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Uniqueness of Periodic Solutions to Periodic Linear Functional Differential Equations with Finite Delay

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Abstract

We investigate criteria for the uniqueness of (mild) periodic solutions to periodic linear functional differential equations with finite delay in Banach spaces. Its arguments are carried out by materializing the theory of semi-Fredholm operators and by using the standard way. In particular, two sufficient conditions ensuring the uniqueness of periodic solutions are obtained : they are independent of each other.

1 Introduction and preliminaries

Let R be a real line and E a Banach space with a norm $|\cdot|$. Let r be a given positive number. If $x : R \to E$, then a function $x_t : [-r, 0] \to E, t \in R$, is defined by $x_t(\theta) = x(t+\theta), \theta \in [-r, 0]$. We deal with the linear functional differential equation with finite delay in the Banach space E of the form

$$\frac{dx(t)}{dt} = Ax(t) + L(t, x_t) + f(t). \tag{L}$$

Denote by $\mathcal{C} := C([-r, 0], E)$ the set of all continuous functions from $[-r, 0]$ to E with the suppremum norm. We assume that Eq.(L) always satisfies the following hypothesis(H) :

(i) $A : \mathcal{D}(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $T(t), t \geq 0$, on E, with the inequality $\|T(t)\| \leq M_w e^{-wt}, t \geq 0$, where $M_w \geq 1$ and $w > 0$;
(ii) $L : R \times \mathcal{C} \to E$ is continuous and $L(t, \cdot) : \mathcal{C} \to E$ is linear ;
(iii) $f : R \to E$ is continuous.

If $L(t, \psi)$ and $f(t)$ in Eq.(L) are periodic functions with a period $\omega > 0$, we denote Eq.(L) by Eq.($P_\omega L$). If $f(t) \equiv 0$, we denote Eq.(L) and Eq.($P_\omega L$) by Eq.(L_0) and Eq.($P_\omega L_0$), respectively.

The purpose of this paper is to investigate criteria for the uniqueness of (mild) periodic solutions to Eq.($P_\omega L$) in the relation between the delay r and the period ω by materializing the theory of semi-Fredholm operators, which is used in [9], and by using the standard way.

In 1974, Chow and Hale [1] gave the following fixed point theorem for linear affine maps to show the existence of periodic solutions to Eq.(P_ωL) with $A = 0$ and $E = R^n$. Let X be a Banach space and T a bounded linear affine map defined by $Tx := Sx + z$ for $x \in X$, where S is a bounded linear operator on X and $z \in X$ is fixed.

Theorem 1.1 *If the range $R(I - S)$, I being the identity, is closed and if there is an $x^0 \in X$ such that $\{x^0, Tx^0, T^2x^0, \cdots\}$ is bounded in X, then T has a fixed point in X.*

For the uniqueness of fixed points, the following result holds true.

Theorem 1.2 *If $1 \in \rho(S)$(the resolvent set of S), then T has a unique fixed point in X.*

More recently, Shin and Naito [9] studied the existence and the uniqueness of periodic solutions to Eq.(P_ωL) with infinite delay, in which the semi-Fredholm operator theory plays an important role.

Denote by $\mathcal{F}_T$ the set of the fixed points of the affine map T on X given in the above theorem. Then $\mathcal{F}_T = y + N(I - S)$ for some $y \in \mathcal{F}_T$: it is an affine space, where $N(B)$ stands for the null space of linear operator B. The dimension of the affine space $\mathcal{F}_T$ is defined as the dimension of the null space of $I - S$; that is, $\dim \mathcal{F}_T = \dim N(I - S)$. Denote by $\Phi_+(X)$ the family of semi-Fredholm operators on X ; that is, the family of linear bounded operators B such that $\dim N(B)$ is finite and the range $R(B)$ is closed. Using such semi-Fredholm operators, Theorem 1.1 is refined as follows, see [9].

Theorem 1.3 *Assume that there is an $x^0 \in X$ such that $\{x^0, Tx^0, T^2x^0, \cdots\}$ is bounded in X. If $I - S \in \Phi_+(X)$, then $\mathcal{F}_T \neq \emptyset$ and $\dim \mathcal{F}_T$ is finite.*

Remark Let S be a bounded linear operator on X. Then the following statements hold true.

1) If S is an α-contration operator on X, then $r_e(S) < 1$.
2) If $r_e(S) < 1$, then 1 is a normal point of S.
3) If 1 is a normal point of S, then $I - S \in \Phi_+(X)$.
4) If $I - S \in \Phi_+(X)$, then $R(I - S)$ is closed.

In order to apply the above result to Eq.(P_ωL), the perturbation theory of semi-Fredholm operators is needed, cf.[3], [7], [9, Proposition 7.3]. Let $F, K : X \to X$ be bounded linear operators.

Theorem 1.4 *Assume that $F \in \Phi_+(X)$ (; and hence, $\dim N(F) = n$ and $\|[x]\| \leq c|Fx|$ for $x \in X$ and for some $c > 0$, where $\|[x]\| = \inf\{|x+y| : y \in N(F)\}$). If $\|K\| \leq 1/2c(1+\sqrt{n})$, then $F + K \in \Phi_+(X)$, and $\dim N(F + K) \leq n$.*

To inverstigate criteria for the uniqueness of periodic solutions to Eq.(P_ωL), we will employ two manners. The first manner(Proposition 3.3) is concerned with the following result, which is the case where $n = 0$ in the above theorem.

Proposition 1.5 *Assume that* $1 \in \rho(F)$ (; *and hence,* $|x| \leq c|(I-F)x|$ *for* $x \in X$ *and for some* $c > 0$). *If* $\|K\| \leq 1/2c$, *then* $I-F+K$ *belongs to* $\Phi_+(X)$, *and* $\dim N(I-F+K) = 0$.

The second manner(Theorem 4.4) is based on the following result, which is an immediate consequence of Theorem 1.2.

Proposition 1.6 *Assume that* $1 \in \rho(F)$. *If* $1 \in \rho((I-F)^{-1}K)$, *then* $1 \in \rho(F+K)$.

We here emphasize that in general, our results (Proposition 3.3 and Theorem 4.4) on the uniqueness of periodic solutions to Eq.(P_ωL) are independent of each other.

2 Estimates of $\|K(t)\|$

We denote by $u(t,\sigma,\phi)$ and $u(t,\sigma,\phi,f)$, respectively, the mild solutions of Eq.(L_0) and Eq.(L) through $(\sigma,\phi) \in R \times \mathcal{C}$. Define the solution operator $U(t,\sigma)$, $t \geq \sigma$, on $\mathcal{C}$ for Eq.(L_0) by $U(t,\sigma)\phi = u_t(\sigma,\phi) := u(t+\cdot,\sigma,\phi)$. Put

$$z(t,\sigma,\phi) = \begin{cases} \displaystyle\int_\sigma^t T(t-s)L(s,u_s(\sigma,\phi))ds, & t \geq \sigma \\ 0, & \sigma - r \leq t \leq \sigma. \end{cases}$$

Then $z(t,\sigma,\phi)$ is continuous for $t \geq \sigma - r$. For $t \geq \sigma$ define an operator $K(t,\sigma)$ by $K(t,\sigma)\phi = z_t(\sigma,\phi)$ for $\phi \in \mathcal{C}$. Then it is a bounded linear operator on $\mathcal{C}$, and $U(t,\sigma)$ is decomposed as

$$U(t,\sigma) = \widehat{T}(t-\sigma) + K(t,\sigma), \tag{1}$$

where

$$(\widehat{T}(t)\phi)(\theta) = \begin{cases} T(t+\theta)\phi(0) & \text{for } t+\theta \geq 0 \\ \phi(t+\theta) & \text{for } t+\theta \leq 0. \end{cases}$$

Set $K(t) = K(t,0)$ and assume that $\|L\|_\infty := \sup\{\|L(t)\| : t \geq 0\} < \infty$, where $\|L(t)\| := \|L(t,\cdot)\|$.

Now we have the following estimates on $\widehat{T}(t)$ and $K(t)$.

Proposition 2.1 *Let* $\|T(t)\| \leq M_w e^{-wt}, w > 0,$ *for* $t \geq 0$. *Then*

$$\|\widehat{T}(t)\| \leq \begin{cases} M_w e^{-w(t-r)} & \text{for } t > r \\ M_w & \text{for } t \leq r. \end{cases} \tag{2}$$

Lemma 2.2 *Let* a, r *and* w *be positive constants, and let* $f, u : [0,\infty) \to R$ *be nonnegative continuous functions. Suppose that* $f(t)$ *is a nondecreasing function in* t *and that* $u(t)$ *satisfies the inequality*

$$u(t) \leq a \sup_{\max\{0,t-r\} \leq \tau \leq t} \int_0^\tau e^{-w(\tau-s)}u(s)ds + f(t).$$

Then the following inequality holds :

$$u(t) \leq wf(t)/(w-a) \quad \text{if} \quad w > a \quad \text{and} \quad t \geq 0.$$

Proposition 2.3 *The following estimate holds true for* $\|K(t)\|$.

$$\|K(t)\| \le M_w^2\|L\|_\infty(1-e^{-wt})/(w-M_w\|L\|_\infty) := k(t)$$

$$\text{if} \quad w > M_w\|L\|_\infty \quad \text{and} \quad t \ge 0.$$

Proof. Set $a = M_w\|L\|_\infty$. Using the decomposition (1) of the solution operator for Eq.(L_0), we have

$$\begin{aligned} |K(t)\phi| &\le M_w \sup_{\max\{0,t-r\}\le\tau\le t} \int_0^\tau e^{-w(\tau-s)}|L(s,U(s,0)\phi)|ds \\ &\le a \sup_{\max\{0,t-r\}\le\tau\le t} \{\int_0^\tau e^{-w(\tau-s)}|K(s)\phi|ds + \int_0^\tau e^{-w(\tau-s)}|\widehat{T}(s)\phi|ds\}. \end{aligned}$$

Furthermore, it follows from the estimate (2) of $\widehat{T}(t)$ that

$$\int_0^\tau e^{-w(\tau-s)}|\widehat{T}(s)\phi|ds \le \int_0^\tau e^{-w(\tau-s)}M_w|\phi|ds \le M_w(1-e^{-w\tau})|\phi|/w.$$

If we set $u(t) = |K(t)\phi|$, then

$$u(t) \le a \sup_{\max\{0,t-r\}\le\tau\le t} \int_0^\tau e^{-w(\tau-s)}u(s)ds + aM_w(1-e^{-wt})|\phi|/w.$$

Using Lemma 2.2, we have the estimate described in the proposition.

3 The uniqueness of periodic solutions (I)

To state criteria for the uniqueness of ω-periodic solutions for Eq.($\mathrm{P}_\omega\mathrm{L}$), a result on the boundedness of solutions of Eq.($\mathrm{P}_\omega\mathrm{L}$) is needed, cf.[9, Proposition 8.1, Theorem 8.2].

Lemma 3.1 *If* $M_w\|L\|_\infty < w$, *then every solution of Eq.*($\mathrm{P}_\omega\mathrm{L}$) *is bounded and every* ω*-periodic solution* $v(t)$ *of Eq.*($\mathrm{P}_\omega\mathrm{L}$) *is estimated as*

$$\|v\|_\infty \le \frac{M_w}{w - M_w\|L\|_\infty}\|f\|_\infty, \tag{3}$$

where $\|f\|_\infty = \sup\{|f(t)| : t \ge 0\}$.

Theorem 3.2 *Let* $c > 0$ *be the constant such that* $|\phi| \le c|(I-\widehat{T}(\omega))\phi|$ *for all* $\phi \in C$. *If*

$$2ck(\omega) < 1 \text{ and } w > M_w\|L\|_\infty,$$

then Eq.($\mathrm{P}_\omega\mathrm{L}$) *has a unique* ω*-periodic solution, where* $k(t)$ *is as in Proposition 2.3.*

Proof. The existence of the constant c in the theorem follows from Theorem 4.8 in [9]. To show the existence of ω-periodic solutions of Eq.(P_ωL), we will estimate $\|K(\omega)\|$. From Proposition 2.3 and the conditions in this theorem, we have

$$\|K(\omega)\| \leq k(\omega) < \frac{1}{2c}.$$

Proposition 1.5 implies that Eq.(P_ωL) has a unique ω-periodic solution. For the remainder see [9, Theorem 8.2]; and hence the proof is complete.

Proposition 3.3 *Let p be a positive integer such that $(p-1)\omega < r \leq p\omega$. Assume that $w\omega > \log M_w$. If*

$$\frac{2(p+M_w)k(\omega)}{1-M_w e^{-w\omega}} < 1 \text{ and } w > M_w\|L\|_\infty, \tag{4}$$

then Eq.(P_ωL) has a unique ω-periodic solution.

Proof. Since $w\omega > \log M_w$, we have $\|T(\omega)\| \leq M_w e^{-w\omega} < 1$, and hence,

$$\|(I-T(\omega))^{-1}\| \leq 1/(1-\|T(\omega)\|) \leq 1/(1-M_w e^{-w\omega}). \tag{5}$$

To compute the value of the constant c in Theorem 3.2, we define an operator $V : D(V) \subset C \to C$ by

$$[V\psi](\theta) = \sum_{j=0}^{k-1} \psi(\theta + j\omega) + T(\theta + k\omega)(I - T(\omega))^{-1}\psi(0), \quad \theta \in I_k,$$

for $k = 1, 2, \cdots, p$, where $I_k = [-k\omega, -(k-1)\omega), I_p = [-r, -(p-1)\omega)$, and $[V\psi](0) = (I-T(\omega))^{-1}\psi(0)$. Notice that $D(V) = \{\psi \in C : \psi(0) \in R(I - T(\omega))\}$. If there exists a positive constant c such that

$$|V\psi| \leq c|\psi| \quad \text{for all} \quad \psi \in D(V),$$

then $|\phi| \leq c|(I-\widehat{T}(\omega))\phi|$ for $\phi \in C$ (for details [9], [12]). Suppose that $\psi \in D(V)$. Then, for $\theta \in I_k, k = 1, 2, \cdots, p$,

$$|V\psi(\theta)| \leq k|\psi| + \sup_{0\leq t\leq\omega} \|T(t)\|\|(I-T(\omega))^{-1}\||\psi(0)| \leq (p + M_w\|(I-T(\omega))^{-1}\|)|\psi|.$$

This implies that $|V\psi| \leq (p + M_w\|(I - T(\omega))^{-1}\|)|\psi|$. Hence, using (5) we get

$$\begin{aligned} c &\leq p + M_w\|(I-T(\omega))^{-1}\| \\ &\leq p + M_w/(1-M_w e^{-w\omega}) \\ &\leq (p+M_w)/(1-M_w e^{-w\omega}). \end{aligned}$$

Therefore it follows from the assumption that all conditons of Theorem 3.2 are satisfied, and the proof is complete.

Remark Let $M_w = 1$ in the Proposition 3.3. Then (4) becomes

$$\frac{2(p+1)\|L\|_\infty}{w - \|L\|_\infty} < 1 \text{ and } w > \|L\|_\infty.$$

4 The uniqueness of periodic solutions (II)

In this section we consider criteria for the uniqueness of periodic solutions to Eq.(P_ωL) by using Proposition 1.6.

Theorem 4.1 *Assume that* $1 \in \rho(T(\omega))$. *If*

$$1 \in \rho((I - \widehat{T}(\omega))^{-1}K(\omega)), \tag{6}$$

then Eq.(P_ωL) *has a unique* ω*-periodic solution.*

Proof. It is not difficult to see that if $1 \in \rho(T(\omega))$, then $1 \in \rho(\widehat{T}(\omega))$. For the solution operator $U(t, 0)$ of *Eq.*($\mathrm{P}_\omega\mathrm{L}_0$), the periodic map $U(\omega, 0)$ is decomposed as $U(\omega, 0) = \widehat{T}(\omega) + K(\omega)$. We here note that $1 \in \rho(U(\omega, 0))$ if and only if $1 \in \rho((I - \widehat{T}(\omega))^{-1}K(\omega))$. Therefore, applying Proposition 1.6 to our situation, we can obtain the proof of the theorem.

We now get a sufficient condition for the condition (6) in the above theorem.

Lemma 4.2 *Assume that* $\|\widehat{T}(\omega)\| < 1$. *If*

$$\|K(\omega)\| < 1 - \|\widehat{T}(\omega)\|, \tag{7}$$

then the condition (6) *holds.*

Proof. From the condition (7) we have

$$\begin{aligned} \|((I - \widehat{T}(\omega))^{-1}K(\omega)\| &\leq \|((I - \widehat{T}(\omega))^{-1}\|\|K(\omega)\| \\ &\leq \frac{\|K(\omega)\|}{1 - \|\widehat{T}(\omega)\|} < 1. \end{aligned}$$

This fact implies the condition (6).

Lemma 4.3 *Assume that* $\omega > r$ *and* $w(\omega - r) > \log M_w$. *Then the inequality* (7) *in Lemma 4.2 can be replaced by the following inequality :*

$$\frac{\|K(\omega)\|}{1 - M_w e^{-w(\omega - r)}} < 1. \tag{8}$$

Proof. From assmptions and the estimate (2) of $\widehat{T}(t)$ we have $\|\widehat{T}(\omega)\| \leq M_w e^{-w(\omega - r)} < 1$. Hence, the inequality (7) follows from the inequality (8). This proves the lemma.

Combining Lemma 4.3 with Proposition 2.3, we can obtain the following result.

Theorem 4.4 *Suppose that* $\omega > r$ *and* $w(\omega - r) > \log M_w$. *If*

$$\frac{k(\omega)}{1 - M_w e^{-w(\omega - r)}} < 1 \quad \textit{and } w > M_w \|L\|_\infty,$$

then Eq.(P_ωL) *has a unique* ω*-periodic solution.*

In general, Proposition 3.3 and Theorem 4.4 are independent of each other, which will be shown in Remark of the next section. Hence, summarizing those results, we have the following result.

Theorem 4.5 *Suppose that $\omega > r, w > M_w \|L\|_\infty e^{wr}$ and $w(\omega - r) > \log M_w$. If*

$$\min\{\frac{2(p+M_w)}{1-M_w e^{-w\omega}}, \frac{1}{1-M_w e^{-w(\omega-r)}}\}k(\omega) < 1, \tag{9}$$

then Eq.$(\mathrm{P}_\omega\mathrm{L})$ *has a unique ω-periodic solution.*

5 An Example

In this section, we shall see, by means of a simple example, how the results of Proposition 3.3 and Theorem 4.4 can be used to prove the unique existence of a periodic solution of a partial differential-integral equation.

Denote by $E = C[-\infty, \infty]$, the space of all continuous real valued functions $u(x)$, defined on $(-\infty, \infty)$, satisfying the condition that $\lim_{x\to-\infty} u(x)$ and $\lim_{x\to+\infty} u(x)$ exist, and take its norm as $\|u\| = \sup_{-\infty<x<+\infty} |u(x)|$. Then E is a Banach space.

We consider the initial value problem for the equation of the form

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \frac{\partial^2 u(t,x)}{\partial x^2} - \alpha u(t,x) + b(t,x)\int_{t-r}^{t} e^{-c(t-s)}u(s,x)ds \\ &\qquad + f(t,x), \\ u(\theta, x) &= \phi(\theta, x), -r \le \theta \le 0, \phi \in \mathcal{C}. \end{aligned} \tag{10}$$

It is well known that the linear operators A and A_0, defined by

$$Au = \frac{d^2u}{dx^2} - \alpha u \ \text{ for } u \in \mathcal{D}(A), \qquad A_0 u = \frac{d^2u}{dx^2} \ \text{ for } u \in \mathcal{D}(A_0)$$

and

$$\mathcal{D}(A) = \mathcal{D}(A_0) = \{u \in E \ : \ \frac{d}{dx}u, \frac{d^2}{dx^2}u \in E\},$$

are infinitesimal generators of C_0-semigroups $T(t)$ and $T_0(t)$ on E, respectively, cf.[2, Chapter VIII]. Hence,

$$\|T_0(t)\| = 1, T(t) = e^{-\alpha t}T_0(t), \ \text{ and } \ \|T(t)\| = e^{-\alpha t}$$

for all $t \ge 0$. Assume that

(C-1) $\alpha > 0$ and $c > 0$.

(C-2) $b(t,x)$ and $f(t,x) : R \times R \to R$ are continuous and ω-periodic functions in t such that $b(t,\cdot), f(t,\cdot) \in E, t \in R$.

Put $\|b(t)\| = \sup_{-\infty<x<\infty} |b(t,x)|, \|b\|_\infty = \sup_{0\le t\le\omega} \|b(t)\|$. Similarly, we define $\|f(t)\|$ and $\|f\|_\infty$ for $f(t,x)$. Set

$$B(t,\phi)(x) = b(t,x)\int_{-r}^{0} e^{c\theta}\phi(\theta,x)\,d\theta, \ \phi \in \mathcal{C}.$$

Then we have
$$|B(t,\phi)(x)| \leq \|b(t)\| \int_{-r}^{0} e^{c\theta}|\phi|\, d\theta \leq \frac{\|b(t)\|}{c}|\phi|$$
and hence, $\|B\|_\infty \leq \|b\|_\infty/c$. Therefore from Proposition 3.3 and the estimate (3) we have the following result.

Theorem 5.1 *Asssume that the conditions* (C-1) *and* (C-2) *are satisfied. Let p be a positive integer such that $(p-1)\omega < r \leq p\omega$. If*
$$\frac{2(p+1)\|b\|_\infty}{c\alpha - \|b\|_\infty} < 1 \textit{ and } c\alpha > \|b\|_\infty, \tag{11}$$
then Eq.(10) *has a unique ω-periodic solution v, and*
$$\|v\|_\infty \leq \frac{c}{c\alpha - \|b\|_\infty}\|f\|_\infty. \tag{12}$$

The following theorem follows immediately from Theorem 4.4.

Theorem 5.2 *Asssume that $\omega > r$ and the conditions* (C-1) *and* (C-2) *are satisfied. If*
$$\frac{\|b\|_\infty(1-e^{-\alpha\omega})}{(1-e^{-\alpha(\omega-r)})(c\alpha - \|b\|_\infty)} < 1 \quad \textit{and } c\alpha > \|b\|_\infty, \tag{13}$$
then Eq.(10) *has a unique ω-periodic solution v with the estimate* (12).

Remark For simplicity, let $\omega > r$, $c = 1$ and $\alpha = 1 > \|b\|_\infty$. Then we compare the condition (11) in Theorem 5.1 with the condition (13) in Theorem 5.2. Since $\omega > r$, we have $p = 1$. Hence, using the condition (9) in Theorem 4.5 we can obtain the following fact : if
$$\|b\|_\infty < \max\{\frac{1}{4+1}, \ \frac{1}{\frac{1-e^{-\omega}}{1-e^{-(\omega-r)}}+1}\},$$
then Eq.(10) has a unique ω-periodic solution. However, we see that

1) there is a periodic ω close to the delay r such that
$$\frac{1-e^{-\omega}}{1-e^{-(\omega-r)}} > 4:$$

2) there is a sufficiently lage periodic ω such that
$$\frac{1-e^{-\omega}}{1-e^{-(\omega-r)}} < 4.$$

Those facts show that the condition (11) in Theorem 5.1 and the condition (13) in Theorem 5.2 are independent of each other.

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