0 + k - 1) \) \((k < k_0)\) and \( u(t_0 + k_0) > u(t_0 + k_0 - 1)\), we suppose there is \( h_0 \) such that \( u(t_0 + k_0 + h) \leq c \) \((h < h_0)\) and \( u(t_0 + k_0 + h_0) > c\). Then we have \( u(t_0 + k_0 + h_0 + 1) > u(t_0 + k_0 + h_0) > c\).

Now we proceed to the following case (ii).

(ii) When \( u(t_0) > c \). Then we have

\[
\alpha\{u(t_0 + 1) - u(t_0)\} > u(t_0 + 2) - u(t_0 + 1).
\]

(ii-a) Suppose \( u(t_0 + 1) > u(t_0) > 0 \). If \( u(t_0 + n) > u(t_0 + n - 1) \) \((n \geq 1)\), we have

\[
c < u(t_0 + n) \uparrow u_0 < \frac{u(t_0 + 1)}{1 - \alpha},
\]

which is a contradiction. Thus there is \( n_0 \) such that

\[
u(t_0 + n) > u(t_0 + n - 1) \) \((n < n_0) \) and \( u(t_0 + n + 1) < u(t_0 + n) \).

Thus we proceed to the following case (ii-b).

(ii-b) Suppose \( u(t_0 + 1) < u(t_0) \). Then \( u(t_0 + 2) < u(t_0 + 1) \) by (2.3). If \( u(t_0 + n) \geq c \), then we get \( u(t_0 + n) \downarrow c \). If \( u(t_0 + n) \geq c \) \((n < n_0)\) and \( u(t_0 + n_0) < c \), then we come back to the case (i).

Thus we see that, when \( 0 < \alpha < 1 \), we have either \( u(t + n) \to 0 \) or \( u(t) \) is stable.

2.3. When \( \alpha = 1 \)

We have in this case

\[
u(t + 2) - u(t + 1) = \frac{c}{u(t)}(u(t + 1) - u(t))
\]

Note that,

\[
\text{if } \lim_{n \to \infty} u(t_0 + n) = u_0 \neq \pm \infty, \text{ then } \lim_{n \to \infty} \frac{u(t_0 + n + 1)}{u(t_0 + n)} = 1.
\]

(i) When \( u(t_0 + 1) > u(t_0) \). If \( u(t_0) \geq c \). Then we have

\[
u(t_0 + n) \uparrow u_0 \leq \frac{(2 - b)u(t_0 + 1) - u(t_0)}{1 - b}, \quad (u_0 > c).
\]
If $u(t_0) < c$. Hereafter $u(t_0 + n)$ behaves as above and we get $u(t_0 + n) \uparrow u_0 < \infty$.

(ii) When $u(t_0 + 1) < u(t_0)$.
If $u(t_0) \leq c$, and $u(t_0 + n) > 0$. $u(t_0 + n)$ oscillate.
If $u(t_0) > c$ and $u(t_0 + n_1) \leq c$ for some $n_1$, then $u(t_0 + n)$ oscillate.
If $u(t_0) > c$ and $u(t_0 + n) > c$ for all $n$, then there is a $u_0$ such that $u(t_0 + n) \downarrow u_0 \geq c$. Therefore, when $\alpha = 1$, we get that

either $u(t_0 + n_0) \leq 0 (\exists n_0)$, or $u(t_0 + n) \downarrow u_0 \geq c$, or $u(t_0 + n) \uparrow u_0 > c$.

3. Analytic Solution

3.1. When $\alpha > 1$

We have showed that, if $u(t_0 + n)$ is stable, then $u(t_0 + n) \rightarrow c$ as $n \rightarrow \infty$.
Then $v(t) = u(t) - c$ satisfies $v(t_0 + n) \rightarrow 0$ and

\[(3.1) \quad v(t + 2) = (\alpha + 1)v(t + 1) - v(t) + F(v(t), v(t + 1)),\]

where $F(v_1, v_2)$ is the sum of terms of higher degree with respect to $v_1, v_2$.
Let $\lambda_1, \lambda_2$, be roots of the characteristic equation

\[(3.2) \quad \lambda^2 - (\alpha + 1)\lambda + 1 = 0, \quad (0 < \lambda_2 < 1 < \lambda_1).\]

If we suppose $u(t + n)$ is stable when $n \rightarrow \infty$. Therefore we consider a particular solution of (1.1) in the form

\[(3.3) \quad u(t_0 + t) = c + \sum_{k=1}^{\infty} a_k \lambda_2^kt.\]

If we suppose $u(t + n)$ is stable when $n \rightarrow -\infty$, then we also consider solutions in the form

\[(3.4) \quad u(t_0 + t) = c + \sum_{k=1}^{\infty} a_{-k} \lambda_2^{-kt}.\]

Furthermore, if $u(t_0 - n) \rightarrow 0 (n \rightarrow \infty)$, then we have

\[\lim_{n \rightarrow -\infty} \frac{u(t + n + 2)}{u(t + n + 1)} = \alpha.\]
And we have a solution \( u(t) = a_1 \alpha^t \) (\( a_1 \) : arbitrary). Suppose \( x(t) = \phi(t) \alpha^t \) be a solution of (1.1) such that \( x(t - n) \to 0 \) as \( n \to +\infty \), and \( \phi \) satisfies \( \lim_{n \to \infty} |\phi(t - n) - a_1| < M \) uniformly on any compact set for some constant \( M \). Then we have \( \lim_{n \to \infty} (\phi(t - n + 1) - \phi(t - n)) = 0 \).

3.2. **When \( \alpha < 1 \)**

There is no asymptotically stable solution.

3.3. **When \( \alpha = 1 \)**

Suppose \( u(t_0 + n) \uparrow u_0 \) or \( u(t_0 + n) \downarrow u_0 \) with \( u_0 > c \), then we have

\[
u(t_0 + t + 1) - u(t_0 + t) = \sum_{k=1}^{\infty} d_k \left( \frac{c}{u_0} \right)^{kt}, \quad \frac{c}{u_0} < 1
\]

and \( u(t_0 + t) \) will be obtained.

4. **Existence of analytic stable solutions in the case \( \alpha > 1 \)**

In this section, we write \( u(t_0 + t) \) simply as \( u(t) \). Time \( t \) is of course a real variable. But in this section, we consider \( t \) to be a complex variable, and we will prove existence of analytic solutions.

When \( \alpha > 1 \), we can determined a formal solution to (3.1) in the form (4.1),

\[
v(t) = \sum_{n=1}^{\infty} a_n \lambda^{nt}, \quad 0 < \lambda = \lambda_2 < 1,
\]

where

\[
a_n \beta \{\lambda^{2n} - (1 + \alpha)\lambda^n + 1\} = \alpha \sum_{k=1}^{n-1} a_k a_{n-k} \lambda^{n-k} (\alpha - \lambda^{n-k}).
\]

4.1. **Make a Map \( T \) and existence of a fixed point of \( T \).**

We rewrite (3.1) as

\[
v(t) = \beta \frac{(1 + \alpha)v(t + 1) - v(t + 2)}{\alpha v(t + 2) - \alpha^2 v(t + 1) + \beta} = f(v(t + 1), v(t + 2)).
\]

Let \( N \) be a positive integer. Put \( P_N(t) = \sum_{n=1}^{N} a_n \lambda^{nt} \) and \( y(t) = v(t) - P_N(t) \). We rewrite (4.2) as

\[
y(t) = g(t, y(t + 1), y(t + 2))
\]
Put
\[
S(\rho) = \{ t \in \mathbb{C} : |\lambda^t| \leq \rho \}
\]
\[
H(A, \rho) = \{ y : y(t) \text{ is holomorphic and } |y(t)| \leq A|\lambda^t|^{N+1} \text{ for } t \in S(\rho) \}.
\]
Take \( A > 0 \) and \( 0 < \rho < 1 \), which will be determined later. For \( y(t) \in H(A, \rho) \), put
\[(4.4) \quad T[y](t) = g(t, y(t+1), y(t+2)).\]
Then we can show that \( T \) has a fixed point \( y(t) = y_N(t) \in H(A, \rho) \) by schauder's fixed point theorem in [5].
Furthermore we can show the uniqueness of the fixed poit and arbitrariness of \( N \) (indepened of \( N \)). Thus we have proved that a solution \( v(t) \) is defined and holormorphic in \( S(\rho) \) for a \( \rho > 0 \), which has the expansion (4.1).

4.2. Analytic General Solutions.
Analytic solutions of some difference equations are investigated in [6]-[9]. In this section, we shall investigate analytic general solutions of (1.1).

From in [6], we have following theorem.

**Theorem 1.** Let \( v(s) \) be the solution of (4.2) obtained in section 3. Suppose \( x(t) \) be an analytic solution of (4.2) such that \( x(t-n) \to 0 \) as \( n \to +\infty \), uniformly on any compact set. Then there is a periodic entire function \( \pi(t) \), \((\pi(t+1) = \pi(t))\), such that
\[
x(t) = v\left( t + \frac{\log \pi(t)}{\log \lambda} \right).
\]

Conversely if we put
\[
x(t) = v\left( t + \frac{\log \pi(t)}{\log \lambda} \right),
\]
where \( \pi \) is a periodic function whose period is one, then \( x(t) \) is a solution of (3.1).

Now we have sought general solutions of the population model which is given by the equation (1.1) such that
\[
u(t) = \frac{\beta}{\alpha} + v\left( t + \frac{\log \pi(t)}{\log \lambda} \right)
\]
where \( \pi(t) \) is an arbitrarily peroidic function whose period is one.
REFERENCES