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ON ABSOLUTE NORMS ON $\mathbb{C}^2$
—the Geometric Aspect

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This is an announcement of some recent results of the authors [10, 11]. Each absolute normalized norm on $\mathbb{C}^2$ corresponds to a continuous convex function on $[0,1]$ satisfying certain conditions (cf. [2]). By virtue of this correspondence we can obtain many concrete norms of non ℓ_p-type on $\mathbb{C}^2$. In this note we discuss some geometrical properties of absolute norms by means of the corresponding convex functions. Also we consider the direct sums of Banach spaces with norms associated with these functions.

1. Absolute norms on $\mathbb{C}^2$

A norm $\|\cdot\|$ on $\mathbb{C}^2$ is called *absolute* if

$$\|(z,w)\| = \|(|z|,|w|)\| \quad \forall\, z,\, w \in \mathbb{C}, \tag{1}$$

and is called *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p$ are typical such examples. Let N_a denote the family of all absolute normalized norms on $\mathbb{C}^2$.

Lemma A (Bonsall-Duncan [2]). *For any* $\|\cdot\| \in N_a$

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1. \tag{2}$$

Indeed, for any $(z,w) \in \mathbb{C}^2$

$$\begin{aligned}
\|(z,\, w)\|_\infty &= \max\{\|(z,\, 0)\|,\, \|(0,\, w)\|\} \\
&\leq \frac{1}{2}\max\{\|(z,\, w)\| + \|(z,\, -w)\|,\, \|(z,\, w)\| + \|(-z,\, w)\|\} \\
&= \|(z,\, w)\| \\
&\leq \|(z,\, 0)\| + \|(0,\, w)\| \\
&= \|(z,\, w)\|_1.
\end{aligned}$$

Theorem A (Bonsall-Duncan [2]). (i) *For any norm* $\|\cdot\| \in N_a$ *put*

$$\psi(t) = \|(1-t,\, t)\| \quad (0 \leq t \leq 1). \tag{3}$$

Then ψ *is continuous and convex on* $[0,1]$, *and*

$$\psi(0)=\psi(1)=1,\ \max\{1-t,t\}\leq\psi(t)\leq 1. \tag{4}$$

(ii) *For any* $\psi\in\Psi$ *let*

$$\|(z,w)\|_\psi=\begin{cases}(|z|+|w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & ((z,w)\neq(0,0)),\\ 0 & ((z,w)=(0,0)).\end{cases} \tag{5}$$

Then $\|\cdot\|_\psi\in N_a$ *and* $\|\cdot\|_\psi$ *satisfies* (3).

Put

$$\Psi:=\{\psi:\text{continuous convex function on [0,1] with (4)}\}. \tag{6}$$

According to Theorem A, N_a and Ψ are in 1-1 correspondence under (3). For instance the ℓ_p-norm $\|\cdot\|_p$ is associated with

$$\psi_p(t):=\begin{cases}\{(1-t)^p+t^p\}^{1/p} & \text{if } 1\leq p<\infty,\\ \max\{1-t,t\} & \text{if } p=\infty.\end{cases}$$

2. Von Neumann-Jordan constant

The *von Neumann-Jordan constant* (NJ-constant in short) of a Banach (or normed) space X, $C_{NJ}(X)$, is defined to be the smallest constant C for which

$$\frac{1}{C}\leq\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)}\leq C\quad(\forall(x,y)\neq(0,0)) \tag{7}$$

holds ([3]). We summarize some basic facts about NJ-constant.

Proposition A. (i) $1\leq C_{NJ}(X)\leq 2$ *for any Banach space* X; $C_{NJ}(X)=1$ *if and only if* X *is a Hilbert space* (Jordan-von Neumann [4]).

(ii) $C_{NJ}(X)<2$ *if and only if* X *is uniformly non-square* (Takahashi-Kato [9]; see also [7]).

(iii) $C_{NJ}(L_p)=C_{NJ}(\ell_p)=2^{(2/t)-1}$, *where* $1\leq p\leq\infty$, $1/p+1/p'=1$ *and* $t=\min\{p,p'\}$ (Clarkson [3]).

We determine and estimate the NJ-constant $C_{NJ}(\|\cdot\|_\psi)$ of an absolute normalized norm $\|\cdot\|_\psi$ by means of ψ. We easily have the following lemma.

Lemma 1. *Let* φ, $\psi\in\Psi$ *and* $\varphi\leq\psi$ *for all* $0\leq t\leq 1$. *Put*

$$M=\max_{0\leq t\leq 1}\frac{\psi(t)}{\varphi(t)}.$$

Then

$$\|\cdot\|_\varphi \le \|\cdot\|_\psi \le M\|\cdot\|_\varphi.$$

Theorem 1 (Saito-Kato-Takahashi [10]). *Let $\psi \in \Psi$.*
(i) *Assume that $\psi \ge \psi_2$. Then*

$$C_{NJ}(\|\cdot\|_\psi) = \max_{0\le t\le 1} \frac{\psi(t)^2}{\psi_2(t)^2}. \tag{8}$$

(ii) *Assume that $\psi \le \psi_2$. Then*

$$C_{NJ}(\|\cdot\|_\psi) = \max_{0\le t\le 1} \frac{\psi_2(t)^2}{\psi(t)^2}. \tag{9}$$

Proof. (i) Put $M_1 = \max_{0\le t\le 1} \psi(t)/\psi_2(t)$. Then by Lemma 1

$$\begin{aligned}\|x+y\|_\psi^2 + \|x-y\|_\psi^2 &\le M_1^2\left(\|x+y\|_2^2 + \|x-y\|_2^2\right)\\ &= 2M_1^2\left(\|x\|_2^2 + \|y\|_2^2\right)\\ &\le 2M_1^2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right).\end{aligned}$$

Let $M = \psi(t_1)/\psi_2(t_1)$ with some $0 \le t_1 \le 1$. Put $x_1 = (1-t_1,\, 0)$, $y_1 = (0,\, t_1)$. Then we have

$$\|x_1+y_1\|_\psi^2 + \|x_1-y_1\|_\psi^2 = 2M_1^2(\|x_1\|_\psi^2 + \|y_1\|_\psi^2),$$

which implies (8).
(ii) Put $M_2 = \max_{0\le t\le 1} \psi_2(t)/\psi(t)$. Then in the same way as above we have

$$\|x+y\|_\psi^2 + \|x-y\|_\psi^2 \le 2M_2^2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right). \tag{10}$$

Assume $M_2 = \psi_2(t_2)/\psi(t_2)$ with some t_2 $(0 \le t_2 \le 1)$. Then equality is attained in (10) with $x_2 = (1-t_2,\, t_2)$ and $y_2 = (1-t_2,\, -t_2)$. Thus we have (9).

According to Theorem 1 the NJ-constant of $\|\cdot\|_\psi$ does not depend on the shape of ψ. The following lemma is helpful to apply Theorem 1 ([10]).

Lemma 2. *Let $\varphi(t) \ge \psi(t) > 0$ on $[a,b]$. Assume that $\varphi - \psi$ has the maximum, resp., ψ has the minimum at $t = c$ in $[a,b]$. Then φ/ψ attains the maximum at $t = c$.*

Corollary 1. (i) *Let $1 \le p \le \infty$ and $1/p + 1/p' = 1$. Let $t = \min\{p, p'\}$. Then*

$$C_{NJ}(\|\cdot\|_p) = 2^{(2/t)-1}. \tag{11}$$

In particular, $C_{NJ}(\|\cdot\|_1) = C_{NJ}(\|\cdot\|_\infty) = 2$ (Clarkson [3]).
(ii) *Let $2 \le p < \infty$. Let $\|\cdot\|_{p,2}$ be the (Lorentz) $\ell_{p,2}$-norm;*

$$\|(z,\, w)\|_{p,2} = \left\{|z|^{*2} + 2^{(2/p)-1}|w|^{*2}\right\}^{1/2},$$

where $\{|z|^*, |w|^*\}$ *is the non-increasing rearrangement of* $\{|z|, |w|\}$*, that is,* $|z|^* \geq |w|^*$. (Note that if $p < 2$, $\|\cdot\|_{p,2}$ is a quasi-norm; cf. [5, Proposition 1], [12, p.126]). *Then*

$$C_{NJ}(\|\cdot\|_{p,2}) = \frac{2}{1+2^{2/p-1}}. \tag{12}$$

Proof. (i) Let $1 \leq p \leq 2$. Then

$$\psi_2(t) \leq \psi_p(t) \leq 2^{(2/p)-1}\psi_2(t) \quad (0 \leq \forall\, t \leq 1),$$

where the constant $2^{(2/p)-1}$ is the best possible. Hence we have (11) by Theorem 1. For the case $2 \leq p \leq \infty$ a similar argument works.

(ii) Clearly $\|\cdot\|_{p,2} \in N_a$ and the corresponding convex function is given by

$$\psi_{p,2}(t) = \begin{cases} \{(1-t)^2 + 2^{2/p-1}t^2\}^{1/2} & \text{if } 0 \leq t \leq 1/2, \\ \{t^2 + 2^{2/p-1}(1-t)^2\}^{1/2} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Since $\psi_{p,2} \leq \psi_2$, and $\psi_2/\psi_{p,2}$ is symmetric with respect to $t = 1/2$, we find the maximum of $\psi_2^2/\psi_{p,2}^2$ in the interval $[0,1/2]$. The difference $\psi_2(t)^2 - \psi_{p,2}(t)^2 = (1 - 2^{2/p-1})t^2$ takes its maximum at $t = 1/2$, and $\psi_{p,2}$ has the minimum at $t = 1/2$. Therefore by Lemma 2 we have

$$\max_{0\leq t\leq 1} \frac{\psi_2(t)^2}{\psi_{p,2}(t)^2} = \frac{\psi_2(1/2)^2}{\psi_{p,2}(1/2)^2} = \frac{2}{1+2^{2/p-1}},$$

which implies (12) by Theorem 1.

Remark 1. The only known way to calculate NJ-constants needs Clarkson's inequalities (cf. [8]), whereas the above discussions to derive (11) and (12) do not require them.

For some further examples we refer to [10]. We have the following estimate for the general case.

Theorem 2 (Saito-Kato-Takahashi [10]). *Let* $\psi \in \Psi$ *and let*

$$M_1 = \max_{0\leq t\leq 1} \frac{\psi(t)}{\psi_2(t)} \quad \text{and} \quad M_2 = \max_{0\leq t\leq 1} \frac{\psi_2(t)}{\psi(t)}. \tag{13}$$

Then

$$\max\{M_1^2, M_2^2\} \leq C_{NJ}(\|\cdot\|_\psi) \leq M_1^2 M_2^2. \tag{14}$$

Further we have

$$1 \leq \max\{M_1^2, M_2^2\} \leq M_1^2 M_2^2 \leq 2. \tag{15}$$

Remark 2. (i) From the proof of Theorem 2 we have that $M_1^2 M_2^2 = 2$ if and only if $\psi = \psi_1$ or $\psi = \psi_\infty$.

(ii) $\max\{M_1, M_2\} = 1$ if and only if $\psi = \psi_2$.
(iii) $\max\{M_1, M_2\} = M_1M_2$ if and only if $\psi \geq \psi_2$ or $\psi \leq \psi_2$: In particular, Theorem 1 is derived from Theorem 2 and this fact.

By Remark 2 (i) we obtain

Corollary 2. *All absolute normalized norms on $\mathbb{C}^2$ are uniformly non-square except the ℓ_1- and ℓ_∞-norms.*

Theorem 1 gives a class of convex functions for which $C_{NJ}(\|\cdot\|_\psi) = \max\{M_1^2, M_2^2\} = M_1^2M_2^2$. In [10] we gave a sufficient condition that $C_{NJ}(\|\cdot\|_\psi) = M_1^2M_2^2$ ($\max\{M_1^2, M_2^2\} < C_{NJ}(\|\cdot\|_\psi)$), and that $C_{NJ}(\|\cdot\|_\psi) < M_1^2M_2^2$, respectively. Some examples of absolute norms of non ℓ_p-type are also given there.

3. Strict convexity and direct sums of Banach spaces

Lemma B ([2, p.36, Lemma 2]). *Let $\|\cdot\| \in N_a$. If $|p| \leq |r|$ and $|q| \leq |s|$, then $\|(p,q)\| \leq \|(r,s)\|$. Further, if $|p| < |r|$ and $|q| < |s|$, then $\|(p,q)\| < \|(r,s)\|$.*

Note that the latter assertion of Lemma B fails to hold if we replace the condition "$|p| < |r|$ *and* $|q| < |s|$" by "$|p| < |r|$ *or* $|q| < |s|$" (consider the ℓ_∞-norm).

Lemma 3 ([11, Corollary 4]). *Let $\psi \in \Psi$. Then the following are equivalent.*
(i) *If $|p| \leq |r|$ and $|q| < |s|$, or $|p| < |r|$ and $|q| \leq |s|$, then $\|(p,q)\| < \|(r,s)\|$.*
(ii) *$\psi(t) > \psi_\infty(t)$ for all $0 < t < 1$.*

In particular we have the following corollary which is needed to obtain Theorems 3 and 4 below.

Corollary 3 ([11]). *Let $\psi \in \Psi$ be strictly convex. Let $|p| \leq |r|$ and $|q| \leq |s|$, and let $|p| < |r|$ or $|q| < |s|$. Then $\|(p,q)\| < \|(r,s)\|$.*

Theorem 3 (Takahashi-Kato-Saito [11]). *Let $\psi \in \Psi$. Then $\|\cdot\|_\psi$ is strictly convex if and only if ψ is strictly convex.*

Let $X \oplus Y$ be the direct sum of Banach spaces X and Y. For any $\psi \in \Psi$, define

$$\|(x,y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi \quad \text{for } (x,y) \in X \oplus Y. \tag{16}$$

Let $X \oplus_\psi Y$ denote $X \oplus Y$ with the norm (16). Then we have

Theorem 4 (Takahashi-Kato-Saito [11]). *Let X, Y be Banach spaces, and let $\psi \in \Psi$. Then $X \oplus_\psi Y$ is strictly convex if and only if X, Y are strictly convex, and ψ is strictly convex.*

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