数理解析研究所講究録
1136 巻 2000 年 162-165

Observation on Set Optimization with Set-Valued Maps*

島根大学総合理工学部　黒岩　大史 (Daishi Kuroiwa)

Department of Mathematics and Computer Science

Interdisciplinary Faculty of Science and Engineering, Shimane University

1060 Nishikawatsu, Matsue, Shimane 690-8504, JAPAN

Abstract

We consider a minimization problem whose objective is presented by a set-valued map, and treat as the problem as a set optimization problem. Also we define improved natural criteria of solutions of such problems, and investigate such solutions; especially we introduce some lower-semicontinuities and we show existence theorems.

1 Introduction and Preliminaries

Let X be a topological space, S a nonempty subset of X, $(Y, \leq_K)$ an ordered topological vector space with an ordering solid convex cone K, and F a map from X to 2^Y with $F(x) \neq \emptyset$ for each $x \in S$. We consider the following problem:

(SP) Minimize $F(x)$
subject to $x \in S$

Ordinary solutions of (SP) is considered as *vector optimization with set-valued maps*, however these are often not suitable for some set-valued optimization. Against the vector optimization, *set optimization with set-valued maps* is introduced at [1] as follows: $x_0 \in S$ is said to be *l-minimal solution* of (SP) if $F(x) \leq^l F(x_0)$ and $x \in S$ implies $F(x_0) \leq^l F(x)$, and *u-minimal solution* of (SP) if $F(x) \leq^u F(x_0)$ and $x \in S$ implies $F(x_0) \leq^u F(x)$.

Such notions of solutions are natural, suitable, and useful for some set optimization problem, however, there some faults as follows:

(i) there are too many solutions;

*This research is partially supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan, No. 11740065

(ii) it is difficult to check whether an element is a solutions or not.

In this paper, we introduce certain improved concepts of solutions which are more natural to consider set optimization and consider relations between such notions and usual ones. Also, we derive some cone-convexities for set-valued optimization, and we prove existence theorems for such problems.

Definition 1.1 Let $M \subset K^+$ be a set of weight. For nonempty subsets A, B of Y,

$$A \leq_M^l B \iff \langle y^*, A+K\rangle \supset \langle y^*, B\rangle, \quad \forall y^* \in M;$$

$$A \leq_M^u B \iff \langle y^*, A\rangle \subset \langle y^*, B-K\rangle, \quad \forall y^* \in M.$$

Proposition 1.1 Let $M \subset K^+$ be a set of weight. For nonempty subsets A, B of Y,

(i) $A \leq^l B \Rightarrow A \leq_M^l B$
(ii) if $M = K^+$ and $A+K$, $B+K$: closed convex, then $A \leq^l B \iff A \leq_M^l B$.

In the rest of the paper, we fix a weight set $M \subset K^+$.

Definition 1.2 $x_0 \in S$ is said to be

(i) l-minimal solution with weight M if $x \in S, F(x) \leq_M^l F(x_0)$ implies $F(x_0) \leq_M^l F(x)$;
(ii) u-minimal solution with weight M if $x \in S, F(x) \leq_M^u F(x_0)$ implies $F(x_0) \leq_M^u F(x)$.

Example 1.1 (solutions with weight)
Let $X = Y = \mathbf{R}^n$, $K = K^+ = \mathbf{R}^n_+$, $M = \{e_1, e_2, \ldots, e_n\}$, $S \subset X$, $F : S \to 2^Y$. Assume that for each $x \in S$, there exists $y \in X$ such that $y \leq^l F(x)$. Then,

(i) x_0 : l-minimal solution $\Longrightarrow \operatorname{Inf} F(x_0) \in \operatorname{Min} \bigcup_{x\in S} \operatorname{Inf} F(x)$
(ii) x_0 : u-minimal solution $\Longrightarrow \operatorname{Sup} F(x_0) \in \operatorname{Min} \bigcup_{x\in S} \operatorname{Sup} F(x)$

The reverse implication is satisfied when F be a compact valued map.

In this example, we feel *active* impression from l-minimal solution and *passive* from u-minimal solution. About this example, we have the following proposition:

Proposition 1.2 (i) If $\bigcup_{x\in S} \operatorname{Inf} F(x)$ is closed, and there exists $y^* \in Y$ such that $\langle y^*, \cdot\rangle$ is bounded below on $\bigcup_{x\in S} \operatorname{Inf} F(x)$, then there exists an l-minimal solution.
(ii) if $\bigcup_{x\in S} \operatorname{Sup} F(x)$ is closed, and there exists $y^* \in Y$ such that $\langle y^*, \cdot\rangle$ is bounded below on $\bigcup_{x\in S} \operatorname{Sup} F(x)$, then there exists an u-minimal solution.

Example 1.2 (check whether $A \leq^l B$ or not)
Let $M = \{e_1, e_2, \ldots, e_n\} \subset \mathbf{R}^n$, and A, $B \subset \mathbf{R}^n$ with $|A| = |B| = m$. When we check if $A \leq^l B$, the worst order of calculus is $m^2 n$. However if we check whether $A \leq^l_M B$ or not, it is only nm.

2 Continuities and Existence Theorems

We redefine cone-continuities of set-valued optimization based on [1], characterize such notions, and show main results.

Definition 2.1 A set-valued map F is said to be

(i) l-lower semicontinuous on S with weight M if for any l-closed subset A of Y,

$$\mathcal{L}^l(A) = \{x \in S | F(x) \leq^l_M A\}$$

is closed.

(ii) l-demi-lower semicontinuous at $x_0 \in S$ with weight M if for each net $\{x_\lambda\}$ with $F(x_{\lambda'}) \leq^l_M F(x_\lambda)$ if $\lambda < \lambda'$ and $x_\lambda \to x_0$,

$$F(x_0) \leq^l_M \operatorname*{Lim\,sup}_{\lambda}(F(x_\lambda) + K)$$

is satisfied. F is said to be l-demi-lower semicontinuous on S with weight M if it is l-demi-lower semicontinuous at each point of S.

Also we define u type lower semicontinuities in the similar way.

We can define another lower-semicontinuities as l and u types, however we omit, see [1].

Proposition 2.1 (i) If F is l-lower semicontinuous then F is also l-lower semicontinuous with weight M,

(ii) If F is l-demi-lower semicontinuous then F is also l-demi-lower semicontinuous with weight M.

Proposition 2.2 If F is l-lower semicontinuous with weight M then F is also l-demi-lower semicontinuous at $x_0 \in S$ with weight M.

In the two propositions above, we can show the similar claims with respect to u type semicontinuities. By using such continuities, we have the following existence theorems:

Theorem 2.1 If S is compact and F is l-demi-lower semicontinuous with weight M, then there exists an l-minimal solution of (SP) with weight M.

Theorem 2.2 If (X, d) is a complete metric space, Y is a locally convex topological vector space, F is l-closed and l-lower semicontinuous with weight M, and the following condition is satisfied:

there exists $y^* \in K^+ \setminus \{\theta^*\}$ such that

- $\inf \langle y^*, F(x) \rangle$ is finite for each $x \in S$, and
- $F(x_1) \leq_M^l F(x_2), x_1, x_2 \in S \Rightarrow \inf \langle y^*, F(x_2) \rangle - \inf \langle y^*, F(x_1) \rangle \geq d(x_2, x_1)$.

Then, there exists an l-minimal solution of (SP) with weight M.

References

[1] D. Kuroiwa, Some Duality Theorems of Set-Valued Optimization with Natural Criteria, *Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis*, World Scientific, 1999, 221–228.