Facial structure of convex sets and some applications

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§1 Introduction

Let $\Omega$ be a measure space and let $S(\Omega)$ be the space of all measurable functions $f$ on $\Omega$ such that $f(t) < \infty \ (a.e.t \in \Omega)$. An operator $F : X \supset D(F) \rightarrow S(\Omega)$ is called a convex operator if $D(F)$ is a convex set in a real vector space $X$, and for each $x, y \in D(F)$ and $0 < \alpha < 1$,

$$F((1 - \alpha)x + \alpha y)(t) \leq (1 - \alpha)F(x)(t) + \alpha F(y)(t) \quad (a.e.t \in \Omega).$$

On the other hand, a function $f : X \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ is called a convex integrand if for each $t \in \Omega$ the function $f(\cdot, t)$ is convex on $\mathbb{R}$. The convex integrand theory is well known and there are many applications. (See [7] for example.) We say that a convex integrand $f$ represents a convex operator $F$ if

$$f(x, t) = \begin{cases} F(x)(t) & \text{for a.e.} \ t \in \Omega \\ \infty & x \notin D(F) \end{cases} \quad x \in D(F)$$

In two of the author’s previous paper [3, 4], many applications of integrand representations of convex operators were demonstrated. However, the existense of integrand representation is nontrivial, and it is known only in some special cases. When $X$ is the $d$-dimensional Euclidian space $\mathbb{R}^d$, the representation theorem has been proved in [3]. In this note, we apply the theory of the faces of convex sets, and give a new method of the proof which is expected to have an advantage in extending the representation theorem to infinite dimensional cases.

§2 Faces of convex sets

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. When $x, y \in \mathbb{R}^d$ are distinct points, then the set $[x, y] = \{(1 - t)x + ty \mid 0 \leq t \leq 1\}$ is called the closed segment between $x$ and $y$. Half open segments $(x, y\], [x, y)$ and open segments $(x, y)$ are defined analogously. Through this section, we fix a nonempty closed convex set $D$ in $\mathbb{R}^d$. A convex subset $C$ of $D$ is called a face of $D$ if
$$\{(x,y) \in C \mid x, y \in D \} \neq \emptyset$$ implies $[x, y] \subset C$.

By $\mathfrak{F}(D)$, we denote the set of all faces of $D$. For $C \in \mathfrak{F}(D)$, dim $C$ is defined to be the dimension of aff $C$ (the affine hull of $C$). It is clear that $x \in D$ is an extreme point of $D$ if and only if $\{x\}$ is a 0-dimensional face of $D$. For preparation, we will state some fundamental properties of faces in the following propositions whose proofs are given in [1].

**Proposition 1.** If $C_\lambda \in \mathfrak{F}(D)$, ($\lambda \in \Lambda$), then $\bigcap_{\lambda \in \Lambda} C_\lambda \in \mathfrak{F}(D)$, and also there exists a smallest face of $D$ containing $\bigcup_{\lambda \in \Lambda} C_\lambda$. Hence $(\mathfrak{F}(D), \subset)$ forms a complete lattice.

**Proposition 2.** Let $C_1$ be a face of $D$ and suppose that $C_2 \subset C_1$. Then $C_2 \in \mathfrak{F}(D)$ if and only if $C_2 \in \mathfrak{F}(C_1)$.

For a convex set $C$ in $\mathbb{R}^d$, $\overset{\circ}{C}$ denotes the relative interior of $C$, which means the interior of $C$ with respect to the relative topology of aff $C$. It is easy to see that every face of $D$ is a closed set. Indeed, if $x$ is a point of the closure of a face $C$ and $x_0 \in \overset{\circ}{C}$, the convexity of $C$ yields $[x_0, x] \subset \overset{\circ}{C} \subset C$. Since $C$ is a face of $D$, $x$ must be in $C$.

**Proposition 3.** If $C_1, C_2 \in \mathfrak{F}(D)$, and $C_1 \nsubseteq C_2$, then $C_1 \cap \overset{\circ}{C}_2 = \emptyset$.

**Proposition 4.** Let $x$ be a point of $D$ and let $C$ be a face of $D$. Then $C$ is the smallest face of $D$ containing $x$ if and only if $x \in \overset{\circ}{C}$.

**Proposition 5.** Let $C_1$ be a face of $D$ and let $x$ be a relative boundary point of $C_1$. If $C_2$ is the smallest face of $D$ containing $x$, then $C_2$ is contained by the relative boundary of $C_1$.

From these propositions we obtain the following decomposition of a convex set by its faces.

**Proposition 6.** For a closed convex set $D$ in $\mathbb{R}^d$,

$$D = \bigcup \{\overset{\circ}{C}_\lambda \mid C_\lambda \in \mathfrak{F}(D)\}$$
and the union is disjoint.

We say that a collection \( \{C_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{F}(D) \) is normal if \( \lambda \in \Lambda \) and \( C_\lambda \subset C_\mu \in \mathcal{F}(D) \) imply \( \mu \in \Lambda \). Now we define

\[
\mathcal{A} = \{ A = \bigcup_{\lambda \in \Lambda} \overset{\circ}{C}_\lambda \mid \{C_\lambda\}_{\lambda \in \Lambda} \text{ is normal} \}.
\]

Since \( \{D\} \subset \mathcal{A} \), \( \mathcal{A} \) is at least nonempty. It is easy to see that if each \( A_\lambda \ (\lambda \in \Lambda) \) is a member of \( \mathcal{A} \), then so are \( \bigcup_{\lambda \in \Lambda} A_\lambda \) and \( \bigcap_{\lambda \in \Lambda} A_\lambda \), and therefore \( (\mathcal{A}, \subset) \) is a complete lattice.

**Lemma 1.** If \( A \in \mathcal{A} \), then \( A \) is a convex set.

**proof.** We write \( A = \bigcup_{\lambda \in \Lambda} \overset{\circ}{C}_\lambda \) and let \( x, y \) be arbitrary points of \( A \). Then there exist \( \lambda \) and \( \mu \) such that \( x \in \overset{\circ}{C}_\lambda \) and \( y \in \overset{\circ}{C}_\mu \). Let \( z \) be an arbitrary point of the open segment \( (x, y) \), and let \( C_\nu \) be the smallest face containing \( z \). Since \( C_\nu \) is a face, we have \([x, y] \subset C_\nu \). By Proposition 4, \( C_\lambda \) is the smallest face containing \( x \), and it follows that \( C_\lambda \subset C_\nu \). Since the collection \( \{C_\lambda\}_{\lambda \in \Lambda} \) is normal, we obtain \( \overset{\circ}{C}_\nu \subset A \). This means that \( z \in A \), and thus \( A \) is convex.

### §3 Representation of Convex Operators

In this section, we prove a representation theorem of convex operators. Let \( D(F) \) be a convex set in \( \mathbb{R}^d \) and let \( F: D(F) \rightarrow S(\Omega) \) be a convex operator. We can assume without losing generality that the interior of \( D(F) \) is nonempty. Through this section, \( D \) denotes the closure of \( D(F) \). First we state the main theorem.

**Theorem 1.** Every convex operator \( F: \mathbb{R}^d \supset D(F) \rightarrow S(\Omega) \) has at least a representation. That is, there exists a convex integrand \( f: \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \{\infty\} \) such that (1) holds.

For \( D = \overline{D(F)} \), we define \( \mathcal{A} \) as in §2. For \( A \in \mathcal{A} \), a convex integrand \( f: A \times \Omega \rightarrow \mathbb{R} \cup \{\infty\} \) is said to represent \( F \) on \( A \), if

\[
f(x, t) = \begin{cases} F(x)(t) & \text{for a.e.} \ t \in \Omega \\ \infty & x \in A \setminus D(F) \end{cases}
\]
Definition. For a convex operator $F$, we define

$$\tilde{\mathfrak{A}} = \{(A, f) \mid A \in \mathfrak{A}, \text{ and } f \text{ represents } F \text{ on } A\}.$$  

Moreover, for $(A_1, f_1), (A_2, f_2) \in \tilde{\mathfrak{A}}$, we write $(A_1, f_1) \leq (A_2, f_2)$ when $A_1 \subset A_2$ and $f_2$ is an extension of $f_1$ to $A_2$.

Lemma 2. $(\tilde{\mathfrak{A}}, \leq)$ is inductively ordered.

proof. Let $\{(A_\lambda, f_\lambda)\}_{\lambda \in \Lambda}$ be a totally ordered subset of $\tilde{\mathfrak{A}}$. Then $A = \bigcup_{\lambda \in \Lambda} A_\lambda$ is an element of $\mathfrak{A}$. Moreover we can define a convex integrand $f$ on $A \times \Omega$ satisfying $f = f_\lambda$ on $A_\lambda \times \Omega$ for every $\lambda \in \Lambda$. Clearly, $(A, f) \in \tilde{\mathfrak{A}}$ and it is an upper bound of $\{(A_\lambda, f_\lambda)\}_{\lambda \in \Lambda}$.

Lemma 3. For $A \in \mathfrak{A}$ such that $A \neq D$, we define $\mathfrak{S}_A = \{C \in \mathfrak{F}(D) \mid C \cap A = \emptyset\}$. Then $(\mathfrak{S}_A, \subset)$ is inductively ordered.

proof. Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a totally ordered subset of $\mathfrak{S}_A$. If we put $C = \bigcup_{\lambda \in \Lambda} C_\lambda$, then $C$ is a convex set and $C \cap A \neq \emptyset$. Moreover $C \in \mathfrak{F}(D)$.

Indeed, if we assume $(x, y) \cap C \neq \emptyset$, then there exists $\lambda \in \Lambda$ such that $(x, y) \cap C_\lambda \neq \emptyset$. Hence it follows that $[x, y] \subset C_\lambda \subset C$. Thus $C \in \mathfrak{S}_A$ and it is an upper bound of $\{C_\lambda\}_{\lambda \in \Lambda}$.

Lemma 4. Let $A$ be an element of $\mathfrak{A}$, and assume that $A \neq D$. Then there exists $C \in \mathfrak{S}_A$ such that $A \cup \overset{\circ}{C} \in \mathfrak{A}$.

proof. By Lemma 3 and Zorn’s lemma, $\mathfrak{S}_A$ has at least a maximal element $C$. It is sufficient to show that $A \cup \overset{\circ}{C} \in \mathfrak{A}$. Put $A = \bigcup_{\lambda \in \Lambda} \overset{\circ}{C}_\lambda$, and take $C_1 \in \mathfrak{F}(D)$, such that $C_1 \supset C$. Since $C$ is a maximal element of $\mathfrak{S}_A$, we have $C_1 \not\in \mathfrak{S}_A$ and hence $C_1 \cap A \neq \emptyset$. Therefore we can choose $\lambda \in \Lambda$ such that $\overset{\circ}{C}_\lambda \cap C_1 \neq \emptyset$. It follows from Proposition 3 that, $C_\lambda \subset C_1$ holds. Since the collection $\{C_\lambda\}_{\lambda \in \Lambda}$ is normal, $\overset{\circ}{C}_1 \subset A \subset A \cup \overset{\circ}{C}$. This shows that the collection $\{C_\lambda\}_{\lambda \in \Lambda} \cup \{C\}$ is also normal, and $A \cup \overset{\circ}{C} \in \mathfrak{A}$.

Lemma 5. $\tilde{\mathfrak{A}}$ is not empty. In other words, there exists $A \in \mathfrak{A}$ such that $F$ has a representation $f$ on $\overset{\circ}{D}$. The method of construction is an analogy of that in [4].
Lemma 6. Suppose that $(A, f) \in \tilde{\mathfrak{U}}$ and $A \neq D$. Let $C \in \mathfrak{S}_A$ is a face such that $A \cup \hat{C} \in \mathfrak{A}$ as in Lemma 4. Then $f$ has an extension $f_1$ defined on $(A \cup \hat{C}) \times \Omega$ such that $(A \cup \hat{C}, f_1) \in \tilde{\mathfrak{U}}$.

The proof of this lemma is an analogy of one provide in a previous paper by the author [3].

proof of Theorem 1. By Lemma 3, Lemma 5 and Zorn’s lemma, $\tilde{\mathfrak{A}}$ has at least a maximal element $(A_0, f_0)$. Moreover, Lemma 6 shows that $A_0 = D$, and this means that $f_0$ represents $F$ on $D$. Defining $f_0 = \infty$ on $D^c \times \Omega$, we complete the construction of a representation of $F$.

§4 Normal Representations

A convex integrand $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ is said to be normal if $f(\cdot, t)$ is lower semicontinuous for every $t \in \Omega$ and there exists a countable family of measurable functions $\xi_n : \Omega \rightarrow \mathbb{R}^d (n = 1, 2, \cdots)$ such that

1. for each $n$, $f(\xi_n(t), t)$ is measurable in $t \in \Omega$,
2. for each $t \in \Omega$, $\{\xi_n(t)\}_{n=1}^{\infty}$ is dense in $D(f(\cdot, t))$,

where $D(f(\cdot, t)) = \{x \in \mathbb{R}^d | f(x, t) < \infty\}$. If a convex integrand $f$ is normal, then $f(\xi(t), t)$ is measurable in $t \in \Omega$ whenever $\xi : \Omega \rightarrow \mathbb{R}^d$ is measurable. A convex operator $F$ is said to have a normal representation if there exists a normal convex integrand which represents $F$. We will find some conditions under which a convex operator has a normal representation. By the conjugate of a convex integrand $f$, we mean the convex integrand $f^* : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$f^*(\xi, t) = \sup_{x \in \mathbb{R}^d} \{<x, \xi> - f(x, t)\}.$$  

Also the biconjugate $f^{**} : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \infty$ is given by

$$f^{**}(x, t) = \sup_{\xi \in \mathbb{R}^d} \{<x, \xi> - f^*(\xi, t)\}.$$  

If a convex integrand $f$ is normal, then so are $f^*$ and $f^{**}$. We note that if a convex integrand $f$ represents a convex operator $F$ then $D(f(\cdot, t))$ does not depend on $t \in \Omega$.
Lemma 7. Let \( f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \{\infty\} \) be a representation of some convex operator. Then \( f \) is normal if and only if \( f(\cdot, t) \) is lower semicontinuous, in other words, \( f^{**} = f \) on \( \mathbb{R}^d \times \Omega \).

*proof.* Let \( D = D(f(\cdot, t)) \) and take a countable subset \( \{a_n\} \) of \( D \). If we put \( \xi_n(t) = a_n \) for all \( t \in \Omega \) and \( n = 1, 2, \cdots \), then the family \( \{\xi_n\} \) satisfies the definition of normality.

*Remark.* If a convex integrand \( f \) satisfies

1. for each \( x \in \mathbb{R}^d \), \( f(x, \cdot) \) is measurable, and
2. \( D(\cdot, t)) \) does not depend on \( t \in \Omega \),

the conclusion of Lemma 7 is also valid.

Let \( L(\mathbb{R}^d, S(\Omega)) \) denotes the space of all linear mapping from \( \mathbb{R}^d \) to \( S(\Omega) \). We identify \( L(\mathbb{R}^d, S(\Omega)) \) with the set \( S(\Omega)^d = \{\xi = (\xi_1, \cdots, \xi_d) \mid \xi_i \in \xi(\Omega), i = 1, \cdots, d\} \) by corresponding \( S(\Omega)^d \ni (\xi_1, \cdots, \xi_d) \) to the mapping \( \varphi : \mathbb{R}^d \ni (x_1, \cdots, x_d) \rightarrow < x, \xi >= x_1\xi_1 + \cdots + x_d\xi_d \in S(\Omega) \). The conjugate operator \( F^* : L(\mathbb{R}^d, S(\Omega)) \supset D(F^*) \rightarrow S(\Omega) \) of \( F \) is defined by

\[
F^*(\xi) = \bigvee_{x \in D(F^*)} (\langle x, \xi \rangle - F(x))
\]

where \( \bigvee \) means the lattice supremum in the space \( S(\Omega) \), and \( D(F^*) \) is the set of all \( \xi \in S(\Omega)^d \) such that the supremum \( F^* \) exists. The bi-conjugate operator \( F^{**} \) is defined on the space \( L(L(\mathbb{R}^d, S(\Omega)), S(\Omega)) = L(S(\Omega)^d, S(\Omega)) \), and we regard \( S(\Omega)^d \) and \( \mathbb{R}^d \) as the subspaces of this by corresponding \( \eta \in S(\Omega)^d \) and \( x \in \mathbb{R}^d \) to \( < \eta, \cdot > \) and \( < x, \cdot > \in L(S(\Omega)^d, S(\Omega)) \) respectively. For \( x \in \mathbb{R}^d \) and \( \eta \in S(\Omega) \), \( F^{**} \) is defined by

\[
F^{**}(x) = \bigvee_{\xi \in D(F^*)} (\langle x, \xi \rangle - F^*(\xi)), \quad \quad F^{**}(\eta) = \bigvee_{\xi \in D(F^*)} (\langle \eta, \xi \rangle - F^*(\xi)).
\]

They are only defined on the domain \( D(F^{**}) \) where these suprema exist.

**Theoem 2.** Let \( F : \mathbb{R}^d \supset D(F) \rightarrow S(\Omega) \) be a convex operator and let \( f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \{\infty\} \) be a representation of \( F \). Then the convex integrand \( f^* \) and \( f^{**} \) are normal representations of \( F^* \) and \( F^{**} \) respectively. Moreover for \( \xi \in D(F^*) \) and \( \eta \in D(F^{**}) \),

\[
(F^*(\xi))(t) = f^*(\xi(t), t)
\]

\[
(F^{**}(\eta))(t) = f^{**}(\xi(t), t)
\]
\[(F^{**}(\eta))(t) = f^{**}(\eta(t), t)\]
holds for almost every \(t \in \Omega\).

This theorem is due to the following lemma.

**Lemma 8.** Let \(F : \mathbb{R}^d \supset D(F) \rightarrow S(\Omega)\) be a convex operator, and let \(f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d \cup \{\infty\}\) be a representation of \(F\). Let \(U\) be a convex subset of \(D(F)\) and suppose that \(\inf_{x \in U} f(x, t) > -\infty\) for almost every \(t \in \Omega\). Then \(\bigwedge_{x \in U} F(x) \in S(\Omega)\) exists and

\[
(\bigwedge_{x \in U} F(x))(t) = \inf_{x \in U} f(x, t).
\]

**Proof.** Let \(E\) be a countable dense set in \(U\). Then we have

\[
\inf_{x \in U} f(x, t) = \inf_{x \in E} f(x, t)
\]

for a.e.\(t \in \Omega\). Hence \(\inf_{x \in U} f(x, t)\) is measurable in \(t\) and

\[
(\bigwedge_{x \in U} F(x))(t) \leq (\bigwedge_{x \in E} F(x))(t) = \inf_{x \in E} f(x, t) = \inf_{x \in U} f(x, t) = (\bigwedge_{x \in U} F(x))(t)
\]

for a.e.\(t \in \Omega\), and the lemma is proved.

**Proof of Theorem 2.** By Lemma 8 we have

\[
(F^*(\xi))(t) = \bigvee_{x \in D(F)} (\langle \xi, x \rangle - F(x))(t) = \sup_{x \in D(F)} (\langle \xi(t), x \rangle - f(x, t)) = f^*(\xi(t), t) \quad (a.e. t \in \Omega),
\]

for every \(\xi \in D(F^*) \subset S(\Omega)^d\). The latter statement can be obtained by analogy.

Combining Lemma 7 and Theorem 2, we obtain the following result.

**Theorem 3.** A convex operator \(F : \mathbb{R}^d \supset D(F) \rightarrow S(\Omega)\) satisfies

\[
F^{**}(x) = F(x)
\]

for every \(x \in D(F)\), if and only if \(F\) has a normal representation.
REFERENCES


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