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Special Functions and River Phenomenon

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Abstract

Phase portrait of Ordinary Differential Equations can have special structures that strongly attract or repeal the solutions. Those structures, the "rivers", are revealed by change of coordinates and singularly perturbed models. We first recall results for approaching this phenomenon in real and complex analysis. Then we study the connexion between rivers and special solutions of classical equations (Airy, Bessel, Kummer, Whittaker, orthogonal polynomials...)

1. River Phenomenon: a Real Analysis Approach

We consider a scalar differential equation

$$\frac{dY}{dX} = Q(X,Y) \tag{E}$$

where Q is a polynomial with real coefficients: $Q(X,Y) = \sum_{i=1}^{p} a_i\ X^{m_i}\ Y^{n_i}$, $a_i \in \mathbb{R}$, $(m_i, n_i) \in \mathbb{Q}^2$.

Definition 1. *For equation (E), and for $r \in \mathbb{Q}$, we define the $r-$degree of $Q(X,Y)$ to be the number $\partial_r Q = \max(m_i + rn_i)$. We set*

$$^{r}Q(X,Y) = \sum_{\substack{i=1 \\ m_i+rn_i=\partial_r Q}}^{p} a_i\ X^{m_i}\ Y^{n_i}$$

The phase portrait of an equation (E) can reveal interesting structures: asymptotic directions of polynomial growth that exponentially attract or repeal some solutions of (E) when X tends to infinity (see for example figure 3.1). Solutions asymptotically equivalent to these structures are called "rivers": the gathering of trajectories is such that they seem to merge like rivers on a map. The earliest characterizations of rivers are due to F. and M. Diener, and lead to the following theorem (see for instance [7], [3] or [6]).

Theorem 1. *(river solutions) Let $(k,r) \in \mathbb{R} \times \mathbb{Q}$ such that*

(i) The value $c(r) = 1 - r + \partial_r Q$ is greater than 0

(ii) The value k is a single root of the polynomial $\lambda \to\ ^{r}Q(1,\lambda)$

Then:

- *If* $({}^{r}Q'_Y)(1,k) > 0$, *there exists a unique river solution* $Y(X)$ *of* (E) *defined on the neighborhood of* $+\infty$ *which is asymptotic to* kX^r
- *If* $({}^{r}Q'_Y)(1,k) < 0$, *there exists an infinite number of river solutions* $Y(X)$ *of* (E) *defined on the neighborhood of* $+\infty$ *which are asymptotic to* kX^r

The values r of this theorem can be obtained through an algorithmic way, using a Newton's polygon technique (see [5]).
The river phenomenon is revealed by a singularly perturbed model. The change of coordinates $x = \varepsilon X$, $y = \varepsilon^r Y$ transforms equation (E) into

$$\varepsilon^{c(r)}\frac{dy}{dx} = {}^{r}Q(x,y) + R(x,y,\varepsilon) \tag{e}$$

with $R(x,y,0) = 0$. Condition (i) implies that (E) is a singularly perturbed equation. Condition (ii) implies that ${}^{r}Q(x,kx^r) = 0$, *i.e.*, $y = kx^r$ is a "slow curve" of the slow-fast equation (E). This slow curve is attracting when $({}^{r}Q'_Y)(1,k) < 0$, because the field is positive below the curve and negative above. It is repelling when $({}^{r}Q'_Y)(1,k) > 0$, because the field is positive above the curve and negative below.
The river solutions of (E) are mapped, with the change of coordinates, onto trajectories that "follow" the slow curve $y = kx^r$. A rigorous sense of "following" the slow curve is given with the notion of infinitesimal proximity, in nonstandard analysis.

2. River Phenomenon: a Complex Analysis Approach

We now consider equation (E) and its model (e) in complex variables. According to Jean-Louis Callot (see [2]), we can build a "hill-and-dale" function for the slow curve $y = kx^r$ of (E) defined by:

$$R(x) = \mathrm{Re}\Big(\int^x ({}^{r}Q)'_Y(s,ks^r)\,ds\Big) = \mathrm{Re}\Big(\int^x ({}^{r}Q)'_Y(1,k)\,s^{c(r)-1}\,ds = \frac{1}{c(r)}\mathrm{Re}\big(({}^{r}Q)'_Y(1,k)\,x^{c(r)}\big)$$

This function is used to determine the attractiveness of the slow curve $y = kx^r$. Let us assume that variable x moves along a complex path $\gamma(t)$. When $\gamma(t)$ is going down the hill-and-dale function, *i.e.*, when the harmonic function $R(\gamma(t))$ is decreasing, the slow curve $y = kx^r$ is attracting neighboring solutions. And when $\gamma(t)$ is going up the hill-and-dale function, *i.e.*, when $R(\gamma(t))$ is increasing, the slow curve $y = kx^r$ is repelling neighboring solutions.
We have

$$R(re^{i\theta}) = \frac{1}{c(r)}\,\mathrm{Re}\left(({}^{r}Q)'_Y(1,k)\cos(c(r)\theta)\right)\,r^{c(r)}$$

For a fixed θ, the hill-and-dale function increases with r when $\mathrm{Re}\left(({}^{r}Q)'_Y(1,k)\cos(c(r)\theta)\right) > 0$, and decreases with r when $\mathrm{Re}\left(({}^{r}Q)'_Y(1,k)\cos(c(r)\theta)\right) < 0$. Therefore, we have sectors of opening $\pi/c(r)$ where the function R increases with the modulus of x: they are the

"hills" of R. And we have sectors of opening $\pi/c(r)$ where the function R decreases with the modulus of x: they are the "dales" of R. We have alternance of hills and dales: each hill is bordered by two dales, and each dale is bordered by two hills.
We consider sets

$$S(A, \theta_-, \theta_+) = \{ X \in \mathbb{C},\ |X| > A,\ \theta_- < \arg(X) < \theta_+ \}$$

with $A \in \mathbb{R}_+$, $(\theta_-, \theta_+) \in \mathbb{R}^2$.
Using the hill-and-dale function, it can be shown (see [9]) that attracting rivers of theorem 1 are asymptotically equivalent to kX^r for $X \in S(A, -\frac{\pi}{2c}, \frac{\pi}{2c})$, with A large enough: they correspond to a dale of the function R. Repelling rivers are asymptotically equivalent to kX^r for $X \in S(A, -\frac{3\pi}{2c}, \frac{3\pi}{2c})$, with A large enough: they correspond to a hill and two adjacent dales of the function R. We say that $Y(X)$ is equivalent to kX^r on $S(A, \theta_-, \theta_+)$ if for all $(a, b) \in \mathbb{R}^2$, $\theta_- < a < b < \theta_+$, we have $Y(X)/(kX^r) \to 1$ when $|X| \to +\infty$ and $\theta \in [a, b]$.
We remark that repelling rivers are "distinguished" among solutions of (E): they are unique (no other solution share the same asymptotic equivalent), and they belong to large sectors of $\mathbb{C}$, because their initial condition correspond to the top of a hill.
We can extend this study to complex solutions of complex equations (E), and determine all distinguished solutions that are defined on large sectors. Assume now that coefficients a_i of $Q(X, Y)$ belong to $\mathbb{C}$. We have the following result (see [4]).

Theorem 1. *(distinguished solutions) Let $(k, r) \in \mathbb{C} \times \mathbb{Q}$ such that*

(i) The value $c(r) = 1 - r + \partial_r Q$ is greater than 0

(ii) The value k is a simple root of the polynomial $\lambda \to^r Q(1, \lambda)$

Let q be the lowest positive integer such that $\bar{c} = c(r)\ q$ belongs to $\mathbb{N}$. Then for all directions $\theta_n = (-\arg((^rQ)'_Y(1, k)) + 2n\pi)/\bar{c}$, $n = 1..\bar{c}$, there exists $A \in \mathbb{R}_+$ large enough such that equation (E) has a solution defined on $S(A, \theta_n - \frac{3\pi}{2c}, \theta_n + \frac{3\pi}{2c})$ and asymptotically equivalent to kX^r on this set.

3. The Example of Airy Equation

Using the example of the Airy equation, we will show in this section how rivers and special functions are related. The special functions considered here are particular solutions of second order linear differential equations. They have been chosen among all the solutions because they have special asymptotic behavior, and sometimes, because of other reasons, as symmetry properties.
If one transforms the second order linear equation onto a non linear Riccati equation, we remark a correspondence between rivers (in particular repelling) of the Riccati's equivalent, and special solutions of the initial equation.
The well-known Airy equation is

$$W''(X) - X\,W(X) = 0 \tag{3.1}$$

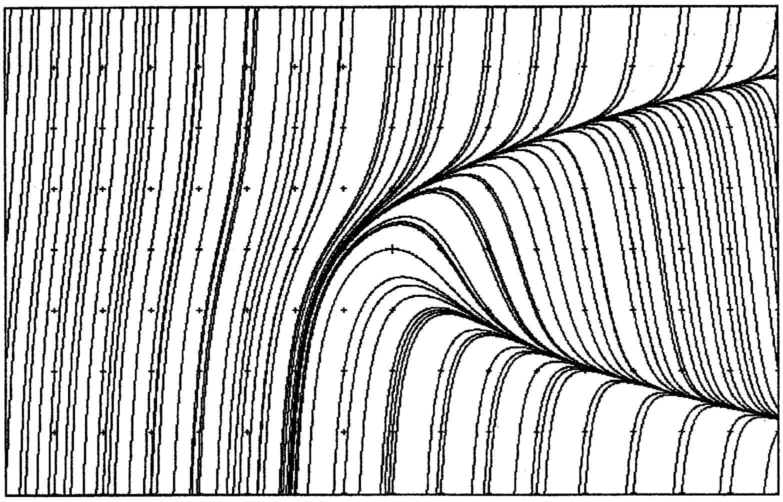


Figure 3.1: Phase portrait of Liouville's equation

The change of variable $Y(X) = -W'(X)/W(X)$ transforms (3.1) in the Liouville's equation

$$Y'(X) = Y^2 - X \tag{3.2}$$

On the phase portrait (see figure 3.1), we can see two river structures for $X \to +\infty$ (a repelling and an attracting one).
A Newton's polygon technique gives the value $r = 1/2$ such that the change of coordinates $x = \varepsilon X$, $y = \varepsilon^{1/2} Y$ transforms (3.2) onto the singularly perturbed model

$$\varepsilon^{3/2}\, y'(x) = y^2 - x \tag{3.3}$$

A real analysis approach reveals two slow curves $\pm\sqrt{x}$. The slow curve $\sqrt{x}$ strongly repeals solutions of (3.2) and corresponds to a unique repelling river (see theorem 1) which is asymptotically equivalent to $\sqrt{X}$ when $X \to +\infty$. The slow curve $-\sqrt{x}$ strongly attracts solutions of (3.2) and corresponds to an infinity of attracting rivers asymptotically equivalent to $-\sqrt{X}$.

In complex analysis, the slow curve is $x^{1/2} = r^{1/2}\, e^{i\theta/2}$, the determination of the square root depends on the value of θ. We compute the hill-and-dale function

$$R(x) = \mathrm{Re}\left(\int^x 2s^{1/2} ds\right) = \mathrm{Re}\left(\frac{4}{3} x^{3/2}\right)$$

$$R(re^{i\theta}) = \frac{4}{3}\, r^{3/2}\, \cos\left(\frac{3}{2}\theta\right)$$

The repelling river corresponds to the top of a hill centered on $\mathbb{R}_+$, bordered by two dales. It is possible to reach each point of $\mathbb{C}$ along a path that always "goes down" the hill-and-dale function. As drawn in figure 3.2, left part, the repelling river solution $Y(re^{i\theta})$ is

asymptotically equivalent to $r^{1/2}e^{i\theta/2}$ for $\theta \in]-\pi, \pi[$. The asymptotical equivalence holds on a large sector of opening 2π.
An attracting river corresponds to the bottom of a dale centered on $\mathbb{R}_+$, and bordered by two hills. It is only possible to go down the hill. As drawn in figure 3.2, right part, a river solution $Y(re^{i\theta})$ is asymptotically equivalent to $r^{1/2}e^{i\theta/2}$ for $\theta \in]2\pi/3, 4\pi/3[$. The asymptotical equivalence holds only on a sector of opening $2\pi/3$.

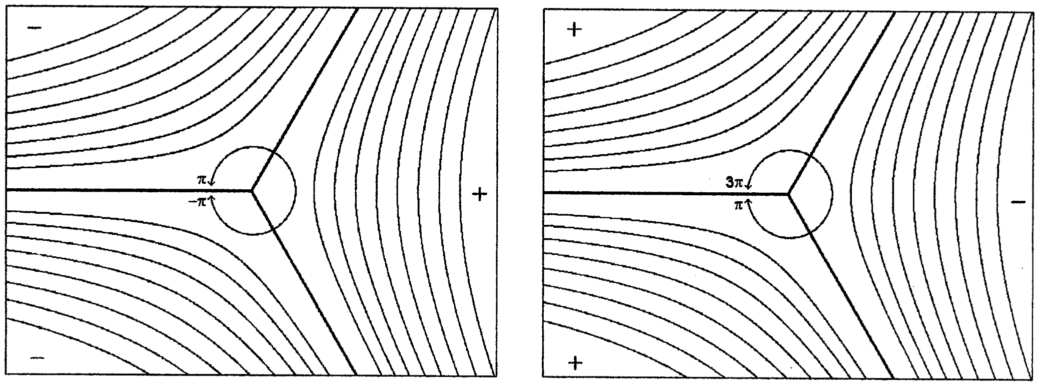


Figure 3.2: Level curves of the hill-and dale function $\frac{4}{3}r^{3/2}\cos(\frac{3}{2}\theta)$. The sign "+" is for increasing level curves ("hills"), the sign "-" is for decreasing level curves ("dales"). Left part: $\theta \in]-\pi, \pi[$, the repelling river correspond to the hill and its two neighboring dales. Right part: $\theta \in]2\pi, 3\pi[$, an attracting river corresponds only to the dale.

There are three solutions of (E) asymptotically equivalents to $r^{1/2}e^{i\theta/2}$ on large sectors. These three distinguished solutions corresponds to the tops of the three hills of the hill-and-dale function. As shown in figure 3.3, the first one is the repelling river solution; the second one is a complex solution, asymptotically equivalent to $r^{1/2}e^{i\theta/2}$ for $\theta \in]\pi/3, 7\pi/3[$; and the third one is another complex solution, asymptotically equivalent to $r^{1/2}e^{i\theta/2}$ for $\theta \in]-7\pi/3, -\pi/3[$.

A special solution has an historical importance for equation (3.1): the Airy function $A_i(x)$. This function is real, oscillating for $x < 0$ and asymptotically decreasing for $x > 0$. Transformation $u_{A_i}(x) = -A_i'(x)/A_i(x)$ maps the Airy function onto the repelling river of (3.2).
Another real solution is defined for equation (3.1): the function $B_i(x)$. Figure 3.4 shows the mapping of the function A_i and B_i on Liouville's plane. The image of $B_i(x)$ is an attracting river, good graphical representative of solutions following the slow curve $-\sqrt{x}$. Two complex solutions are classically pointed out as special solutions of (3.1): the function $A_i(ze^{i\frac{2\pi}{3}})$ and the function $A_i(ze^{-i\frac{2\pi}{3}})$. We remark these two complex solutions, joined with the Airy function itself, correspond to the three distinguished solutions of the equation.

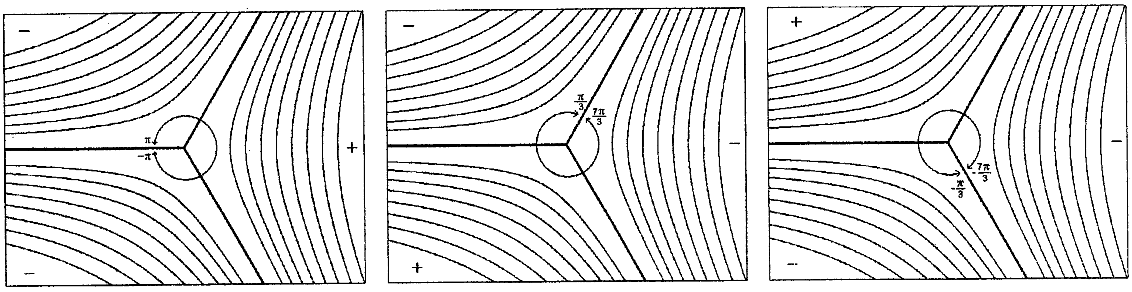


Figure 3.3: Level curves of the hill–and-dale function. First draw: $\theta \in]\pi, \pi[$. Second draw: $\theta \in]\pi/3, 7\pi/3[$. Third draw: $\theta \in]-7\pi/3, -\pi/3[$. These three domain corresponds to three distinguished solutions, asymptotically equivalent to $r^{1/2}e^{i\theta/2}$

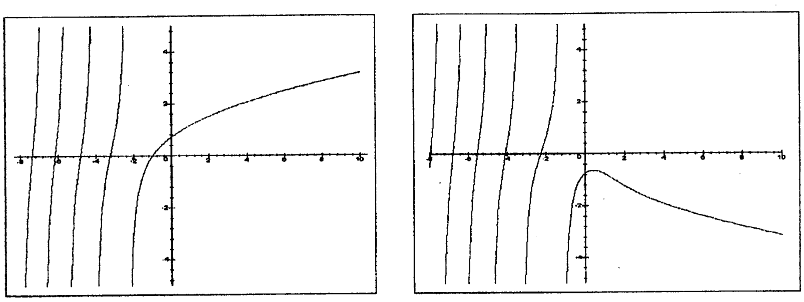


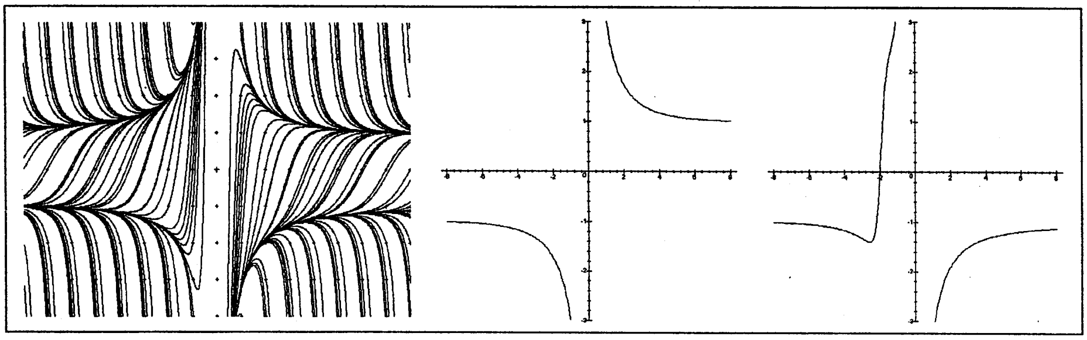
Figure 3.4: Left part: function $-A_i'(x)/A_i(x)$. Right part: function $-B_i'(x)/B_i(x)$

4. Other Special Functions

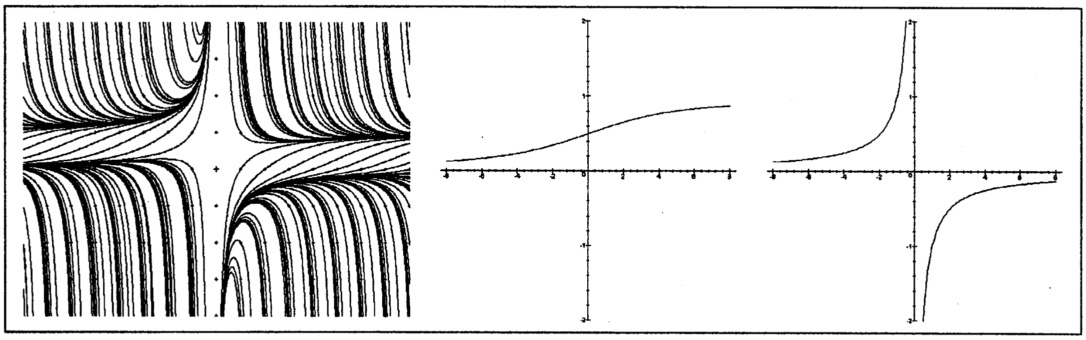
The study of relations between rivers and special solutions can be extend to other classical equations. The results are summarized in the following table, and illustrated in a collection of figures.

We consider equations with real coefficients, that have real river structures in $+\infty$. The definition and properties of special solutions considered here, as well as details about these equations, can be found in classical books on special functions (see in particular [1]). All functions plotted here are real along the real axis, except $K_3'(x)$, $M'(1,2,x)$ and $W_{1,1}'(x)$ for $x < 0$. In these three cases, we plotted the real part of these functions.

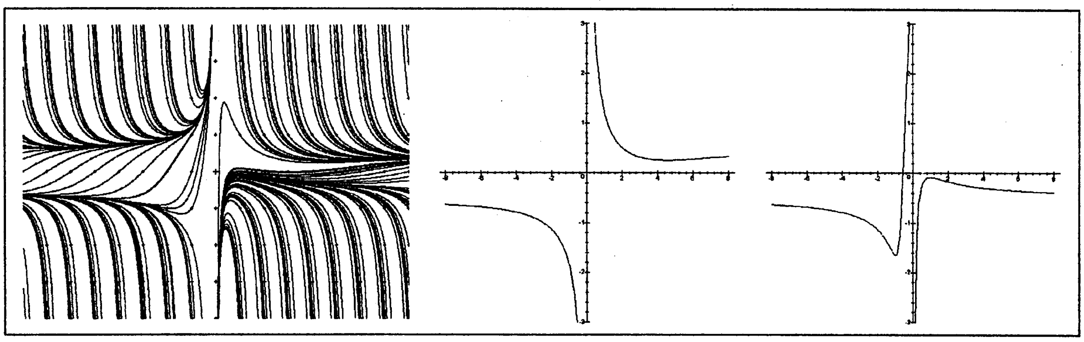
In all those examples, we notice that repelling rivers are usually distinguished as special functions. Other special functions, that are solutions of these equations, have particular properties of symmetry. All the special functions considered in this table correspond to rivers of some Riccati equations. As rivers exist as well for any differential equation (E) (even it is not a Ricccati's equation), we can consider they are a form of generalization of special functions.



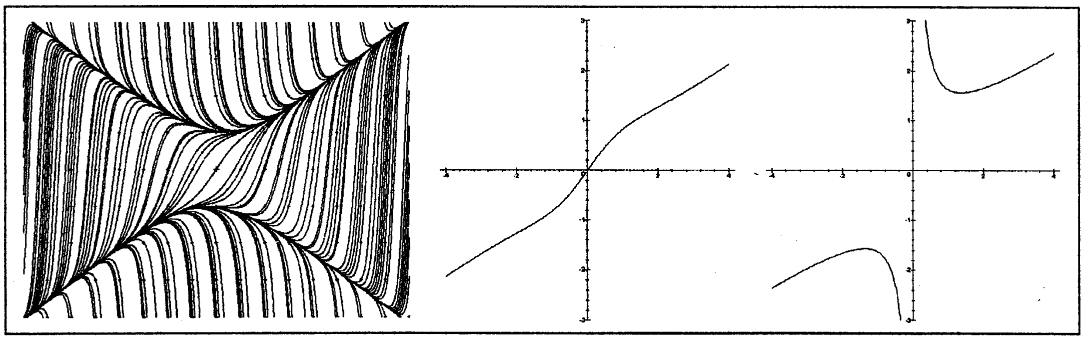
Modified Bessel's equation for $v = 3$. Function $I_3'(x)/I_3(x)$. Function $K_3'(x)/K_3(x)$



Kummer's equation for $a = 1$ and $b = 2$. Function $M'(1, 2, x)/M(1, 2, x)$. Function $U'(1, 2, x)/U(1, 2, x)$



Whittaker's equation for $\kappa = 1$ and $\mu = 2$. Function $M_{1,1}'(x)/M_{1,1}(x)$. Function $W_{1,1}'(x)/W_{1,1}(x)$



Cylinder Parabolic Equation for $a = 1$. Function $y_1'(1, x)/y_1(1, x)$. Function $y_2'(1, x)/y_2(1, x)$.

	Equation	Special solutions
Modified Bessel's Equation	$x^2\, w''(x) + x\, w'(x) - (z^2 + v^2)\, w(x) = 0$	Bessel functions $I_v(x)$ and $K_v(x)$
Riccati's equivalent	$u'(x) = -u^2(x) - \frac{1}{x}\, u(x) + (1 + \frac{v^2}{x^2})$	Attracting rivers ~ 1 Repelling river ~ -1
Mapping	Function $I_v(x)$ with an attracting river; function $K_v(x)$ with the repelling river	
Kummer's Equation	$x\, w''(x) + (b - x)\, w'(x) - a\, w(x) = 0$	Kummer's functions $M(a,b,x)$ and $U(a,b,x)$
Riccati's equivalent	$u'(x) = -u^2(x) + \frac{x-b}{x}\, u(x) + \frac{a}{x}$	Attracting rivers ~ 1 Repelling river $\sim -a/x$
Mapping	Function M with an attracting river; function U with the repelling river	
Whittaker's Equation	$w''(x) + (-\frac{1}{4} + \frac{\kappa}{x} + \frac{1-4\mu^2}{4x^2})\, w(x) = 0$	Whittaker's functions $M_{\kappa,\mu}(x)$ and $W_{\kappa,\mu}(x)$
Riccati's equivalent	$u'(x) = -u^2(x) + (\frac{1}{4} - \frac{\kappa}{x} + \frac{4\mu^2-1}{4x^2})$	Attracting rivers $\sim 1/2$ Repelling river $\sim -1/2$
Mapping	Function $M_{\kappa,\mu}$ with an attracting river; function $U_{\kappa,\mu}$ with the repelling river	
Parabolic Cylinder Equation	$w''(x) + (\frac{x^2}{2} + a)\, w(x) = 0$	Functions $y_1(a,x)$ and $y_2(a,x)$
Riccati's equivalent	$u'(x) = -u^2(x) + (\frac{x^2}{4} + a)$	Attracting rivers $\sim x/2$ Repelling river $\sim -x/2$
Mapping	Functions $y_1(a,x)$ and $y_2(a,x)$ with attracting rivers	
Laguerre	$x\, w''(x) + (\alpha + 1 - x)\, w'(x) + n\, w(x) = 0$	Laguerre's polynomials $L_n^{(\alpha)}(x)$
Riccati's equivalent	$u'(x) = -u^2(x) + \frac{x-1-\alpha}{x}\, u(x) - \frac{n}{x}$	Attracting rivers ~ 1 Repelling river $\sim n/x$
Mapping	Polynomial $L_n^{(\alpha)}(x)$ with the repelling river	
Hermite	$w''(x) - w'(x) + n\, w(x) = 0$	Hermite's polynomials $He_n(x)$
Riccati's equivalent	$u'(x) = -u^2(x) + x\, u(x) - n$	Attracting rivers $\sim x$ Repelling river $\sim n/x$
Mapping	Polynomial $He_n(x)$ with the repelling river	

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