数理解析研究所講究録
1136 巻 2000 年 139-146

DIRICHLET METHOD OF SUMMABILITY AND NONLINEAR ERGODIC THEOREMS IN HILBERT SPACE

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My purpose in this exposé is to give a brief summary of some recent results concerning the summation method of Dirichlet's type and nonlinear ergodic theorems of Dirichlet's type in Hilbert spaces. The exposé is mainly a report on the author's personal work on the subject, by a general survey. Most of the results mentioned below were discussed in [6] and [7].

Let X be a complex Banach space and let B[X] be the Banach algebra of bounded linear operators from X to itself. For a given $T\varepsilon B[X]$, the resolvent set of T denoted by $\rho(T)$ is the set of $\lambda\varepsilon C$ for which $(\lambda I-T)^{-1}$ exists as an operator in B[X] with domain X. The spectrum of T is the complement of $\rho(T)$ and denoted by $\sigma(T)$. $\rho(T)$ is an open subset of C and $\sigma(T)$ is a nonempty bounded closed subset of C. So, the spectral radius $\gamma(T)$ of T is well-defined: in fact, $\gamma(T) = \sup|\sigma(T)| = \lim_{n\to\infty} \|T^n\|^{1/n}$. The function $R(\lambda;T)$ defined by $R(\lambda;T)= (\lambda I-T)^{-1}$ for $\lambda\varepsilon\rho(T)$ is called the resolvent of T. It is well known [3] that $R(\lambda;T)$ is analytic in $\rho(T)$ and if $T\varepsilon B[X]$ and $|\lambda| > \gamma(T)$ then $\lambda\varepsilon\rho(T)$ and

$$R(\lambda;T) = (\lambda I-T)^{-1}= \sum_{n=0}^{\infty} \lambda^{-(n+1)}T^n,$$

the series converging in the uniform operator topology. It is also known that if $d(\lambda)$ denotes the distance from $\lambda\varepsilon C$ to $\sigma(T)$, then $\|R(\lambda;T)\| \geq 1/d(\lambda)$. If we take $\lambda = e^z$, $z = s+it$ $(s,t\varepsilon R)$, then $|\lambda| > \gamma(T)$ implies $s > \log\gamma(T)$ whenever $\gamma(T) > 0$. This characterization is a matter of great interest in connection with the question of what is the abscissa of uniform convergence of $R(\lambda;T)$ as a series.

Given $T \varepsilon B[X]$ let $\Phi(T)$ denote the class of all functions of complex variables which are analytic in some open set containing $\sigma(T)$. The following theorem is fundamental in the theory of linear ergodic theorems.

THEOREM 1 (Dunford [2]). Let $T \varepsilon B[X]$ and let $f_n \varepsilon \Phi(T)$ satisfy $\lim_{n\to\infty} f_n(1)=1$ and $(so)\lim_{n\to\infty} (I-T)f_n(T) = \theta$ (the null operator). Then the following statements

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1991 Mathematics Subject Classification. 47H09, 47H10.

are equivalent.

(1) $(so)\lim_{n\to\infty} f_n(T) = E$, $E^2 = E$, $EX = Ker(I-T)$.

(2) $\{f_n(T)x\}$ is weakly sequentially compact for each $x \in X$.

(3) $X = Ker(I-T) \oplus \overline{(I-T)X}$, $\sup_n \| f_n(T) \| < \infty$.

Suppose $f_n, f \in B[X]$, where $T \in B[X]$. If $f_n(T)$ converges strongly (or uniformly) then $f(T)f_n(T)$ converges strongly (or uniformly). We are interested in the converse problem. Under what conditions on f_n, f and T does the convergence of $f(T)f_n(T)$ imply the convergence of $f_n(T)$? Theorems describing this situation are in the nature of Tauberian theorems. The condition $(so)\lim_{n\to\infty} (I-T)f_n(T) = \theta$ appearing in Theorem 1 is just the case $f(T) = I-T$. Such Tauberian conditions are indispensable in discussing ergodic theorems. It should be noted that the strong regularity of summation methods has a very close connection with Tauberian conditions.

Now we consider the Dirichlet series of the following type

$$D[f,\mu;\, z](T) = \sum_{n=0}^{\infty} e^{-\mu_n z} f_n(T),$$

where $z \in C$, $f = \{f_n\} \subset \Phi(T)$ and $\mu = \{\mu_n\}$, $0 \le \mu_0 < \mu_1 < \cdots < \mu_n \to \infty$ as $n\to\infty$.

THEOREM 2 ([6]). Let $T \in B[X]$ and $f_n \in \Phi(T)$ be such that $\sup_n \| \Sigma_{k=0}^{n} f_k(T) \| > 0$ and define

$$a_\mu(f;\, T) = \limsup_{n\to\infty} \frac{\log \| \sum_{k=0}^{n} f_k(T) \|}{\mu_n}$$

with $f = \{f_n\}$ and $\mu = \{\mu_n\}$. Then the following statements hold.

(1) If $s > 0$ and $D[f,\mu;\, z](T)$ converges in the uniform operator topology for any $z \in C$ with $z = s+it$, $t \in R$, then $s \ge a_\mu(f;T)$.

(2) The Dirichlet series $D[f,\mu;\, z](T)$ converges in the uniform operator topology for any $z \in C$ with $Re(z) > \max(0, a_\mu(f;T))$ when $a_\mu(f;T) < \infty$.

If $0 \le a_\mu(f;T) < \infty$ in Theorem 2, we say that the number $a_\mu(f;T)$ is the abscissa of uniform convergence of the Dirichlet series $D[f,\mu;\, z](T)$. This theorem plays a fundamental role in investigating ergodic theorems of Dirichlet's type.

COROLLARY 3. Let $T \in B[X]$ satisfy the conditions $\sup_n \| \Sigma_{k=0}^{n} T^k \| > 0$ and $(uo)\lim_{n\to\infty} T^n/n^\omega = \theta$ for some $0 < \omega \le 1$. Let $f = \{f_n\}$, $f_n(T) = T^n$ $(n \ge 0)$ and $\mu = \{\mu_n\}$, $\mu_n = n+1$ $(n \ge 0)$. Then $a_\mu(f;T) \le 0$ and $R(\lambda;T) = \Sigma_{n=0}^{\infty} \lambda^{-(n+1)} T^n$ converges in the uniform operator topology for $\lambda \in C$ with $\log|\lambda| > 0$.

Proof. In view of Theorem 2, all that is required is to show that $a_\mu(f;T) \leq 0$. For a sufficiently small $\varepsilon > 0$, there exists by assumption an integer $N \geq 1$ such that

$$\|T^n\| < \varepsilon n^\omega \quad \text{for all} \quad n > N.$$

Then we have

$$a_\mu(f;T) = \limsup_{n\to\infty} \frac{\log\|\sum_{k=0}^{n} T^k\|}{n+1}$$

$$\leq \limsup_{n\to\infty} \frac{\log(n+1) + \log\{\max_{0\leq k\leq N} \|T^k\| + \varepsilon n^\omega\}}{n+1} = 0,$$

as desired.

Let $\mu = \{\mu_n\}$ $(n \geq 0)$ be a sequence of real numbers satisfying the following conditions :

(i) $0 \leq \mu_0 < \mu_1 < \cdots < \mu_n \to \infty$ as $n\to\infty$;

(ii) $\inf_{n\geq 0} \{\mu_{n+1} - \mu_n\} = \delta$ for some $\delta > 0$;

(iii) $\lim_{n\to\infty} \{\mu_{n+1} / \mu_n\} = 1$;

(iv) $\lim_{s\to 0+} g(s) = +\infty$;

(v) $\sup_{s>0} \frac{1}{g(s)} \sum_{n=0}^{\infty} n\{e^{-\mu_n s} - e^{-\mu_{n+1} s}\} < \infty$,

where $g(s) = \Sigma_{n=0}^{\infty} e^{-\mu_n s}$ which converges for $s > 0$. Such a sequence $\mu = \{\mu_n\}$ determines a strongly regular method of summability. This new summation method will be called the (D,μ)-method (Dirichlet method of summability). Let H be a real Hilbert space and let C be a nonempty bounded closed convex subset of H. A mapping $T: C \to C$ is called asymptotically nonexpansive with Lipschitz constants $\{\alpha_n\}$ if

$$\| T^n x - T^n y \| \leq (1+\alpha_n) \|x-y\| \text{ for all } n \geq 0 \text{ and all } x,y \in C,$$

where $\alpha_n \geq 0$ for all $n \geq 0$ and $\alpha_n \to 0$ as $n\to\infty$ (see Goebel and Kirk [4]). In particular, if $\alpha_n = 0$ for all $n \geq 0$ then T is said to be nonexpansive. If T is an asymptotically nonexpansive mapping on C, then for any $x \in C$

$$\| \frac{1}{g(s)} \sum_{n=p}^{p+q} e^{-\mu_n s} T^n x \| \leq M_C \left(\frac{1}{g(s)} \sum_{n=p}^{p+q} e^{-\mu_n s}\right) \to 0$$

as $p,q \to\infty$, where $M_C = \sup(\|x\|: x \in C)$. This means that for any $x \in C$,

$$D_s^{(\mu)}[T]x = \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} T^n x$$

is well defined for $s > 0$. Furthermore, for each $x \varepsilon C$ there exists a unique point $x_0 \varepsilon C$ such that

$$\limsup_{n\to\infty} \|T^n x - x_0\| = \inf_{y\varepsilon C} [\limsup_{n\to\infty} \|T^n x - y\|].$$

Such a point x_0 is called the asymptotic center of the sequence $\{T^n x\}$ (see Lim [5] and Brézis and Browder [1]). We are particularly interested in the weak and strong convergence of $D_s^{(\mu)}[T]x$ when $s \to 0+$.

THEOREM 4 ([7]). Let C be a nonempty bounded closed convex subset of H and let T be an asymptotically nonexpansive mapping of C into itself. Let $\mu = \{\mu_n\}$ be the (D,μ)-method. Then for any $x \varepsilon C$, $D_s^{(\mu)}[T]x$ converges weakly to the asymptotic center of $\{T^n x\}$ as $s \to 0+$.

Following the idea of Brézis and Browder [1], we say that the (D,μ)-method is proper if for each $\{\beta(n)\} \varepsilon \ell^\infty$ with $\beta(\cdot) \geq 0$, $[g(s)]^{-1} \Sigma_{n=0}^{\infty} e^{-\mu_n s} \beta(n)$ converges to some δ as $s \to 0+$, then

$$\lim_{s\to 0+} \left(\frac{1}{g(s)}\right)^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} e^{-(\mu_n+\mu_k)s} \beta(|n-k|) = \delta.$$

For example, the (D,μ)-method $\mu = \{\mu_n\}$ given by $\mu_n = an+b$, where $a > 0$ and $b > 0$, satisfies the properness condition just mentioned.

THEOREM 5 ([7]). Let C be a nonempty bounded closed convex subset of H and let T be a nonexpansive nonlinear mapping of C into itself. Let $\mu=\{\mu_n\}$ be the proper (D,μ)-method. Suppose that

(i) $0 \varepsilon C$ and $T(0) = 0$;

(ii) For some $c > 0$, T satisfies for all $u,v \varepsilon C$ the inequality

$$| < Tu, Tv > - < u,v > | \leq c\{ \|u\|^2 - \|Tu\|^2 + \|v\|^2 - \|Tv\|^2 \} ;$$

(iii) there is an element $\{\beta(n)\} \varepsilon \ell^\infty$ with $\beta(\cdot) \geq 0$ such that for any $x \varepsilon C$

$$| < T^p x, T^q x > - \beta(|p-q|) | \leq \gamma_{\min(p,q)} ,$$

where $\gamma_{\min(p,q)} \to \infty$ as $\min(p,q) \to \infty$.

Then for each $x \varepsilon C$, $D_s^{(\mu)}[T]x$ converges strongly as $s \to 0+$ to the asymptotic center of $\{T^n x\}$.

Next we consider the convergence of the sequences $\{x_n\} \subset C$ generated by the iteration procedures (called Mann's type and Halpern-Wittmann's type) by the Dirichlet method.

THEOREM 6 ([7]). Let C be a nonempty bounded closed convex subset of H and let T be a nonexpansive nonlinear mapping of C into itself. Let $\mu = \{\mu_n\}$ be the (D,μ)-method. Define (the Mann's type sequence)

$$x_1 = x \in C$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) D_{s_n}^{(\mu)}[T]x_n \quad \text{for } n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $[0, a]$ for some $0 < a < 1$ and $s_n \to 0+$ as $n \to \infty$. Then the sequence $\{x_n\}$ so defined converges weakly to the asymptotic center of $\{T^n x\}$.

THEOREM 7. Let C be a nonempty bounded closed convex subset of H and let T be a nonexpansive nonlinear mapping of C into itself. Let $\mu = \{\mu_n\}$ be the (D,μ)-method. Define (the Halpern-Wittmann's type sequence)

$$x_0 = x \in C$$
$$x_{n+1} = \beta_n x + (1-\beta_n) D_{s_n}^{(\mu)}[T]x_n \quad \text{for } n \geq 0,$$

where $s_n \to 0+$ as $n \to \infty$ and $\{\beta_n\}$ is a sequence in $[0, 1]$ satisfying the conditions

$$\lim_{n\to\infty} \beta_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \beta_n = \infty.$$

Then the sequence $\{x_n\}$ so defined converges strongly to Px, where P is the metric projection of H onto Fix(T). Moreover, Px coincides with the asymptotic center of $\{T^n x\}$.

Proof. Note that $\mathrm{Fix}(T) \neq \phi$ by Theorem 4 and let $z \in \mathrm{Fix}(T)$. Then

$$\begin{aligned} \| x_1 - z \| &= \| \beta_0 x + (1-\beta_0) D_{s_0}^{(\mu)}[T]x - z \| \\ &\leq \beta_0 \| x-z \| + (1-\beta_0) \| D_{s_0}^{(\mu)}[T]x - z \| \\ &\leq \beta_0 \| x-z \| + (1-\beta_0) \| x-z \| \\ &= \| x-z \|, \end{aligned}$$

and so, by the induction argument, $\|x_n - z\| \leq \| x-z \|$ for all $n \geq 0$. This implies that $\{x_n\}$ and $\{D_{s_n}^{(\mu)}[T]x_n\}$ are both bounded. Next we claim that

$$\limsup_{n\to\infty} < x - Px,\ D_{s_n}^{(\mu)}[T]x_n - Px > \le 0.$$

Since $\{D_{s_n}^{(\mu)}[T]x_n\}$ is bounded, there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty} < x - Px,\ D_{s_n}^{(\mu)}[T]x_n - Px > = \lim_{i\to\infty} < x - Px,\ D_{s_{n_i}}^{(\mu)}[T]x_{n_i} - Px >.$$

We may assume that $(w)\lim_{i\to\infty} D_{s_{n_i}}^{(\mu)}[T]x_{n_i} = z_0$ for some $z_0 \in C$ (through a subsequence of $\{n_i\}$, if necessary). Using Lemma 2 in [7] we have

$$\lim_{i\to\infty} \| D_{s_{n_i}}^{(\mu)}[T]x_{n_i} - T\,D_{s_{n_i}}^{(\mu)}[T]x_{n_i} \| = 0,$$

and thus, the demiclosedness of I-T at 0 yields $z_0 \in \mathrm{Fix}(T)$. Hence

$$\lim_{i\to\infty} < x - Px,\ D_{s_{n_i}}^{(\mu)}[T]x_{n_i} - Px > \ = \ < x - Px,\ z_0 - Px > \ \le 0.$$

Now, given $\varepsilon > 0$ sufficiently small, we can choose an integer $n_0 \ge 1$, no matter how large, such that for all $n \ge n_0$

$$\beta_n \| x - Px \|^2 \le \frac{\varepsilon}{2} \quad \text{and} \quad 2 < x - Px,\ D_{s_n}^{(\mu)}[T]x_n - Px > \ \le \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned}
\|x_{n+1} - Px\|^2 &= \| \beta_n x + (1-\beta_n) D_{s_n}^{(\mu)}[T]x_n - Px \|^2 \\
&= \beta_n^2 \| x - Px \|^2 + (1-\beta_n)^2 \| D_{s_n}^{(\mu)}[T]x_n - Px \|^2 \\
&\quad + 2\beta_n(1-\beta_n) < x - Px,\ D_{s_n}^{(\mu)}[T]x_n - Px > \\
&\le \beta_n \varepsilon + (1-\beta_n) \|x_n - Px\|^2,
\end{aligned}$$

and inductively

$$\begin{aligned}
\| x_{n+} - Px \|^2 &\le \{ 1 - \prod_{i=n_0}^{n} (1-\beta_i) \} \varepsilon + \prod_{i=n_0}^{n} (1-\beta_i) \| x_{n_0} - Px \|^2 \\
&\le \varepsilon + \exp\{ - \sum_{i=n_0}^{n} \beta_i \} \| x_{n_0} - Px \|^2.
\end{aligned}$$

Hence since $\Sigma_{n=0}^{\infty} \beta_n = \infty$ we have

$$\limsup_{n\to\infty} \| x_n - Px \|^2 \le \varepsilon.$$

The final stage of the proof is to show that Px coincides with the asymptotic center of the sequence $\{T^n x\}$. From the definition of the sequence $\{x_n\}$ it follows that

$$Px \in \mathrm{Fix}(T) \cap \left(\bigcap_{n\ge 0} \overline{co} \{T^k x : k \ge n\} \right).$$

Let u be the asymptotic center of $\{T^n x\}$. Then $u \in Fix(T)$ (cf. Brézis and Browder [1]). We claim that $Px = u$. Suppose, for a contradiction, that $Px \neq u$. We define

$$\rho(x : z) = \liminf_{n \to \infty} \| T^n x - z \|$$

for $z \in C$. Then there exists a subsequence $\{n_i\}$ of $\{n\}$ for which $\lim_{i\to\infty} \| T^{n_i} x - Px \| = \rho(x : Px)$. So, for any $\varepsilon > 0$ we can find an integer $i_0 = i_0(x, Px, \varepsilon)$, no matter how large, such that $\| T^{n_{i_0}} x - Px \| < \rho(x : Px) + \varepsilon$. Therefore, since T is nonexpansive, we have

$$\| T^{n+n_{i_0}} x - Px \| \leq \| T^{n_{i_0}} x - Px \| < \rho(x : Px) + \varepsilon$$

for all $n \geq 0$. This gives $\lim_{n\to\infty} \| T^n x - Px \| = \rho(x : Px)$. Similarly we get $\lim_{n\to\infty} \| T^n x - u \| = \rho(x : u)$. Hence

$$\begin{aligned} \rho(x : u) &= \lim_{n\to\infty} \| T^n x - u \| \\ &= \inf [\limsup_{n \to \infty} \| T^n x - y \| : y \in C] \\ &< \limsup_{n \to \infty} \| T^n x - Px \| \\ &= \lim_{n\to\infty} \| T^n x - Px \| = \rho(x : Px). \end{aligned}$$

Taking into account that H is a Hilbert space, let K be the closed convex set of all $z \in H$ such that $\| z - u \| \leq \| z - Px \|$. Then one can find an integer m_0 for which $\{T^n x : n \geq m_0\} \subset K$, and hence

$$\overline{co}\{T^n x : n \geq m_0\} \subset K.$$

Whereas K does not contain Px, in reality Px belongs to $\overline{co}\{T^n x : n \geq m_0\}$. This is a contradiction and $Px = u$. This completes the proof of the theorem.

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