DIRICHLET METHOD OF SUMMABILITY AND NONLINEAR ERGODIC THEOREMS IN HILBERT SPACE

TAKESHI YOSHIMOTO

Department of Mathematics, Toyo University
Kawagoe, Saitama 350-8585, Japan

My purpose in this exposé is to give a brief summary of some recent results concerning the summation method of Dirichlet's type and nonlinear ergodic theorems of Dirichlet's type in Hilbert spaces. The exposé is mainly a report on the author's personal work on the subject, by a general survey. Most of the results mentioned below were discussed in [6] and [7].

Let \( X \) be a complex Banach space and let \( B[X] \) be the Banach algebra of bounded linear operators from \( X \) to itself. For a given \( T \in B[X] \), the resolvent set of \( T \) denoted by \( \rho(T) \) is the set of \( \lambda \in \mathbb{C} \) for which \( (\lambda I - T)^{-1} \) exists as an operator in \( B[X] \) with domain \( X \). The spectrum of \( T \) is the complement of \( \rho(T) \) and denoted by \( \sigma(T) \). \( \rho(T) \) is an open subset of \( \mathbb{C} \) and \( \sigma(T) \) is a nonempty bounded closed subset of \( \mathbb{C} \). So, the spectral radius \( \gamma(T) \) of \( T \) is well-defined; in fact, \( \gamma(T) = \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{n \to \infty} \|T^n\|^{1/n} \).

The function \( R(\lambda; T) \) defined by \( R(\lambda; T) = (\lambda I - T)^{-1} \) for \( \lambda \in \rho(T) \) is called the resolvent of \( T \). It is well known [3] that \( R(\lambda; T) \) is analytic in \( \rho(T) \) and if \( T \in B[X] \) and \( |\lambda| > \gamma(T) \) then \( \lambda \in \rho(T) \) and

\[
R(\lambda; T) = (\lambda I - T)^{-1} \sum_{n=0}^{\infty} \lambda^{-n} n! T^n,
\]

the series converging in the uniform operator topology. It is also known that if \( d(\lambda) \) denotes the distance from \( \lambda \in \mathbb{C} \) to \( \sigma(T) \), then \( \|R(\lambda; T)\| \geq 1/d(\lambda) \). If we take \( \lambda = e^z \), \( z = s + it \) \( (s, t \in \mathbb{R}) \), then \( |\lambda| > \gamma(T) \) implies \( s > \log \gamma(T) \) whenever \( \gamma(T) > 0 \). This characterization is a matter of great interest in connection with the question of what is the abscissa of uniform convergence of \( R(\lambda; T) \) as a series.

Given \( T \in B[X] \) let \( \Phi(T) \) denote the class of all functions of complex variables which are analytic in some open set containing \( \sigma(T) \). The following theorem is fundamental in the theory of linear ergodic theorems.

**THEOREM 1** (Dunford [2]). Let \( T \in B[X] \) and let \( f_n \in \Phi(T) \) satisfy \( \lim_{n \to \infty} f_n(1) = 1 \) and \( (\text{so}) \lim_{n \to \infty} (I - T)f_n(T) = 0 \) (the null operator). Then the following statements

---

1991 Mathematics Subject Classification. 47H09, 47H10.
are equivalent.

(1) \( \lim_{n \to \infty} f_n(T) = E, \quad E^2 = E, \quad EX = \text{Ker}(I - T) \).

(2) \( \{f_n(T)x\} \) is weakly sequentially compact for each \( x \in X \).

(3) \( X = \text{Ker}(I - T) \oplus (I - T)X, \quad \sup_n \|f_n(T)\| < \infty \).

Suppose \( f_n, f \in B[X] \), where \( T \in B[X] \). If \( f_n(T) \) converges strongly (or uniformly) then \( f(T)f_n(T) \) converges strongly (or uniformly). We are interested in the converse problem. Under what conditions on \( f_n, f \) and \( T \) does the convergence of \( f(T)f_n(T) \) imply the convergence of \( f_n(T) \)? Theorems describing this situation are in the nature of Tauberian theorems. The condition \( \lim_{n \to \infty} (I - T)f_n(T) = 0 \) appearing in Theorem 1 is just the case \( f(T) = I - T \). Such Tauberian conditions are indispensable in discussing ergodic theorems. It should be noted that the strong regularity of summation methods has a very close connection with Tauberian conditions.

Now we consider the Dirichlet series of the following type

\[
D[f, \mu; z](T) = \sum_{n=0}^{\infty} e^{-\mu_n z} f_n(T),
\]

where \( z \in C \), \( f = \{f_n\} \subseteq \Phi(T) \) and \( \mu = \{\mu_n\} \), \( 0 \leq \mu_0 < \mu_1 < \cdots < \mu_n + \infty \) as \( n \to \infty \).

THEOREM 2 ([6]). Let \( T \in B[X] \) and \( f_n \in \Phi(T) \) be such that \( \sup_n \|\sum_{k=0}^{n} f_k(T)\| > 0 \) and define

\[
a_\mu(f; T) = \lim_{n \to \infty} \sup_{n \to \infty} \log \|\sum_{k=0}^{n} f_k(T)\| \mu_n
\]

with \( f = \{f_n\} \) and \( \mu = \{\mu_n\} \). Then the following statements hold.

(1) If \( s > 0 \) and \( D[f, \mu; z](T) \) converges in the uniform operator topology for any \( z \in C \) with \( z = s + it \), \( t \in R \), then \( s \geq a_\mu(f; T) \).

(2) The Dirichlet series \( D[f, \mu; z](T) \) converges in the uniform operator topology for any \( z \in C \) with \( \text{Re}(z) > \max(0, a_\mu(f; T)) \) when \( a_\mu(f; T) < \infty \).

If \( 0 \leq a_\mu(f; T) < \infty \) in Theorem 2, we say that the number \( a_\mu(f; T) \) is the abscissa of uniform convergence of the Dirichlet series \( D[f, \mu; z](T) \). This theorem plays a fundamental role in investigating ergodic theorems of Dirichlet's type.

COROLLARY 3. Let \( T \in B[X] \) satisfy the conditions \( \sup_n \|\sum_{k=0}^{n} t^k\| > 0 \) and \( \lim_{n \to \infty} T^n/n^\omega = 0 \) for some \( 0 < \omega \leq 1 \). Let \( f = \{f_n\}, \quad f_n(T) = T^n (n \geq 0) \) and \( \mu = \{\mu_n\}, \mu_n = n+1 (n \geq 0) \). Then \( a_\mu(f; T) < 0 \) and \( R(\lambda; T) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n \) converges in the
uniform operator topology for $\lambda \in \mathbb{C}$ with $\log|\lambda| > 0$.

Proof. In view of Theorem 2, all that is required is to show that $a_{\mu}(f;T) \leq 0$. For a sufficiently small $\varepsilon > 0$, there exists by assumption an integer $N \geq 1$ such that

$$||T^n|| < \varepsilon n^\omega \text{ for all } n > N.$$  

Then we have

$$a_{\mu}(f;T) = \lim \sup_{n \to \infty} \frac{\log \|\sum_{k=0}^{n} T^k\|}{n+1} \leq \lim \sup_{n \to \infty} \frac{\log(n+1) + \log\left(\max_{0 \leq k \leq N} ||T^k|| + \varepsilon n^\omega\right)}{n+1} = 0,$$

as desired.

Let $\mu = \{\mu_n\} (n \geq 0)$ be a sequence of real numbers satisfying the following conditions:

(i) $0 \leq \mu_0 < \mu_1 < \ldots < \mu_n \to \infty$ as $n \to \infty$;

(ii) $\inf_{n \geq 0} (\mu_{n+1} - \mu_n) = \delta$ for some $\delta > 0$;

(iii) $\lim_{n \to \infty} \frac{\mu_{n+1}}{\mu_n} = 1$;

(iv) $\lim_{s \to 0^+} g(s) = +\infty$;

(v) $\sup_{s > 0} \frac{1}{g(s)} \sum_{n=0}^{\infty} n \{e^{-\mu_n s} - e^{-\mu_{n+1} s}\} < \infty$,

where $g(s) = \sum_{n=0}^{\infty} e^{-\mu_n s}$ which converges for $s > 0$. Such a sequence $\mu = \{\mu_n\}$ determines a strongly regular method of summability. This new summation method will be called the $(D,\mu)$-method (Dirichlet method of summability). Let $H$ be a real Hilbert space and let $C$ be a nonempty bounded closed convex subset of $H$. A mapping $T : C \to C$ is called asymptotically nonexpansive with Lipschitz constants $\{\alpha_n\}$ if

$$||T^n x - T^n y|| \leq (1 + \alpha_n) ||x - y|| \text{ for all } n \geq 0 \text{ and all } x, y \in C,$$

where $\alpha_n \geq 0$ for all $n \geq 0$ and $\alpha_n \to 0$ as $n \to \infty$ (see Goebel and Kirk [4]). In particular, if $\alpha_n = 0$ for all $n \geq 0$ then $T$ is said to be nonexpansive. If $T$ is an asymptotically nonexpansive mapping on $C$, then for any $x \in C$

$$\left|\left|\frac{1}{g(s)} \sum_{n=p}^{p+q} e^{-\mu_n s} T^n x\right|\right| \leq M_C \left(\frac{1}{g(s)} \sum_{n=p}^{p+q} e^{-\mu_n s}\right) \to 0$$

as $p, q \to \infty$, where $M_C = \sup(||x|| : x \in C)$. This means that for any $x \in C$,
\[ D^S(T)x = \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} T^nx \]

is well defined for \( s > 0 \). Furthermore, for each \( x \in C \) there exists a unique point \( x_0 \in C \) such that

\[ \limsup_{n \to \infty} \| T^nx - x_0 \| = \inf_{y \in C} \{ \limsup_{n \to \infty} \| T^nx - y \| \} . \]

Such a point \( x_0 \) is called the asymptotic center of the sequence \( \{ T^nx \} \) (see Lim [5] and Brézis and Browder [1]). We are particularly interested in the weak and strong convergence of \( D^S(T)x \) when \( s \to 0^+ \).

**THEOREM 4 ([7]).** Let \( C \) be a nonempty bounded closed convex subset of \( H \) and let \( T \) be an asymptotically nonexpansive mapping of \( C \) into itself. Let \( \mu = \{ \mu_n \} \) be the \((D,\mu)\)-method. Then for any \( x \in C \), \( D^S(T)x \) converges weakly to the asymptotic center of \( \{ T^nx \} \) as \( s \to 0^+ \).

Following the idea of Brézis and Browder [1], we say that the \((D,\mu)\)-method is proper if for each \( \{ \beta(n) \} \in K^\infty \) with \( \beta(\cdot) \geq 0 \), \( [g(s)]^{-1} \sum_{n=0}^{\infty} e^{-\mu_n s} \beta(n) \)

converges to some \( \delta \) as \( s \to 0^+ \), then

\[ \lim_{s \to 0^+} \left( \frac{1}{g(s)} \right)^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} e^{-(\mu_n + \mu_k)s} \beta(|n-k|) = \delta. \]

For example, the \((D,\mu)\)-method \( \mu = \{ \mu_n \} \) given by \( \mu_n = an+b \), where \( a > 0 \) and \( b > 0 \), satisfies the properness condition just mentioned.

**THEOREM 5 ([7]).** Let \( C \) be a nonempty bounded closed convex subset of \( H \) and let \( T \) be a nonexpansive nonlinear mapping of \( C \) into itself. Let \( \mu = \{ \mu_n \} \) be the proper \((D,\mu)\)-method. Suppose that

(i) \( 0 \in C \) and \( T(0) = 0 \);

(ii) For some \( c > 0 \), \( T \) satisfies for all \( u, v \in C \) the inequality

\[ | < Tu, Tv > - < u, v > | \leq c( \| u \|^2 + \| Tu \|^2 + \| v \|^2 + \| Tv \|^2 ) ; \]

(iii) there is an element \( \{ \beta(n) \} \in K^m \) with \( \beta(\cdot) \geq 0 \) such that for any \( x \in C \)

\[ | < T^px, T^q x > - \beta(|p-q|) | \leq \gamma_{\min(p,q)} , \]

where \( \gamma_{\min(p,q)} \to \infty \) as \( \min(p,q) \to \infty \).

Then for each \( x \in C \), \( D^S(T)x \) converges strongly as \( s \to 0^+ \) to the asymptotic center of \( \{ T^nx \} \).
Next we consider the convergence of the sequences \( \{x_n\} \subset C \) generated by the iteration procedures (called Mann's type and Halpern-Wittmann's type) by the Dirichlet method.

**THEOREM 6 ([7]).** Let \( C \) be a nonempty bounded closed convex subset of \( H \) and let \( T \) be a nonexpansive nonlinear mapping of \( C \) into itself. Let \( \mu = \{\mu_n\} \) be the \((D, \mu)\)-method. Define (the Mann's type sequence)

\[
x_1 = x \in C \\
x_{n+1} = \alpha_n x_n + (1-\alpha_n)D_s^{(u)}[T]x_n \quad \text{for } n \geq 1,
\]

where \( \{\alpha_n\} \) is a sequence in \([0, a]\) for some \( 0 < a < 1 \) and \( s_n \to 0^+ \) as \( n \to \infty \). Then the sequence \( \{x_n\} \) so defined converges weakly to the asymptotic center of \( \{T^n x\} \).

**THEOREM 7.** Let \( C \) be a nonempty bounded closed convex subset of \( H \) and let \( T \) be a nonexpansive nonlinear mapping of \( C \) into itself. Let \( \mu = \{\mu_n\} \) be the \((D, \mu)\)-method. Define (the Halpern-Wittmann's type sequence)

\[
x_0 = x \in C \\
x_{n+1} = \beta_n x + (1-\beta_n)D_s^{(u)}[T]x_n \quad \text{for } n \geq 0,
\]

where \( s_n \to 0^+ \) as \( n \to \infty \) and \( \{\beta_n\} \) is a sequence in \([0, 1]\) satisfying the conditions

\[
\lim_{n \to \infty} \beta_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \beta_n = \infty.
\]

Then the sequence \( \{x_n\} \) so defined converges strongly to \( P x \), where \( P \) is the metric projection of \( H \) onto \( \text{Fix}(T) \). Moreover, \( P x \) coincides with the asymptotic center of \( \{T^n x\} \).

**Proof.** Note that \( \text{Fix}(T) \neq \emptyset \) by Theorem 4 and let \( z \in \text{Fix}(T) \). Then

\[
\|x_1 - z\| = \|\beta_0 x + (1-\beta_0)D_s^{(u)}[T]x - z\| \\
\leq \beta_0 \|x - z\| + (1-\beta_0)\|D_s^{(u)}[T]x - z\| \\
\leq \beta_0 \|x - z\| + (1-\beta_0)\|x - z\| \\
= \|x - z\|,
\]

and so, by the induction argument, \( \|x_n - z\| \leq \|x - z\| \) for all \( n \geq 0 \). This implies that \( \{x_n\} \) and \( D_s^{(u)}[T]x_n \) are both bounded. Next we claim that
\[
\limsup_{n \to \infty} <x - Px, D_{s_n}^{(u)}[T]x_n - Px > \leq 0.
\]

Since \( \{D_{s_n}^{(u)}[T]x_n\} \) is bounded, there exists a subsequence \( \{n_i\} \) of \( \{n\} \) such that
\[
\limsup_{n \to \infty} <x - Px, D_{s_{n_i}}^{(u)}[T]x_{n_i} - Px > = \lim_{i \to \infty} <x - Px, D_{s_{n_i}}^{(u)}[T]x_{n_i} - Px >.
\]

We may assume that \( (w)\lim_{i \to \infty} D_{s_{n_i}}^{(u)}[T]x_{n_i} = z_0 \) for some \( z_0 \in C \) (through a subsequence of \( \{n_i\} \), if necessary). Using Lemma 2 in [7] we have
\[
\lim_{i \to \infty} \|D_{s_{n_i}}^{(u)}[T]x_{n_i} - TD_{s_{n_i}}^{(u)}[T]x_{n_i} \| = 0,
\]
and thus, the demiclosedness of \( I - T \) at 0 yields \( z_0 \in \text{Fix}(T) \). Hence
\[
\lim_{i \to \infty} <x - Px, D_{s_{n_i}}^{(u)}[T]x_{n_i} - Px > = <x - Px, z_0 - Px > \leq 0.
\]

Now, given \( \varepsilon > 0 \) sufficiently small, we can choose an integer \( n_0 \geq 1 \), no matter how large, such that for all \( n \geq n_0 \)
\[
\beta_n \|x - Px\|^2 \leq \frac{\varepsilon}{2} \quad \text{and} \quad 2 <x - Px, D_{s_n}^{(u)}[T]x_n - Px > \leq \frac{\varepsilon}{2}.
\]

Therefore
\[
\|x_{n+1} - Px\|^2 = \|\beta_n x + (1 - \beta_n) D_{s_n}^{(u)}[T]x_n - Px\|^2
\]
\[
= \beta_n^2 \|x - Px\|^2 + (1 - \beta_n)^2 \|D_{s_n}^{(u)}[T]x_n - Px\|^2
\]
\[
+ 2\beta_n (1 - \beta_n) <x - Px, D_{s_n}^{(u)}[T]x_n - Px >
\]
\[
\leq \beta_n \varepsilon + (1 - \beta_n) \|x_n - Px\|^2,
\]
and inductively
\[
\|x_{n+1} - Px\|^2 \leq \left\{\frac{1}{n} \sum_{i=n_0}^{n} (1 - \beta_i)\right\} \varepsilon + \frac{1}{n} \sum_{i=n_0}^{n} (1 - \beta_i) \|x_{n_0} - Px\|^2
\]
\[
\leq \varepsilon + \exp \left\{ - \sum_{i=n_0}^{n} \beta_i \right\} \|x_{n_0} - Px\|^2.
\]

Hence since \( \sum_{n=0}^{\infty} \beta_n = \infty \) we have
\[
\limsup_{n \to \infty} \|x_n - Px\|^2 \leq \varepsilon.
\]

The final stage of the proof is to show that \( Px \) coincides with the asymptotic center of the sequence \( \{T^n x\} \). From the definition of the sequence \( \{x_n\} \) it follows that
\[
Px \in \text{Fix}(T) \cap \bigcap_{n \geq 0} \overline{\text{co}} \{T^k x : k \geq n\}.
\]
Let $u$ be the asymptotic center of $\{T^nx\}$. Then $u \in \text{Fix}(T)$ (cf. Brézis and Browder [1]). We claim that $Px = u$. Suppose, for a contradiction, that $Px \neq u$. We define

$$\rho(x : z) = \lim \inf_{n \to \infty} \|T^nx - z\|$$

for $z \in \mathbb{C}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ for which $\lim_{i \to \infty} \|T^{n_i}\alpha x - Px\| = \rho(x : Px)$. So, for any $\varepsilon > 0$ we can find an integer $i_0 = i_0(x, Px, \varepsilon)$, no matter how large, such that $\|T^{n_i}\alpha x - Px\| < \rho(x : Px) + \varepsilon$. Therefore, since $T$ is nonexpansive, we have

$$\|T^{n+i_0}\alpha x - Px\| \leq \|T^{n_i}\alpha x - Px\| < \rho(x : Px) + \varepsilon$$

for all $n \geq 0$. This gives $\lim_{n \to \infty} \|T^nx - Px\| = \rho(x : Px)$. Similarly we get

$$\lim_{n \to \infty} \|T^nx - u\| = \rho(x : u).$$

Hence

$$\rho(x : u) = \lim_{n \to \infty} \|T^nx - u\|$$

$$= \inf \left[ \lim \sup_{n \to \infty} \|T^nx - y\| : y \in \mathbb{C} \right]$$

$$< \lim \sup_{n \to \infty} \|T^nx - Px\|$$

$$= \lim_{n \to \infty} \|T^nx - Px\| = \rho(x : Px).$$

Taking into account that $H$ is a Hilbert space, let $K$ be the closed convex set of all $z \in H$ such that $\|z - u\| \leq \|z - Px\|$. Then one can find an integer $m_0$ for which $\{T^nx : n \geq m_0\} \subseteq K$, and hence

$$\overline{co}\{T^nx : n \geq m_0\} \subseteq K.$$

Whereas $K$ does not contain $Px$, in reality $Px$ belongs to $\overline{co}\{T^nx : n \geq m_0\}$.

This is a contradiction and $Px = u$. This completes the proof of the theorem.

REFERENCES


