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The Nonstationary Infinite Horizon Production-inventory Problem with Uncertain Capacity and Uncertain Demand

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1 Introduction

In this paper we consider a non-stationary periodic review dynamic production-inventory model with uncertain production capacity and uncertain demand. The demands occur independently, but they are not necessarily identically distributed. Also, the maximum production capacity varies stochastically. This is because of uncertainties in production processes. for instance, unexpected breakdown and unplanned maintenance (Ciarallo, Akella and Morton[3]). Therefore, if the realized production capacity is below the planned production quantity, only part of it can be produced.

The non-stationary stochastic periodic review inventory models have been studied by many researchers; see Karlin[11, 12], Veinott[16, ?], Morton[14], Zipkin[19], Morton and Pentico[15], and Iida[9]. However all of theses literature do not consider the production capacity constraint.

The stationary models with the production capacity constraint have been studied (Federgruen and Zipkin[4, 5], Ciarallo, Akella and Morton[3], Güllü[8]). A multi-stage model with the production capacity constraint has been studied (Glasserman and Tayur[6, 7]). A periodic(cyclic) demand model with the production capacity constraint is studied in Kapuściński and Tayur[13]. The stationary models with random yields have been studied (Yano and Lee[18], Wang and Gerchak[17]).

It is known that myopic (or near myopic) policies are nearly optimal for the inventory problems without the production capacity. However, it has not been known whether the near myopic policies are still nearly optimal for the production-inventory problems with the production capacity. In this paper we show that the similar result holds for the problems with the production capacity.

We consider a single-stage non-stationary production-inventory model with uncertain production capacity and uncertain demand. The objective of our model is to minimize the total discounted expected costs which include production, inventory holding and penalty costs. Production, inventory holding and penalty costs are assumed to be linear respectively. We deal with both finite horizon problems and infinite horizon problems for which order up-to (or base-stock, critical number) policies are optimal, and obtain upper and lower bounds of optimal order up-to levels. Furthermore we show that for an infinite horizon problem the upper and the lower bounds of optimal order up-to levels for the finite horizon counterparts converge respectively as the planning horizons considered become longer. Furthermore, under mild conditions the differences between the upper and the lower bounds of optimal order up-to levels for the finite horizon counterparts converge to zero.

We first formulate the finite horizon problem as a dynamic program along standard steps, and then introduce a new problem which is equivalent to the original problem with respect to the optimal expected cost and the optimal ordering policies. Then we obtain upper and lower bounds of the optimal order up-to levels for the finite horizon problem. For the infinite horizon problem, the convergence results of the upper and the lower bounds of the optimal order up-to levels for the finite horizon counterparts are shown. Numerical examples illustrate computation of the bounds and convergence.

The paper is organized as follows. Section 2 presents the problem considered in this paper and introduce an equivalent problem. Section 3 explores the infinite horizon problem: we develop upper and lower bounds of optimal order up-to levels and show the convergence results. Section 4 presents numerical examples. Section 5 concludes this paper.

All proofs of propositions and lemmas in this paper are shown in [10].

2 Problem Formulation

We consider a periodic review dynamic production-inventory model with uncertain production capacity and uncertain demand. The demands and the production capacities are assumed to be independent but not necessarily be identical. Leadtime is zero and unsatisfied demands are backlogged. The objective of the problem is to minimize the total discounted expected costs which include three types of costs: production, inventory holding and backlog penalty costs. The costs are assumed to be linear respectively.

The activities take place in the following manner: at the beginning of a period a new ordering decision is made, during the period the production capacity realizes and the customer demand occurs, and at the end of the period inventory holding and backlog penalty costs are charged.

We define

Z_t : customer demand at period t,

$Q_t(z)$, $q_t(z)$: cumulative distribution and density functions for Z_t,

A_t : production capacity at period t,

$F_t(a)$, $f_t(a)$: cumulative distribution and density functions for A_t,

x_t : inventory position at the beginning of period t before ordering,

u_t : planned production quantity at period t,

c_t, h_t, p_t : unit production, holding and penalty costs at period t (h_t, $p_t > 0$, $c_t \geq 0$ and $p_t > c_t$). We assume that the unit holding, penalty and production costs are bounded, that is, $p_t < p < \infty$, $h_t < h < \infty$ and $0 \leq \underline{c} < c_t < \bar{c} < \infty$ for all t.

2.1 Dynamic Programming Formulation

We first consider the expected one-period cost. The production cost is determined by the actual production quantity which is limited by the production capacity A_t. If the planned production quantity is u_t and the realized production capacity is a_t, then the actual production quantity is $\min\{u_t, a_t\}$. The inventory holding and backlog penalty costs are determined by the inventory position at the end of the period which depends on the realized customer demand Z_t. Thus the one-period expected cost $g_t(x, u)$, where the inventory position at the beginning of period t before ordering is x and the planned production quantity is u, is as follows,

$$g_t(x,u) = E\Big[c_t \min\{u, A_t\} + h_t\big(x + \min\{u, A_t\} - Z_t\big)^+ + p_t\big(x + \min\{u, A_t\} - Z_t\big)^-\Big].$$

For an n-period problem, we minimize the total discounted expected cost. Let $\alpha < 1$ be the discount factor per period. The periods are numbered $1, 2, \dots, n$. Let $G_{t,n}(x)$ denote the total discounted expected cost from period t through n using optimal policies, given that the inventory position at the beginning of period t before ordering is x. Then, $G_{t,n}(x)$ satisfies the following recursive equation,

$$G_{t,n}(x) = \min_{u \geq 0}\Big\{g_t(x,u) + \alpha E G_{t+1,n}(x + \min\{u, A_t\} - Z_t)\Big\}, \quad \text{for } t = 1, 2, \dots, n,$$

where $G_{n+1,n}(x) \equiv 0$. In the above recursive equation the expectation is taken with respect to the random variables A_t and Z_t. We shall hereafter assume that all relevant functions are differentiable.

We define

$$\begin{aligned} H_{t,n}(x,u) &= g_t(x,u) + \alpha E G_{t+1,n}(x + \min\{u, A_t\} - Z_t) \\ &= g_t(x,u) + \alpha \int_0^u \int_0^\infty G_{t+1,n}(x + a - z) f_t(a) q_t(z)\, da\, dz \\ &\quad + \alpha(1 - F(u)) \int_0^\infty G_{t+1,n}(x + u - z) q_t(z)\, dz. \end{aligned} \tag{1}$$

Now we denote the solution of the following equation by $x^*_{t,n}$,

$$(h_t + p_t) Q_t(x) + \alpha \int_0^\infty G'_{t+1,n}(x - z)\, dQ_t(z) - (p_t - c_t) = 0,$$

and define a base-stock ordering policy $u^*_{t,n}(x)$ using $x^*_{t,n}$ as follows,

$$u^*_{t,n}(x) = \begin{cases} x^*_{t,n} - x & \text{if } x \leq x^*_{t,n}, \\ 0 & \text{if } x > x^*_{t,n}. \end{cases}$$

Then the following proposition holds(Ciarallo, Akella and Morton[3], Wang and Gerchak[17]).

Proposition 1 *1. $G_{t,n}(x)$ is convex.*

2. $G_{t,n}(x) = \begin{cases} H_{t,n}(x, u^*_{t,n}(x)) & \textit{when } x \leq x^*_{t,n}, \\ H_{t,n}(x, 0) & \textit{when } x > x^*_{t,n}. \end{cases}$

3. The base-stock policy $u^*_{t,n}(x)$ minimizes (1).

2.2 An Equivalent Problem

We consider a problem equivalent to the original problem defined in the previous subsection. In the original problem the available production capacity realizes after ordering, therefore we have to make a production order considering the uncertainty of the production capacity. We here consider a problem in which the available production capacity realizes before ordering so that we can make a production order with the deterministic production capacity constraint. We will find that this problem is equivalent to the original one with respect to both optimal ordering policies and optimal expected costs, and as a result of this modification the analysis of the problem becomes simpler than that of the original one. We denote the original problem by P1 and the new one by P2.

In problem P2 the activities in a period take place in the following manner: at the beginning of a period production capacity realizes and a new ordering decision is made, during the period customer demand occurs, and at the end of the period penalty and holding costs are charged.

We denote the inventory position after ordering by y_t. Since the production capacity is realized before ordering, we can determine y_t explicitly when we make an order. We consider the expected one-period cost where the inventory position at the beginning of the period after ordering is y $(y = x + u)$. Let

$$L_t(y) = E\big[h_t(y - Z_t)^+ + p_t(y - Z_t)^-\big].$$

Note that $L_t(y)$ is a convex function. Then the expected one-period cost where the inventory position at the beginning of the period after ordering is y is $c_t(y - x_t) + L_t(y)$.

Next we consider an n-period problem. Since the ordering decision in P2 is limited by the realized production capacity a_t, we consider a pair of the inventory position at the beginning of a period before ordering and the realized production capacity as a state, that is, (x_t, a_t). Let $I_{t,n}(x, a)$ denote the total discounted expected cost from period t through n using optimal policies, given that the inventory position at the beginning of period t before ordering is x and the realized production capacity in period t is a. We also define

$$I_{t,n}(x) = EI_{t,n}(x, A_t). \tag{2}$$

Then $I_{t,n}(x, a)$ satisfies the following recursive equation,

$$I_{t,n}(x, a) = \min_{x \le y \le x+a} \Big\{c_t(y - x) + L_t(y) + \alpha EI_{t+1,n}(y - Z_t)\Big\}, \tag{3}$$

where $I_{n+1,n+1}(x) \equiv 0$. We define additional functions $J_{t,n}$ as follows,

$$J_{t,n}(y) = c_t y + L_t(y) + \alpha EI_{t+1,n}(y - Z_t).$$

Then $I_{t,n}(x, a) = -c_t x + \min_{x \le y \le x+a} J_{t,n}(y)$, and the following proposition holds.

Proposition 2 *$J_{t,n}(y)$ is convex.*

From the convexity of $J_{t,n}(y)$ an order up-to policy is optimal. We call $y^*_{t,n}$ optimal order up-to level. It is shown in the proof that $y^*_{t,n}$ solves the following equation,

$$J'_{t,n}(y) \equiv (h_t + p_t)Q_t(y) - (p_t - c_t) + \alpha EI'_{t+1,n}(y - Z_t) = 0.$$

This means that the optimal order up-to levels are same for all a.

Next we come to the main result of this section. The following proposition allows us to investigate problem P2 instead of problem P1.

Proposition 3 *Problems P1 and P2 are equivalent with respect to both the optimal ordering policies and the optimal expected costs.*

In order to derive upper and lower bounds of the optimal order up-to levels in the next section, we now make a cost transformation, which was used in Morton and Pentico[15]. Let $\tilde{I}_{t,n}(x, a) = c_t x + I_{t,n}(x, a)$ and $\tilde{I}_{t,n}(x) = c_t x + I_{t,n}(x)$. Then we obtain the following recursive equations for $\tilde{I}_{t,n}(x, a)$ and $\tilde{I}_{t,n}(x)$,

$$\tilde{I}_{t,n}(x) = E\tilde{I}_{t,n}(x, A_t), \tag{4}$$

and

$$\tilde{I}_{t,n}(x,a) = \min_{x \le y \le x+a} \{\tilde{L}_t(y) + \alpha E \tilde{I}_{t+1,n}(y - Z_t)\} \tag{5}$$

where transformed one period cost function $\tilde{L}_t$ and cost parameters are defined as follows,

$$\tilde{L}_t(y) = \tilde{h}_t E(y - Z_t)^+ + \tilde{p}_t E(y - Z_t)^- + c_t E[Z_t]$$

and

$$\tilde{h}_t = h_t + c_t - \alpha c_{t+1} \text{ and } \tilde{p}_t = p_t - c_t + \alpha c_{t+1}.$$

Note that equations (4) and (5) are identical to equations (2) and (3). Therefore optimal order up-to levels $y^*_{t,n}$ are same. We define additional functions $\tilde{J}_{t,n}(y)$ as follows,

$$\tilde{J}_{t,n}(y) = \tilde{L}_t(y) + \alpha E \tilde{I}_{t+1,n}(y - Z_t).$$

Also, let $\tilde{h} = h + \bar{c} - \alpha \underline{c}$ and $\tilde{p} = p - \underline{c} + \alpha \bar{c}$. Then $\tilde{p}_t < \tilde{p}$ and $\tilde{h}_t < \tilde{h}$.

3 The Infinite Horizon Problem

3.1 Upper and Lower Bounds of the Optimal Order Up-to Levels

We first consider upper and lower bounds of the optimal order up-to levels for an n-period problem. We provide a lemma which shows monotonicity of derivatives of the optimal expected costs and the optimal order up-to levels. The shorthand $f \downarrow$ is used to mean that the function f is everywhere decreased, and $f \uparrow$ is similar.

Lemma 4 $\tilde{J}'_{t,n} \downarrow \implies \tilde{I}'_{t,n} \downarrow$, $y^*_{t,n} \uparrow$ *and* $\tilde{J}'_{t-1,n} \downarrow$.

All results remain true if all arrows are inverted. Recall that $-(\tilde{p} - \alpha \underline{c}) \le \tilde{J}'_{n,n}(y) \le \tilde{h} + \alpha \bar{c}$. We now define

$$\tilde{I}^U_{n+1,n}(x) \equiv -\frac{\tilde{p}\,x}{1-\alpha} \quad \text{and} \quad \tilde{I}^L_{n+1,n}(x) \equiv \frac{\tilde{h}\,x}{1-\alpha}.$$

Let $\tilde{I}^U_{t,n}$ and $\tilde{I}^L_{t,n}$ denote the corresponding additional functions derived from $\tilde{I}^U_{n+1,n}$ and $\tilde{I}^L_{n+1,n}$, respectively. Similarly we define $\tilde{J}^U_{t,n}$, $\tilde{J}^L_{t,n}$, $y^{U*}_{t,n}$ and $y^{L*}_{t,n}$. Then the following proposition is shown by using Lemma 4.

Proposition 5 *1.* $\tilde{J}^{U\prime}_{t,n}(y) \le \tilde{J}^{U\prime}_{t,n+1}(y)$ *for all* y.

2. $\tilde{I}^{U\prime}_{t,n}(x) \le \tilde{I}^{U\prime}_{t,n+1}(x)$ *for all* x.

3. $y^{U*}_{t,n+1} \le y^{U*}_{t,n}$.

The results for the lower bounds also hold similarly. Then we can define

$$\lim_{n\to\infty} y^{U*}_{t,n} = y^{U*}_t \quad \text{and} \quad \lim_{n\to\infty} y^{L*}_{t,n} = y^{L*}_t,$$

since a bounded monotonic sequence converges to a point. Thus from proposition 5 the following proposition holds

Proposition 6 *For any n and $t < n$* $y^{L*}_{t,n} \le y^{L*}_t \le y^{U*}_t \le y^{U*}_{t,n}$.

Remark The inventory problem considered here can be formulated as a non-homogeneous Markov decision process. Under the conditions that the demands are integral, the means of the demands are bounded from above and the inventory position is limited, it is shown from the theory of non-homogeneous Markov decision processes that a set of optimal ordering policies for an infinite horizon problem includes the limiting policy of the optimal policies for the finite horizon counterparts (Bes and Sethi[2] and Bean, Smith and Lasserre[1]).

Hereafter we consider the limiting policy of the optimal policies for the finite horizon counterparts as the optimal policy for the infinite horizon problem.

3.2 The Difference between the Upper and the Lower Bounds

In this subsection we investigate the conditions under which a sequence of the differences between the upper and the lower bounds of the optimal order up-to levels for the finite horizon counterparts converges to zero. We first consider the relations among the second derivatives of $\tilde{L}_t$, $\tilde{J}_{t,n}$, $\tilde{J}^U_{t,n}$ and so on. Let $0 \le m_t \equiv \inf_y \tilde{L}''_t(y)$. and

$$\Delta\tilde{J}'_t \equiv \tilde{J}^{L\prime}_t - \tilde{J}^{U\prime}_t.$$

Similarly, $\Delta\tilde{I}'_t$, Δy^*_t and so on are defined. Next, we show the results for the relation among m_t, $\Delta\tilde{J}'_{t,n}$ and so on.

Lemma 7 *1.* $\Delta\tilde{I}'_{t,n}(x) \le \max_y \Delta\tilde{J}'_{t,n}(y)$.

2. $m_t\Delta y^*_t \le \max_y \Delta\tilde{J}'_{t,n}(y)$.

Then the following proposition is shown by using Lemma 7.

Proposition 8 $\Delta\tilde{J}'_{t,n}(y) \le \alpha^{n-t+1}(\tilde{h}+\tilde{p})/(1-\alpha)$.

Cororally 9 *If* $m_t > 0$, $\Delta y^*_{t,n}$ *converges to zero as* $n \to \infty$.

The convergence rate of $\Delta y^*_{t,n}$ to zero is $O(\alpha^{n-t+1})$. Therefore we may expect that the convergence is rapid for most problems. In the next section we investigate the speed of the convergence with numerical examples.

4 Numerical Examples

In this section we illustrate the upper and the lower bounds of the optimal order up-to levels and the convergence results with numerical examples. The examples include three types of demand pattern: (a) increasing-decreasing, (b) decreasing-increasing and (c) almost stable. Let the distributions of the demands be the normal distributions which are truncated to become non-negative. The means of the demands for the three types of demand pattern are shown in Table 1. Let the standard deviation of the demand at each period be half of the mean of the demand. Let the distributions of the production capacity be also the normal distributions, and their standard deviation be 10. We change the mean of the production capacity among several values.

Let each parameter of the model like below : $c_t = \underline{c} = \bar{c} =$ production cost $= 30$, $h_t = h =$ holding cost $= 2$, $p_t = p =$ penalty cost $= 55$, $\alpha =$ discount rate $= 0.90$.

Table 1: Means of demands

period	1	2	3	4	5	6	7	8	9	10	$\cdots$
(a)	100	130	180	200	220	220	180	150	130	100	$\cdots$
(b)	220	180	150	130	100	130	180	200	220	220	$\cdots$
(c)	150	190	150	190	150	190	150	190	150	190	$\cdots$

We define the deviation between the upper and the lower bounds of the optimal order up-to levels as follows,

$$\delta(n) \equiv \frac{y_{1,n}^{U*} - y_{1,n}^{L*}}{y_{1,n}^{L*}}.$$

From computational convenience we limit the inventory position from above. For this we assume the following.

Assumption 1 *The optimal order up-to levels $y_{t,n}^*$ are bounded, that is, there exists $\bar{y}$ such that $y_{t,n}^* < \bar{y}$ for all n and $t < n$.*

We define

$$\tilde{I}_{n+1,n}^U(x) = \begin{cases} 0 & \text{when } x > \bar{y}, \\ -\tilde{p}x/(1-\alpha) & \text{when } x \leq \bar{y}. \end{cases}$$

Then the following proposition holds.

Proposition 10 *Under assumption 1 and using new $\tilde{I}_{n+1,n}^U(x)$*

1. $y_{t,n}^{U*} \geq y_{t,n}^*$.
2. $y_{t,n}^{U*} \geq y_{t,n+1}^{U*}$.

Let $\bar{y} = 300$ for the numerical examples. Table 2 shows the minimum planning horizons for which $\delta(n)$ is less than 0.05. From Table 2 we find that for the case that the demand pattern is increasing-decreasing and the production capacity constraint is tight, the minimum planning horizon gets longer than ones for other cases. This means that for the increasing-decreasing demand case we have to consider the demands in much further future because of the possibility that shortages may occur. On the other hand, when the production capacity constraint is not tight, the minimum planning horizon for the increasing-decreasing demand case gets shorter. This is consistent with the results of Veinott[?] for the inventory model without the production capacity. The minimum planning horizons for the cases of the tight production capacity constraint become longer than ones for other cases. This reflects the effect that the sufficient production capacity contributes to reducing the influence of future uncertainties on the optimal ordering policy.

Next we investigate the effects of variance of the production capacity on the minimum planning horizons. Table 3 shows the minimum planning horizons for several variances of the production capacity. Let the mean of the production capacity be 220. From Table 3 we find that as the variance of the production capacity becomes larger, the minimum planning horizon also becomes longer. That is, the uncertainty of the production capacity in the future

Table 2: Minimum planning horizons corresponding to several values of mean of production capacity

	200	210	220	230	240	⋯	∞
(a)	10	7	4	3	3	⋯	2
(b)	5	5	5	4	4	⋯	3
(c)	8	6	5	4	4	⋯	3

makes the minimum planning horizon much longer. However the effects of the variance of the production capacity on the lengths of the minimum planning horizons are sufficiently small when the variance is not large.

Table 3: Minimum planning horizons corresponding to several values of standard deviation of production capacity

	0	10	20	30	40	50
(a)	4	4	5	7	9	10
(b)	4	5	5	5	5	6
(c)	5	5	6	6	7	8

5 Conclusions

In this paper we developed an equivalent formulation of the original production-inventory problem with uncertain production capacity and uncertain demand. From the formulation the upper and the lower bounds of the optimal order up-to levels were derived. Then it was shown that the upper and the lower bounds of the optimal order up-to levels converge respectively, and under mild conditions the differences between the upper and the lower bounds converge exponentially to zero.

Extensions to convex holding and penalty costs with bounded derivatives and fixed leadtime are possible. Since the results in this paper depend on the convexity of cost functions and the boundedness of their derivatives, the model can be extended to convex holding and penalty costs with bounded derivatives. Also since the equivalent formulation developed in this paper uses the inventory position as state, fixed leadtime can be incorporated into the model along the standard way.

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