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ON GENERALIZED FRACTIONAL INTEGRALS

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It is known that the fractional integral I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $0 < \alpha < n$, $1 < p < n/\alpha$ and $n/q = n/p - \alpha$ as the Hardy-Littlewood-Sobolev theorem. We introduce generalized fractional integrals and extend the Hardy-Littlewood-Sobolev theorem to the Orlicz spaces. We show that, for example, a generalized fractional integral I_ϕ is bounded from $\exp L^p$ to $\exp L^q$ (see Example 1.2).

It is also known that the modified fractional integral $\tilde{I}_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ when $0 < \alpha < n$ and $p = n/\alpha$, from $L^p(\mathbb{R}^n)$ to $\mathrm{Lip}_{\alpha-n/p}(\mathbb{R}^n)$ when $0 < \alpha < n$ and $0 < \alpha - n/p < 1$, from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ when $0 < \alpha < 1$, and from $\mathrm{Lip}_\beta(\mathbb{R}^n)$ to $\mathrm{Lip}_{\alpha+\beta}(\mathbb{R}^n)$ when $0 < \alpha < \alpha + \beta < 1$. We also investigate the boundedness of generalized fractional integrals from the Orlicz space to BMO_ψ and from BMO_{ψ_1} to BMO_{ψ_2}, where BMO_ψ is the function space defined using the mean oscillation and a weight function $\psi : (0, +\infty) \to (0, +\infty)$. If $\psi(r) \equiv 1$, then $\mathrm{BMO}_\psi = \mathrm{BMO}$. If $\psi(r) = r^\alpha$ $(0 < \alpha \le 1)$, then $\mathrm{BMO}_\psi = \mathrm{Lip}_\alpha$.

1. Generalized fractional integrals on the Orlicz spaces

For a function $\phi : (0, +\infty) \to (0, +\infty)$, let

$$I_\phi f(x) = \int_{\mathbb{R}^n} f(y) \frac{\phi(|x-y|)}{|x-y|^n}\, dy.$$

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We consider the following conditions on ϕ:

$$\frac{1}{A_1} \le \frac{\phi(s)}{\phi(r)} \le A_1 \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{1.1}$$

$$\frac{\phi(r)}{r^n} \le A_2 \frac{\phi(s)}{s^n} \quad \text{for} \quad s \le r, \tag{1.2}$$

$$\int_0^1 \frac{\phi(t)}{t}\, dt < +\infty, \tag{1.3}$$

where $A_i > 0$ $(i = 1, 2)$ are independent of $r, s > 0$. If $\phi(r) = r^\alpha$, $0 < \alpha < n$, then I_ϕ is the fractional integral or the Riesz potential denoted by I_α.

A function $\Phi : [0, +\infty) \to [0, +\infty]$ is called a Young function if Φ is convex, $\lim_{r \to +0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r \to +\infty} \Phi(r) = +\infty$. Any Young function is increasing. For a Young function Φ, the complementary function is defined by

$$\Psi(r) = \sup\{rs - \Phi(s) : s \ge 0\}, \quad r \ge 0.$$

For example, if $\Phi(r) = r^p/p$, $1 < p < \infty$, then $\Psi(r) = r^{p'}/p'$, $1/p + 1/p' = 1$. If $\Phi(r) = r$, then $\Psi(r) = 0 (0 \le r \le 1), = +\infty (r > 1)$.

For a Young function Φ, let

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\epsilon |f(x)|)\, dx < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_\Phi = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \le 1 \right\},$$

$$L^\Phi_{weak}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \sup_{r>0} \Phi(r)\, m(r, \epsilon f) < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{\Phi, weak} = \inf \left\{ \lambda > 0 : \sup_{r>0} \Phi(r)\, m\left(r, \frac{f}{\lambda}\right) \le 1 \right\},$$

where $m(r, f) = |\{x \in \mathbb{R}^n : |f(x)| > r\}|$.

If a Young function Φ satisfies

$$0 < \Phi(r) < +\infty \quad \text{for} \quad 0 < r < +\infty, \tag{1.4}$$

then Φ is continuous and bijective from $[0, +\infty)$ to itself. The inverse function Φ^{-1} is also increasing and continuous.

A function Φ said to satisfy the ∇_2-condition, denoted $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k} \Phi(kr), \quad r \ge 0,$$

for some $k > 1$.

Let $Mf(x)$ be the maximal function, i.e.

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x.

We assume that Φ satisfies (1.4). Then M is bounded from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Phi}_{weak}(\mathbb{R}^n)$. If $\Phi \in \nabla_2$, then M is bounded on $L^{\Phi}(\mathbb{R}^n)$.

Our main results are as follows:

Theorem 1.1. *Let ϕ satisfy (1.1)$\sim$(1.3). Let Φ_i $(i = 1, 2)$ be Young functions with (1.4). Assume that there exist constants $A, A', A'' > 0$ such that, for all $r > 0$,*

$$\int_r^{+\infty} \Psi_1 \left(\frac{\phi(t)}{A \int_0^r (\phi(s)/s)\, ds\ {\Phi_1}^{-1}(1/r^n) t^n} \right) t^{n-1}\, dt \le A', \tag{1.5}$$

$$\int_0^r \frac{\phi(t)}{t}\, dt\ {\Phi_1}^{-1} \left(\frac{1}{r^n} \right) \le A''\ {\Phi_2}^{-1} \left(\frac{1}{r^n} \right), \tag{1.6}$$

where Ψ_1 is the complementary function with respect to Φ_1. Then, for any $C_0 > 0$, there exists a constant $C_1 > 0$ such that, for $f \in L^{\Phi_1}(\mathbb{R}^n)$,

$$\Phi_2 \left(\frac{|I_\phi f(x)|}{C_1 \|f\|_{\Phi_1}} \right) \le \Phi_1 \left(\frac{Mf(x)}{C_0 \|f\|_{\Phi_1}} \right). \tag{1.7}$$

Therefore I_ϕ is bounded from $L^{\Phi_1}(\mathbb{R}^n)$ to $L^{\Phi_2}_{weak}(\mathbb{R}^n)$. Moreover, if $\Phi_1 \in \nabla_2$, then I_ϕ is bounded from $L^{\Phi_1}(\mathbb{R}^n)$ to $L^{\Phi_2}(\mathbb{R}^n)$.

For functions $\theta, \kappa : (0, +\infty) \to (0, +\infty)$, we denote $\theta(r) \sim \kappa(r)$ if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \le \kappa(r) \le C\theta(r), \quad r > 0.$$

A function $\theta : (0, +\infty) \to (0, +\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that $\theta(r) \le C\theta(s)$ $(\theta(r) \ge C\theta(s))$ for $r \le s$.

Remark 1.1. From (1.1) it follows that

$$\phi(r) \le C \int_0^r \frac{\phi(t)}{t}\, dt. \tag{1.8}$$

If $\phi(r)/r^\varepsilon$ is almost increasing for some $\varepsilon > 0$ and $\phi(t)/t^n$ is almost decreasing, then ϕ satisfies (1.1)$\sim$(1.3) and $\int_0^r (\phi(t)/t)\, dt \sim \phi(r)$. Let, for example, $\phi(r) = r^\alpha(\log(1/r))^{-\beta}$ for small r. If $\alpha = 0$ and $\beta > 1$, then $\int_0^r(\phi(t)/t)\,dt \sim (\log(1/r))^{-\beta+1}$. If $\alpha > 0$ and $-\infty < \beta < +\infty$, then $\int_0^r(\phi(t)/t)\,dt \sim \phi(r)$.

Remark 1.2. In the case $\Phi_1(r) = r$, (1.5) is equivalent to

$$\frac{\phi(t)}{t^n} \le \frac{A\int_0^r(\phi(s)/s)\,ds}{r^n}, \quad 0 < r \le t.$$

This inequality follows from (1.2) and (1.8).

The following corollaries are stated without the complementary function.

Corollary 1.2. *Let ϕ satisfy* (1.1)$\sim$(1.3). *Let Φ_i $(i = 1, 2)$ be Young functions with* (1.4). *Assume that*

$$\int_0^r \frac{\phi(t)}{t}\,dt\;\Phi_1{}^{-1}\left(\frac{1}{r^n}\right)$$

is almost decreasing and that there exist constants $A, A' > 0$ such that, for all $r > 0$,

$$\int_r^{+\infty} \frac{\phi(t)}{t}\Phi_1{}^{-1}\left(\frac{1}{t^n}\right)\,dt \le A\int_0^r \frac{\phi(t)}{t}\,dt\;\Phi_1{}^{-1}\left(\frac{1}{r^n}\right), \tag{1.9}$$

$$\int_0^r \frac{\phi(t)}{t}\,dt\;\Phi_1{}^{-1}\left(\frac{1}{r^n}\right) \le A'\,\Phi_2{}^{-1}\left(\frac{1}{r^n}\right). \tag{1.10}$$

Then (1.7) *holds. Therefore I_ϕ is bounded from $L^{\Phi_1}(\mathbb{R}^n)$ to $L^{\Phi_2}_{weak}(\mathbb{R}^n)$. Moreover, if $\Phi_1 \in \nabla_2$, then I_ϕ is bounded from $L^{\Phi_1}(\mathbb{R}^n)$ to $L^{\Phi_2}(\mathbb{R}^n)$.*

Remark 1.3. If $r^\varepsilon\phi(r)\Phi_1{}^{-1}(1/r^n)$ is almost decreasing for some $\varepsilon > 0$, then

$$\int_r^{+\infty} \frac{\phi(t)}{t}\Phi_1{}^{-1}\left(\frac{1}{t^n}\right)\,dt \le C\phi(r)\;\Phi_1{}^{-1}\left(\frac{1}{r^n}\right).$$

This inequality and (1.8) yield (1.9).

Remark 1.4. We cannot replace (1.6) or (1.10) by

$$\phi(r)\;\Phi_1{}^{-1}\left(\frac{1}{r^n}\right) \le A\;\Phi_2{}^{-1}\left(\frac{1}{r^n}\right) \quad \text{for all } r > 0$$

(see Section 5 in [5]).

Corollary 1.3. *Let $\phi(r) = r^\alpha$ with $0 < \alpha < n$. Let Φ_i $(i = 1, 2)$ be Young functions with* (1.4)*. Assume that there exist constants $A, A' > 0$ such that, for all $r > 0$,*

$$\int_r^{+\infty} t^{\alpha-1} {\Phi_1}^{-1}\left(\frac{1}{t^n}\right) dt \le A r^\alpha\, {\Phi_1}^{-1}\left(\frac{1}{r^n}\right), \tag{1.11}$$

$$r^\alpha\, {\Phi_1}^{-1}\left(\frac{1}{r^n}\right) \le A'\, {\Phi_2}^{-1}\left(\frac{1}{r^n}\right). \tag{1.12}$$

Then (1.7) *holds. Therefore I_α is bounded from $L^{\Phi_1}(\mathbb{R}^n)$ to $L^{\Phi_2}_{weak}(\mathbb{R}^n)$. Moreover, if $\Phi_1 \in \nabla_2$, then I_α is bounded from $L^{\Phi_1}(\mathbb{R}^n)$ to $L^{\Phi_2}(\mathbb{R}^n)$.*

Remark 1.5. Results similar to this corollary are in [2] and [10]. Kokilashvili and Krbec [2] considered the boundedness of I_α with weights, and gave a necessary and sufficient condition on the weights so that weighted inequalities hold. Torchinsky [10] treated sublinear operators with weak type (p_i, q_i) $(i = 1, 2)$ and used interpolation.

We state examples given by the theorem and corollaries as follows:

Example 1.1. Let ϕ satisfy (1.1) and

$$\phi(r) = \begin{cases} r^{\alpha_1} & \text{for small } r, \\ r^{\alpha_2} & \text{for large } r, \end{cases}$$

where $0 < \alpha_1 < n$ and $-\infty < \alpha_2 < n$. Let Φ_1 and Φ_2 be convex and

$$\Phi_1(\xi) = \begin{cases} \xi^{p_2}, & \text{for small } \xi, \\ \xi^{p_1}, & \text{for large } \xi, \end{cases}$$

$$\Phi_2(\xi) = \begin{cases} \xi^{q_2}, & \text{for small } \xi, \\ \xi^{q_1}, & \text{for large } \xi, \end{cases}$$

where

$$1 < p_1 < n/\alpha_1,\ q_1 > 1,\ n/q_1 \ge n/p_1 - \alpha_1,$$

$$\begin{cases} 1 < p_2 < n/\alpha_2,\ n/q_2 \le n/p_2 - \alpha_2, & \text{when } 0 < \alpha_2 < n, \\ 1 < p_2 < q_2 < \infty, & \text{when } \alpha_2 = 0, \\ 1 < p_2 \le q_2 < \infty, & \text{when } \alpha_2 < 0. \end{cases}$$

Then (1.7) holds and I_ϕ is bounded from $L^{\Phi_1}(\mathbb{R}^n)$ to $L^{\Phi_2}(\mathbb{R}^n)$.

Remark 1.6. The case $(\alpha_1, p_1, q_1) = (\alpha_2, p_2, q_2) = (\alpha, p, q)$ is the Hardy-Littlewood-Sobolev theorem.

Example 1.2. Let ϕ satisfy (1.1) and

$$\phi(r) = \begin{cases} (\log(1/r))^{-(\alpha_1+1)} & \text{for small } r, \\ (\log r)^{\alpha_2 - 1} & \text{for large } r, \end{cases}$$

where $\alpha_1 > 0$ and $-\infty < \alpha_2 < +\infty$. Let Φ_1 and Φ_2 be convex and

$$\Phi_1(\xi) = \begin{cases} 1/\exp(1/\xi^{p_2}), & \text{for small } \xi, \\ \exp(\xi^{p_1}), & \text{for large } \xi, \end{cases}$$

$$\Phi_2(\xi) = \begin{cases} 1/\exp(1/\xi^{q_2}), & \text{for small } \xi, \\ \exp(\xi^{q_1}), & \text{for large } \xi, \end{cases}$$

where

$$0 < p_1 < 1/\alpha_1, \; 1/q_1 \geq 1/p_1 - \alpha_1, \tag{1.13}$$

$$\begin{cases} 0 < p_2 < 1/\alpha_2, \; 1/q_2 \leq 1/p_2 - \alpha_2, & \text{when } \alpha_2 > 0, \\ 0 < p_2 < q_2 < \infty, & \text{when } \alpha_2 = 0, \\ 0 < p_2 \leq q_2 < \infty, & \text{when } \alpha_2 < 0. \end{cases} \tag{1.14}$$

Then (1.7) holds and I_ϕ is bounded from $L^{\Phi_1}(\mathbb{R}^n)$ to $L^{\Phi_2}(\mathbb{R}^n)$.

Example 1.3. Let ϕ satisfy (1.1) and

$$\phi(r) = \begin{cases} (\log(1/r))^{-1}(\log\log(1/r))^{-(\alpha_1+1)} & \text{for small } r, \\ (\log r)^{-1}(\log\log r)^{\alpha_2 - 1} & \text{for large } r, \end{cases}$$

where $\alpha_1 > 0$ and $-\infty < \alpha_2 < +\infty$. Let Φ_1 and Φ_2 be convex and

$$\Phi_1(\xi) = \begin{cases} 1/\exp\exp(1/\xi^{p_2}), & \text{for small } \xi, \\ \exp\exp(\xi^{p_1}), & \text{for large } \xi, \end{cases}$$

$$\Phi_2(\xi) = \begin{cases} 1/\exp\exp(1/\xi^{q_2}), & \text{for small } \xi, \\ \exp\exp(\xi^{q_1}), & \text{for large } \xi, \end{cases}$$

where p_1, p_2, q_1 and q_2 satisfy (1.13) and (1.14). Then (1.7) holds and I_ϕ is bounded from $L^{\Phi_1}(\mathbb{R}^n)$ to $L^{\Phi_2}(\mathbb{R}^n)$.

Example 1.4. Let ϕ satisfy (1.1) and

$$\phi(r) = \begin{cases} (\log(1/r))^{-(\alpha_1+1)} & \text{for small } r, \\ (\log r)^{\alpha_2 - 1} & \text{for large } r, \end{cases}$$

where $\alpha_i > 0$ $(i = 1, 2)$. Let Φ_1 and Φ_2 be convex and

$$\Phi_1(\xi) = \xi^p, \quad \Phi_2(\xi) = \begin{cases} \xi^p(\log(1/\xi))^{-p\alpha_2} & \text{for small } \xi, \\ \xi^p(\log \xi)^{p\alpha_1} & \text{for large } \xi, \end{cases}$$

where $1 \le p < \infty$. Then (1.7) holds and I_ϕ is bounded from $L^1(\mathbb{R}^n)$ to $L^{\Phi_2}_{weak}(\mathbb{R}^n)$ for $p = 1$ and from $L^p(\mathbb{R}^n)$ to $L^{\Phi_2}(\mathbb{R}^n)$ for $1 < p < \infty$.

Example 1.5. Let ϕ satisfy (1.1) and

$$\phi(r) = \begin{cases} r^n(\log(1/r))^{\alpha_1} & \text{for small } r, \\ r^n(\log r)^{-\alpha_2} & \text{for large } r, \end{cases}$$

where $\alpha_i > 0$ $(i = 1, 2)$. Let Φ_1 and Φ_2 be convex and

$$\Phi_1(\xi) = \xi, \quad \Phi_2(\xi) = \begin{cases} 1/\exp((1/\xi)^{1/\alpha_2}) & \text{for small } \xi, \\ \exp(\xi^{1/\alpha_1}) & \text{for large } \xi. \end{cases}$$

Then (1.7) holds and I_ϕ is bounded from $L^{\Phi_1}(\mathbb{R}^n)$ to $L^{\Phi_2}_{weak}(\mathbb{R}^n)$.

2. Generalized fractional integrals on the Orlicz spaces and BMO_ψ

Let $B(a, r) = \{x \in \mathbb{R}^n : |x - a| < r\}$. We define the modified version of I_ϕ as follows:

$$\tilde{I}_\phi f(x) = \int_{\mathbb{R}^n} f(y) \left(\frac{\phi(|x - y|)}{|x - y|^n} - \frac{\phi(|y|)(1 - \chi_{B(O,1)}(y))}{|y|^n} \right) dy,$$

where $\chi_{B(O,1)}$ is the characteristic function of $B(O, 1)$. We consider the following conditions on ϕ: (1.1), (1.3) and

$$\left| \frac{\phi(r)}{r^n} - \frac{\phi(s)}{s^n} \right| \le A_3 |r - s| \frac{\phi(r)}{r^{n+1}} \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{2.1}$$

$$\frac{\phi(r)}{r^{n+1}} \le A_4 \frac{\phi(s)}{s^{n+1}} \quad \text{for} \quad s \le r, \tag{2.2}$$

$$\int_r^{+\infty} \frac{\phi(t)}{t^2}\, dt \le A_5 \frac{\phi(r)}{r}, \tag{2.3}$$

where $A_i > 0$ $(i = 3, 4, 5)$ is independent of $r, s > 0$. If $\phi(r) r^\alpha$ is increasing for some $\alpha \ge 0$ and $\phi(r)/r^\beta$ is decreasing for some $\beta \ge 0$, then ϕ satisfies (1.1) and (2.1). If $\phi(r) = r^\alpha$, $0 < \alpha \le n + 1$, then $\tilde{I}_\phi = \tilde{I}_\alpha$ which is the modified version of the fractional integral I_α. If $\tilde{I}_\phi f$ and $I_\phi f$ are well defined, then $\tilde{I}_\phi f - I_\phi f$ is a constant.

For a function $\psi : (0, +\infty) \to (0, +\infty)$, let

$$\mathrm{BMO}_\psi(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \sup_{B=B(a,r)} \frac{1}{\psi(r)} \frac{1}{|B|} \int_B |f(x) - f_B|\, dx < +\infty \right\},$$

$$\|f\|_{\mathrm{BMO}_\psi} = \sup_{B=B(a,r)} \frac{1}{\psi(r)} \frac{1}{|B|} \int_B |f(x) - f_B|\, dx,$$

$$\text{where} \qquad f_B = \frac{1}{|B|} \int_B f(x)\, dx.$$

If $\psi(r) \equiv 1$, then $\mathrm{BMO}_\psi(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $\psi(r) = r^\alpha$, $0 < \alpha \le 1$, then $\mathrm{BMO}_\psi(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

It is known that $\tilde{I}_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ when $0 < \alpha < n$ and $p = n/\alpha$, and from $L^p(\mathbb{R}^n)$ to $\mathrm{Lip}_{\alpha - n/p}(\mathbb{R}^n)$ when $0 < \alpha < n$ and $0 < \alpha - n/p < 1$. We extend these as follows:

Theorem 2.1. *Let ϕ satisfy (1.1), (1.3), (2.1) and (2.2). Let Φ be Young function with (1.4), ψ be almost increasing and $\psi(r) \sim \psi(2r)$. Assume that there exist constants $A, A', A'' > 0$ such that, for all $r > 0$,*

$$\int_r^{+\infty} \Psi\left(\frac{r\phi(t)}{A \int_0^r (\phi(s)/s)\, ds\, \Phi^{-1}(1/r^n) t^{n+1}} \right) t^{n-1}\, dt \le A', \tag{2.4}$$

$$\int_0^r \frac{\phi(t)}{t}\, dt\, \Phi^{-1}\left(\frac{1}{r^n}\right) \le A''\psi(r), \tag{2.5}$$

where Ψ is the complementary function with respect to Φ. Then $\tilde{I}_\phi$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $\mathrm{BMO}_\psi(\mathbb{R}^n)$.

It is known that $\tilde{I}_\alpha$ is bounded from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ when $0 < \alpha < 1$, and from $\mathrm{Lip}_\beta(\mathbb{R}^n)$ to $\mathrm{Lip}_{\alpha+\beta}(\mathbb{R}^n)$ when $0 < \alpha < \alpha + \beta < 1$. We extend these as follows:

Theorem 2.2. *Let ϕ satisfy (1.1), (1.3), (2.1) and (2.3). Let ψ_i be almost increasing and $\psi_i(r) \sim \psi_i(2r)$ $(i = 1, 2)$. Assume that there exist constants $A, A' > 0$ such that, for all $r > 0$,*

$$\int_r^{+\infty} \frac{\phi(t)\psi_1(t)}{t^2}\, dt \le A \frac{\phi(r)\psi_1(r)}{r}, \tag{2.6}$$

$$\int_0^r \frac{\phi(t)}{t}\, dt\, \psi_1(r) \le A'\psi_2(r). \tag{2.7}$$

Then $\tilde{I}_\phi$ is bounded from $\mathrm{BMO}_{\psi_1}(\mathbb{R}^n)$ to $\mathrm{BMO}_{\psi_2}(\mathbb{R}^n)$.

The results in Figure 1 are known. By Theorems 2.1 and 2.2 we have the results in Figure 2.

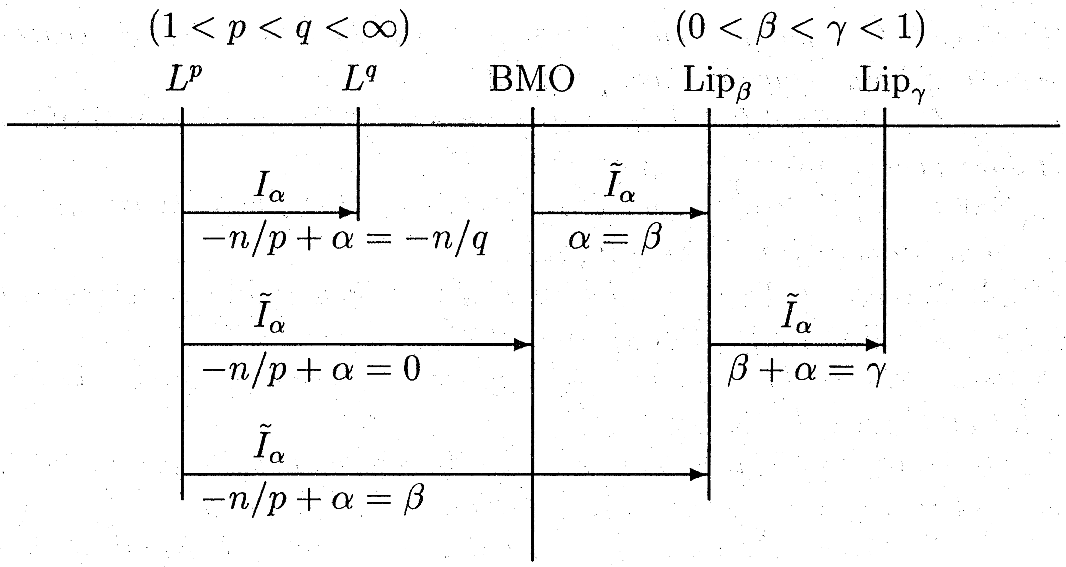

FIGURE 1. Boundedness of fractional integrals

$\phi(r) = (\log(1/r))^{-(\alpha+1)}$ for small $r > 0$ $(\alpha > 0)$

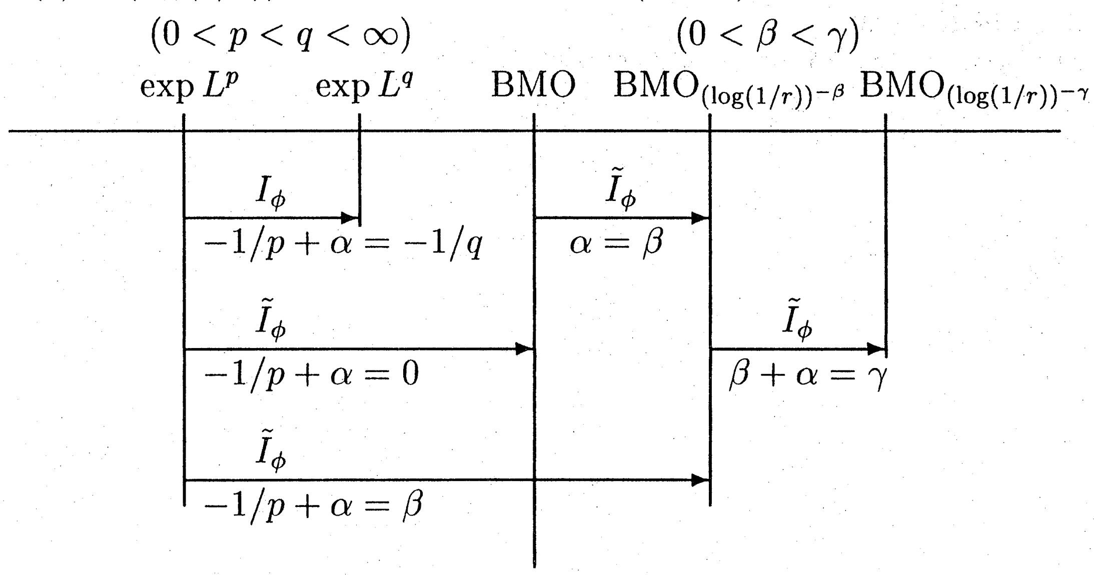

FIGURE 2. Boundedness of generalized fractional integrals

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