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定量的社会学に現れる非線形偏微分積分方程式についての初期値問題の解の爆発について

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1. Introduction

The present paper deals with the *master equation*, which is a nonlinear integro-partial differential equation. The equation plays a very important role in *quantitative sociodynamics* (see, e.g., [1-5] and [8-11]). For example, the equation can describe migration of human population.

The master equation has the following form:

$$\partial v(t,x)/\partial t = -w(t,x)v(t,x) + \int_{y\in D} W(t;x|y)v(t,y)dy, \tag{1.1}$$

$$w(t,x) \equiv \int_{y\in D} W(t;y|x)dy, \tag{1.2}$$

where D is the *state space* (see [4], pp. 8-11 and p. 22). We assume that D is a bounded Lebesgue measurable set $\subset \mathbb{R}^n$, where n is an integer. By $v = v(t,x)$ we denote an unknown function which represents the density of certain sociodynamic quantity at time $t\in[0,+\infty)$ and at a point $x\in D$. For example, if the equation (1.1) describes migration of human population, then the total population in a subset $d\subseteq D$ is equal to $\int_{y\in d} v(t,y)dy$. By $W = W(t;x|y)$ we denote the *transition rate* at time $t \in[0,+\infty)$ and from a point $y\in D$ to a point $x\in D$. In the next section, we will impose certain conditions on the transition rate. In particular, it will be assumed that $W = W(t;x|y)$ contains the unknown function $v = v(t,x)$, i.e., that (1.1) is nonlinear.

By making use of the methods developed in [6] and [7], we can prove that the Cauchy problem for (1.1) has a unique local positive-valued solution (see Proposition 2.7). The purpose of the present paper is to investigate how solutions to the Cauchy problem behave as the time variable increases. The main results of this paper are Theorems 3.3 and 3.5, which will be stated in Section 3.

Remark 1.1. (i) In general the integer n is equal to 1 or 2 in quantitative sociody namics. Hence, in the present paper, we assume that n = 2, i.e., that $D \subset \mathbb{R} \times \mathbb{R}$. We can apply the method developed in this paper also when $n \neq 2$. Hence there is no loss of generality.

(ii) In [6] and [7] we assume that the equation (1.1) describes migration of human population. However, we have no need to impose such a restriction on the present paper.

(iii) See, e.g., papers and books cited in References of [1-5] and [8-11] for migration of human population.

2. Preliminaries

In the same way as [4], pp. 137-138, and [9], pp. 81-100, we will assume that the transition rate has the following form in the present paper:

$$W = W(t;x|y) \equiv \nu(t)\exp(U(t,x) - U(t,y) - E(x,y)), \tag{2.1}$$

where $\nu = \nu(t)$ denotes the *flexibility* at time $t \in [0,+\infty)$, $U = U(t,x)$ is the *utility* at time $t \in [0,+\infty)$ and at a point $x \in D$, and $E = E(x,y)$ denotes the *effort* from a point $y \in D$ to a point $x \in D$. See [4], pp. 137-157, for the flexibility, the utility, and the effort.

In [6] and [7], we assume that the flexibility is a positive-valued essentially bounded known function of the time variable and that the effort is an essentially bounded real-valued known function of the space variable. In place of these as sumptions, for simplicity, we will impose the following assumption on the present paper:

Assumption 2.1. (i) The flexibility is identically equal to a positive constant.

(ii) The effort is identically equal to a real constant.

Remark 2.2. It follows from Assumption 2.1 and (2.1) that the transition rate is represented as the product of a function of (t,x) and a function of (t,y).

In [6] and [7], we assume that the utility is an essentially bounded known function of the time variable, the space variable, and $v(t,x)/\|v(t,\cdot)\|_{L^1(D)}$, where we denote the norm of $L^1(D)$ by $\|\cdot\|_{L^1(D)}$. In place of this assumption, we will impose the following assumption on the present paper:

Assumption 2.3. The utility is an affine function of $v(t,x)/\|v(t,\cdot)\|_{L^1(D)}$ with posi tive constant coefficients, i.e., $U = U(t,x)$ has the following form:

$$U = U(t,x) \equiv c_{2.1}v(t,x)/\|v(t,\cdot)\|_{L^1(D)} + c_{2.2},$$

where $c_{2.j}$, $j = 1,2$, are positive constants.

Remark 2.4. (i) In quite the same way as [6] and [7], we can define solutions to the Cauchy problem for (1.1).

(ii) In [6] and [7], we assume that the utility has its own limit, i.e., that the utility can neither increase nor decrease to an unlimited extent. Making use of this result, in [6] and [7] we deduce that the Cauchy problem for (1.1) has a unique uniformly bounded solution (see [6], Sections 1 and 3). However, from Assumption 2.3, we see that the utility tends to infinity as $v(t,x)/\|v(t,\cdot)\|_{L^1(D)}$ tends to infinity. It follows from this result that some solutions to the Cauchy problem blow up in a fi nite time interval or tend to infinity as the time variable tends to infinity (see Theo rems 3.3 and 3.5 for the details). This is the difference between the result of the present paper and that obtained in [6] and [7].

In the same way as [6] and [7], we can deduce that

$$\|v(t,\cdot)\|_{L^1(D)} = \|v(0,\cdot)\|_{L^1(D)}, \quad \text{for each } t \geqq 0. \tag{2.2}$$

Applying this result, Assumption 2.3, and Assumption 2.1 to (2.1), we see that the transition rate $W = W(t;x|y)$ has the following form:

$$W(t;x|y) = c_{2.3}\exp\{c_{2.1}(v(t,x)-v(t,y))/\|v(0,\cdot)\|_{L^1(D)}\}, \tag{2.3}$$

where $c_{2.3}$ is a positive constant.

Let us rewrite (1.1) by introducing the following new unknown function $u = u(t,x)$ in place of $v = v(t,x)$:

$$u = u(t,x) \equiv c_{2.1}v(t/c_{2.3}|D|, |D|^{1/2}x)/\|v(0,\cdot)\|_{L^1(D)}, \tag{2.4}$$

where $|S|$ denotes the Lebesgue measure of a Lebesgue measurable set $S \subseteq \mathbb{R}\times\mathbb{R}$. Differentiating (2.4) with respect to t, and applying (2.3) and (1.1), we see that $u =$

u(t,x) satisfies the following integro-partial differential equation:

$$\partial u(t,x)/\partial t = M(u(t,x);u(t,\cdot)), \qquad \text{(M)}$$

where

$$M(z;u(t,\cdot)) \equiv -a(u(t,\cdot))ze^{-z} + b(u(t,\cdot))e^{z}, \qquad (2.5)$$

$$a = a(u(t,\cdot)) \equiv \int_{y\in\Omega} e^{u(t,y)}dy,$$

$$b = b(u(t,\cdot)) \equiv \int_{y\in\Omega} u(t,y)e^{-u(t,y)}dy,$$

$$\Omega \equiv \{x = |D|^{-1/2}z;\ z\in D\}. \qquad (2.6)$$

We easily obtain the following equality from (2.6):

$$|\Omega| = 1. \qquad (2.7)$$

We consider (M) in place of (1.1) in what follows throughout the paper. We denote by (CP) the Cauchy problem for (M) with the initial data,

$$u(0,x) = u_0(x), \qquad \text{(ID)}$$

where $u_0 = u_0(x)$ is an essentially bounded, Lebesgue-measurable function of $x\in\Omega$ such that $\operatorname*{ess\,inf}_{x\in\Omega} u_0(x)>0$. We write $\|\cdot\|_p$ as the norm of $L^p(\Omega)$, $p = 1, +\infty$. From (2.4) we easily see that $\|u_0(\cdot)\|_1 = c_{2.1}/|D|$.

In the same way as the Cauchy problem for (1.1), we can define a solution to the Cauchy problem (CP) as follows:

Definition 2.5. Let T be a positive constant. If $u = u(t,x) \in L^\infty([0,T]_t\times\Omega_x)$, if $u = u(t,x)$ satisfies (M) almost everywhere in $[0,T]_t\times\Omega_x$, and if $u = u(t,x)$ satisfies (ID), then we say that $u = u(t,x)$ is a solution to (CP) in [0,T]. If $u = u(t,x)$ is a so lution to (CP) in [0,T] for each $T>0$, then we say that $u = u(t,x)$ is a global solution to (CP).

Remark 2.6. It follows from (M) and the above definition that $\partial u(t,x)/\partial t \in L^\infty([0,T]_t \times \Omega_x)$. Hence, $u = u(t,x)$ is absolutely continuous with respect to $t \geqq 0$ for a.e. $x \in \Omega$.

Proposition 2.7. (i) The Cauchy problem (CP) has a unique solution $u = u(t,x)$ in $[0,T]$, where T is a positive constant dependent on $u_0 = u_0(x)$.

(ii) If $u = u(t,x)$ is a solution to (CP) in $[0,T]$ for some $T > 0$, then the follow ing (1-3) hold:

(1) $\operatorname*{ess\,inf}_{(t,x)\in[0,T]\times\Omega} u(t,x) > 0$.

(2) $\|u(t,\cdot)\|_1 = \|u_0\|_1$ for each $t \in [0,T]$.

(3) If $u(t,x_1) = u(t,x_2)$ for some $t \in [0,T]$ and for some $x_j \in \Omega$, $j = 1,2$, then $u(t,x_1) = u(t,x_2)$ for each $t \in [0,T]$. If $u(t,x_1) < u(t,x_2)$ for some $t \in [0,T]$ and for some $x_j \in \Omega$, $j = 1,2$, then $u(t,x_1) < u(t,x_2)$ for each $t \in [0,T]$.

Remark 2.8. By Remark 2.2, in Proof of Proposition 2.7, (ii), (3) we can regard (M) as an ordinary differential equation with the parameter x. If we do not make Assumption 2.1, (ii), then there is a possibility that $W = W(t;x|y)$ contains a function of (x,y) which cannot be expressed as the product of a function of x and a function of y. In such a case we cannot regard (M) as an ordinary differential equation with the parameter x.

3. The main result

Let us introduce some symbols which will be employed in presenting the main theorems. Consider the following equation:

$$F(z) = f(\theta), \tag{3.1}$$

where z denotes the unknown value, $\theta \in [0,+\infty)$ is the parameter, and

$$F = F(z) \equiv z\exp(-2z),$$

$$f = f(\theta) \equiv F(\theta) \text{ if } 0 \leqq \theta \leqq 1,$$

$$f = f(\theta) \equiv e^{-(\theta+1)} \text{ if } \theta \geqq 1.$$

We consider only positive solutions of (3.1). Investigating the graphs of $F = F(z)$

and $f = f(z)$ (note that if $z>1$, then $F(z)<f(z)$), we obtain the following lemma:

Lemma 3.1. (i) If $\theta \neq 1/2$, then the equation (3.1) has only two positive solutions different from each other.

(ii) Write $\zeta_j = \zeta_j(\theta)$, $j = 1,2$, $\zeta_1(\theta)<\zeta_2(\theta)$, as the solutions of (3.1). If $0<\theta<1/2$, then $\zeta_1(\theta)=\theta$ and $1/2<\zeta_2(\theta)<+\infty$. If $1/2<\theta\leqq 1$, then $0<\zeta_1(\theta)<1/2$ and $\zeta_2(\theta)=\theta$. If $\theta>1$, then $0<\zeta_1(\theta)<1/2$ and $\theta>\zeta_2(\theta)>1$.

(iii) $\zeta_1(\theta) \to 1/2-0$ and $\zeta_2(\theta) \to 1/2+0$ as $\theta \to 1/2$.

For $u_0 = u_0(x)$ (see (ID)), we decompose Ω as follows: $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_2 = \Omega_2(u_0) \equiv \{x\in\Omega;\ u_0(x) = \operatorname*{ess\,sup}_{x\in\Omega} u_0(x)\}, \tag{3.2}$$

$$\Omega_1 = \Omega_1(u_0) \equiv \Omega\setminus\Omega_2(u_0). \tag{3.3}$$

If $\Omega_2(u_0)$ is not a null set, i.e., if $|\Omega_2(u_0)|>0$, then we can define the following function:

$$G = G(z) \equiv -F(z)+F(g(\|u_0\|_1,z)),\quad z\geqq 0, \tag{3.4}$$

where

$$g = g(r,z) \equiv (r-z|\Omega_1(u_0)|)/|\Omega_2(u_0)|,\quad r\geqq 0. \tag{3.5}$$

If $|\Omega_2(u_0)|>0$, then we can define the following step function:

$$u_\infty = u_\infty(u_0;x) \equiv k_j \quad \text{if } x\in\Omega_j(u_0),\quad j=1,2, \tag{3.6}$$

where k_1 is defined in the lemma below, and k_2 is defined by $k_1|\Omega_1|+k_2|\Omega_2| = \|u_0\|_1$, i.e., $k_2 = g(\|u_0\|_1,k_1)$.

Lemma 3.2. If $\Omega_2(u_0)$ is not a null set, and

$$\|u_0\|_1>1/2, \tag{3.7}$$

then there exists $k_1 \in (0,1/2)$ such that

$$G(z) > 0 \text{ if } 0 \leqq z < k_1,$$

$$G(k_1) = 0,$$

$$G(z) < 0 \text{ if } k_1 < z \leqq 1/2.$$

Proof. From (3.4) we easily see that

$$G(0) > 0. \tag{3.8}$$

It follows from (2.7) that

$$|\Omega_1| + |\Omega_2| = 1. \tag{3.9}$$

Hence,

$$G(\|u_0\|_1) = 0. \tag{3.10}$$

Making use of (3.7) and (3.9), we see that $g(\|u_0\|_1, 1/2) > 1/2$. Applying this inequality to (3.4) with $z = 1/2$, and noting that $F = F(z)$ attains the maximum value at $z = 1/2$, we have

$$G(1/2) < 0. \tag{3.11}$$

It is not easy to directly inspect $G = G(z)$. Dividing (3.4) by $|\Omega_2|\exp(2|\Omega_1|\mathbf{z})$, we consider the following function in place of $G = G(z)$:

$$\mathbf{G} = \mathbf{G}(\mathbf{z}) \equiv G(|\Omega_2|\mathbf{z})/|\Omega_2|\exp(2|\Omega_1|\mathbf{z}),$$

where $\mathbf{z}$ is a variable defined as follows: $\mathbf{z} \equiv z/|\Omega_2|$. If $\mathbf{G}(\mathbf{z}) > 0$ (< 0, respec tively), then $G(z) > 0$ (< 0, respectively). We deduce that (3.10), (3.11), (3.8) are equivalent to the following equality and inequalities respectively:

$$\mathbf{G}(\mathbf{u}_0) = 0,\ \mathbf{G}(1/2|\Omega_2|) < 0,\ \mathbf{G}(0) > 0, \tag{3.12}$$

where $\mathbf{u}_0 \equiv \|u_0\|_1/|\Omega_2|$. It follows from (3.7) and (3.9) that

$$\mathbf{u}_0 > 1/2|\Omega_2| > 1/2. \tag{3.13}$$

Dividing (3.4) by $|\Omega_2|\exp(2|\Omega_1|\mathbf{z})$, and making use of (3.9), we can decom pose $\mathbf{G} = \mathbf{G}(\mathbf{z})$ as follows:

$$\mathbf{G}(\mathbf{z}) = -\mathrm{F}(\mathbf{z}) + \mathrm{h}(\mathbf{z}),$$

where $\mathrm{h} = \mathrm{h}(\mathbf{z})$ is an affine function such that

$$\mathrm{h} = \mathrm{h}(\mathbf{z}) \equiv (\mathbf{u}_0 - |\Omega_1|\mathbf{z})/|\Omega_2|\exp(2\mathbf{u}_0).$$

We deduce that $\mathrm{h}(0) > 0$ and $\partial\,\mathrm{h}(\mathbf{z})/\partial\,\mathbf{z} < 0$. Furthermore we see that the graph of $\mathbf{w} = \mathrm{F}(\mathbf{z})$ is strictly concave in $0 \leqq \mathbf{z} < 1$ and is strictly convex in $1 < \mathbf{z} < +\infty$. We deduce that $\mathrm{F}(0) = \mathrm{F}(+\infty) = 0$ and that $\mathrm{F} = \mathrm{F}(\mathbf{z})$ increases with $\mathbf{z} \in [0,1/2]$ and de creases with $\mathbf{z} \in [1/2,+\infty)$. Making use of these results, (3.12), and (3.13), we see that the equation $\mathbf{G}(\mathbf{z}) = 0$ has only three positive solutions $\mathbf{z} = \mathbf{p}, \mathbf{q}, \mathbf{r}$ such that

$$0 < \mathbf{p} < 1/2|\Omega_2| < \mathbf{q} \leqq \mathbf{r}, \tag{3.14}$$

$$\mathbf{u}_0 = \mathbf{q} \text{ or } \mathbf{r},$$

$$\mathbf{G}(\mathbf{z}) > 0 \text{ if } 0 \leqq \mathbf{z} < \mathbf{p}, \tag{3.15}$$

$$\mathbf{G}(\mathbf{z}) < 0 \text{ if } \mathbf{p} < \mathbf{z} < \mathbf{q}, \tag{3.16}$$

$$\mathbf{G}(\mathbf{z}) > 0 \text{ if } \mathbf{q} < \mathbf{z} < \mathbf{r},$$

$$\mathbf{G}(\mathbf{z}) < 0 \text{ if } \mathbf{r} < \mathbf{z}.$$

(3.14-16) imply that $k_1 \equiv \mathbf{p}|\Omega_2|$ satisfies the present lemma.

Theorem 3.3. (I) If $0 < \|u_0\|_1 < 1/2$ and $\operatorname*{ess\,sup}_{x\in\Omega} u_0(x) < \zeta_2(\|u_0\|_1)$, then the Cauchy problem (CP) has a unique positive-valued global solution $u = u(t,x)$ which satisfies that $\|u(t,\cdot)-\|u_0\|_1\|_\infty \to 0$ as $t \to \infty$.

(II) If $u_0 = u_0(x)$ satisfies the following inequality:

$$\|u_0\|_1 > 1, \tag{3.17}$$

then the following (i) and (ii) hold:

(i) If $u_0 = u_0(x)$ is such that

$$|\Omega_2(u_0)| > 0, \tag{3.18}$$

then the Cauchy problem (CP) has a unique positive-valued global solution $u = u(t,x)$ which converges to $u_\infty = u_\infty(u_0;x)$ for a.e. $x \in \Omega$ as $t \to \infty$ (see (3.2) and (3.6)).

(ii) If $u_0 = u_0(x)$ satisfies

$$|\Omega_2(u_0)| = 0, \tag{3.19}$$

then the Cauchy problem (CP) has a unique positive-valued solution $u = u(t,x)$ which satisfies the following (3.20-22):

$$\operatorname*{ess\,sup}_{x \in \omega_+(r)} u(t,x) \to +\infty, \tag{3.20}$$

$$\int_{y \in \omega_+(r)} u(t,y)dy \to \|u_0\|_1, \tag{3.21}$$

$$u(t,x) \to 0+0 \text{ for a.e. } x \in \omega_-(r), \tag{3.22}$$

as $t \uparrow t_\infty$ for each $r \geqq 0$, where t_∞ is a positive constant or $t_\infty = +\infty$. $\{\omega_\pm(r)\}_{r\geqq 0}$ is a family of Lebesgue measurable sets such that

$$\Omega = \omega_+(r) \cup \omega_-(r) \text{ and } \omega_+(r) \cap \omega_-(r) \text{ is empty for each } r, \tag{3.23}$$

$$\omega_+(r_1) \supseteq \omega_+(r_2) \text{ and } \omega_-(r_1) \subseteq \omega_-(r_2) \text{ if } r_1 \leqq r_2, \tag{3.24}$$

$$\omega_+(r) \text{ is not a null set for each } r, \tag{3.25}$$

$$|\omega_+(r)| \downarrow 0 \text{ and } |\omega_-(r)| \uparrow 1 \text{ as } r \uparrow +\infty. \tag{3.26}$$

Remark 3.4. If t_∞ is a positive constant in Theorem 3.3, (II), (ii), then the solution blows up as $t \uparrow t_\infty$. If $t_\infty = +\infty$, then the solution is global. It depends on $u_0 = u_0(x)$ whether $t_\infty = +\infty$ or $t_\infty < +\infty$.

The above theorem does not cover the case where $1/2 \leqq \|u_0\|_1 \leqq 1$. If we try to numerically solve the Cauchy problem (CP) in such a case, then we find that the behavior of solutions is extremely complicated. Hence it is very difficult to take a purely theoretical approach in trying to describe how solutions to (CP) behave when $1/2 \leqq \|u_0\|_1 \leqq 1$. However we can obtain the following theorem:

Theorem 3.5. Let $c_0 \in (1/2,1]$ be a constant. For each $\varepsilon > 0$, there exists some $u_0 = u_0(x)$ which satisfies the following three conditions:

$$1/2 < \|u_0\|_1 \leqq 1, \tag{3.27}$$

$$\|u_0(\cdot) - c_0\|_\infty \leqq \varepsilon, \tag{3.28}$$

a solution to (CP) with the initial data $u_0 = u_0(x)$ satisfies (3.20-26).

Remark 3.6. (i) If $u_0(x) \equiv c_0$, where c_0 is a positive constant, then the Cauchy problem (CP) has a unique global solution $u = u(t,x) \equiv c_0$. By (2.7), we see that $c_0 = \|u_0\|_1$. Theorem 3.5 means that if $1/2 < c_0 \leqq 1$, then even the constant solution $u = u(t,x) \equiv c_0$ is unstable.

(ii) If (3.19) holds, then $u_0 = u_0(x)$ is not identically equal to a constant.

(iii) Theorems 3.3 and 3.5 do not cover the case where $\|u_0\|_1 = 1/2$. We cannot apply the method developed in the present paper to such a case.

(iv) Numerical solutions to the Cauchy problem (CP) will be fully studied in another paper.

(v) See [7] for the details of the proof of the main result.

From Remark 3.6, (ii), and Theorems 3.3 and 3.5, we can obtain the following corollary:

Corollary 3.7. If $0 < c_0 < 1/2$, then the constant solution $u = u(t,x) \equiv c_0$ is as ymptotic stable. If $c_0 > 1/2$, then the constant solution is unstable.

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