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The arbitrary precision calculation of logarithms with continued fraction expansions

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1 Introduction

We have thought about the arbitrary precision calculation of logarithms, for example the error evaluation equations and their improvement and "divided calculation" as one of the calculation methods. In this paper we would like to show the result of the calculation time of logarithmic function using the properties of logarithms.

2 Continued Fraction

First we summarize some basics of continued fraction and their properties. There are some definitions. The following formula is called continued fraction:

$$q_0+\cfrac{p_1}{q_1+\cfrac{p_2}{q_2+\cfrac{p_3}{\ddots\,\cfrac{}{q_{n-1}+\cfrac{p_n}{q_n+\ddots,}}}}} \tag{1}$$

where, $p_1, p_2, \cdots, p_n, q_1, q_2, \cdots, q_n$ are integers. It is called a finite continued fraction when n is finite. Otherwise, it is called an infinite continued fraction. It has become customary to write continued fraction in a typographically more convenient form like the following:

$$q_0+\frac{p_1|}{|q_1}+\frac{p_2|}{|q_2}+\cdots+\frac{p_n|}{|q_n}+\cdots. \tag{2}$$

For $n = 1, 2, 3, \cdots$ the following formula is called nth approximants of continued fraction:

$$q_0 + \frac{p_1|}{|q_1} + \frac{p_2|}{|q_2} + \cdots + \frac{p_n|}{|q_n}. \tag{3}$$

In the above formula the numbers p_n and q_n are called the nth **numerator** and the nth **denominator** of the continued fraction (2).

If the approximants of continued fraction (3) is converted to a fraction $\frac{P_n}{Q_n}$, then following theorem is obtained [2, 3]:

Theorem 1 *Let* $P_{-1} = 1, Q_{-1} = 0$.

$$\begin{cases} P_n = q_n P_{n-1} + p_n P_{n-2} \\ Q_n = q_n Q_{n-1} + p_n Q_{n-2} \end{cases} \quad (\text{for } n = 1, 2, 3, \cdots) \tag{4}$$

Theorem 2 *Let* $p_1, p_2, \cdots, p_n, q_1, q_2, \cdots, q_n > 0$. *If*

$$\lim_{n\to\infty} \frac{\prod_{i=1}^{n} p_i}{Q_n Q_{n-1}} = 0,$$

then P_n/Q_n *is convergent. Put the convergence* α, *then*

$$\left| \frac{P_n}{Q_n} - \alpha \right| < \frac{\prod_{i=1}^{n} p_i}{Q_n Q_{n-1}} \quad (p_i > 0) \tag{5}$$

The above recursive equation (4) is expressed as following [2]:

Corollary 3

$$\begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} = \begin{pmatrix} P_{n-2} & P_{n-1} \\ Q_{n-2} & Q_{n-1} \end{pmatrix} \begin{pmatrix} 0 & p_n \\ 1 & q_n \end{pmatrix} \tag{6}$$

Let

$$\begin{aligned} M_k &= \begin{pmatrix} 0 & p_k \\ 1 & q_k \end{pmatrix}, \\ M_{1,k} &= \begin{pmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{pmatrix}. \end{aligned}$$

Then

$$M_{1,n} = M_1 M_2 \cdots M_n = M_{1,n-1} M_n.$$

This notation is useful to calculate a value of a continued fraction. We call the calculation method using the above equation, `diviced calculation`[4].

3 Calculation of logarithm using continued fraction

3.1 Change the equations

The following formula for natural logarithmic function is well known.

$$\log\frac{1+z}{1-z} = \frac{2z|}{|1} + \frac{-z^2|}{|3} + \frac{-4z^2|}{|5} + \cdots + \frac{-n^2z^2|}{|2n+1} + \cdots . (z \in \mathbf{Z}) \tag{7}$$

Here let z be $\frac{p}{q}$ (p, q are integers). Then the above formula (7) is rewritten as following:

$$\log\frac{1+z}{1-z} = \frac{2p|}{|1} + \frac{-p^2|}{|3q} + \frac{-4p^2|}{|5q} + \cdots + \frac{-n^2p^2|}{|(2n+1)q} + \cdots . \tag{8}$$

By this replacement of the variable z the recursive equation (4) is also rewritten as following:

$$\begin{cases} P_n = (2n-1)qP_{n-1} - (n-1)^2p^2P_{n-2}, \\ Q_n = (2n-1)qQ_{n-1} - (n-1)^2p^2Q_{n-2}. \end{cases} \quad (n = 1, 2, 3, \cdots) \tag{9}$$

3.2 Error evaluation equation

We need an error evaluation for the continued fraction in order to calculate the functions value. And we want to get the functions value according to the required precision by calculating fewer terms of continued fraction expansions. We put some condition for the parameter p and q, namely z. Then we derived the following theorem[5]:

Theorem 4 *For $p, q \in \mathbf{Z}, 1 < \alpha \in \mathbf{R}, 0 < k \in \mathbf{R}, \frac{p}{q} \leq \frac{1}{\alpha}$ and $k = 2\alpha^2 + 1$, the following formula holds.*

$$\frac{nq}{k}Q_{n-1} - (n-1)^2p^2Q_{n-2} > 0 \quad \text{for } n = 2, 3, \cdots .$$

Moreover,

$$\left|\frac{P_n}{Q_n} - \log\frac{1+z}{1-z}\right| < \frac{2}{n\alpha^2}\left(\frac{2\alpha^2-1}{4\alpha^2-3}\right)^{2n-1} z^{2n-1}.$$

In below section we call the above error evaluation equation $GEEE$, which means "General Error Evaluation Equation". About the case of the parameter $\alpha = 2$ we examined $GEEE$ gives us "good evaluation". Here the term "good evaluation" means that the $GEEE$ gives us a smaller term number which we still have to calculate for continued fraction expansions. But we are restricted to the domain of z if we calculate using the $GEEE$ because it has some conditions. So we examined how to calculate using the following properties of logarithm: $\log ab = \log a + \log b$ and $\log a^n = n \log a$.

4 Calculation of logarithms

4.1 Overview

We calculate the logarithms by using "divided calculation" and *GEEE*. The following equation is used in order to get the value of logarithms because *GEEE* has some conditions:

$$\log x = \log \frac{x}{a^n} + n \log a. \tag{10}$$

Especially the following condition is important in this case:

$$z < \frac{1}{\alpha}. \tag{11}$$

So we have to determine the parameters α and a on the right-hand side of the above equation (10). We selected an integer 2 for a and 3 for α because there is a non negative integer n such that

$$1 < \frac{a}{2^n} < \frac{\alpha+1}{\alpha-1}, \tag{12}$$

then

$$2^n < a < \frac{\alpha+1}{\alpha-1} 2^n = 2^{n+1}. \tag{13}$$

That is, let a be any prime number, then the number satisfies the last condition. If you select other numbers for α, it is hard to find the equation to satisfy the above condition (13) for any numbers. The advantage of the determination $\alpha = 3, a = 2$ is that we can calculate the number $\log 2$ by using the same *GEEE*. So we are going to use the parameters $\alpha = 3, a = 2$.

4.2 Algorithm

We show the algorithm in order to calculate logarithms using the equations , *GEEE* and (10) under the condition (13).

[INPUT] a, N,`width`

a	-	The substituted value for log
N	-	The required accuracy
`width`	-	The unit width of divided calculation

[OUTPUT] $\log(a)$

1. Find n which satisfy the condition (13).
2. Calculate the loop number $L1$ using the argument N and $a/2^n$.
   (Use the *GEEE*)

3. Calculate the loop number $L2$ using the argument N and 2.
   (Use the $GEEE$)

4. Calculate Log_1 and Log_2.
   $Log_1 = \log(a/2^n)$, $Log_2 = n\log(2)$. (Use the `divided calculation`)

5. `return` $Log_1 + Log_2$.

5 Result and consideration

We implemented our algorithm by Asir[1] and measured the calculation times $AsirTime$ of some logarithms. We compared the times with the calculation times $PariTime$ obtained by PARI-GP[2].

The results of the experiment show that

- $AsirTime$ is dependent on the time to calculate Log_1.
- If $a = 2^k + 1$, namely $p = 1$ in the equation (9), $AsirTime$ is short.
- If $a = 2^k - 1$, $AsirTime$ is long. Especially, if k is greater than about 40, then it seems to be $AsirTime > PariTime$.

On the right-hand side of the equation (10), because the second term (Log_2) is a constant n multiplied by $\log 2$, its calculation time is about constant. So $AsirTime$ was dependent on the calculation time of the first time(Log_1). Next let $\frac{a}{2^n} = \frac{1+z}{1-z}$, then $z = \frac{a-2^n}{a+2^n}$. Thus if $a = 2^n + 1$, then $p = 1$. On the other hand, if $a = 2^{n+1} - 1$, then $p = 2^n - 1$. Thus $AsirTime$ is long when $a = 2^k - 1$, and $AsirTime$ is short when $a = 2^k + 1$. This shows the following things. Let a be $2^k + m$. The smaller the number m is, the shorter the $AsirTime$ is.

6 Summary

We have shown the availability of the our theorem, GEEE. And we have shown the calculation by using the properties of logarithm. We ware able to calculate the values of $\log(a)$ where $a > \frac{\alpha+1}{\alpha-1}$ by using it. And if a be until about 2^{40} then Asir is able to calculate faster than PARI-GP. But PARI-GP is able to calculate faster than Asir when the value of a is greater than about 2^{40}.

A Timing chart

We examined how long it costs to calculate logarithms with 10000 digits. To do it we coded by Asir, and compared it with PARI-GP. The version of PARI-GP is 2.0.4, and the version of Risa/Asir is 991006. The machine environment we used is the following:

[1] Risa/Asir Version 991006

[2] PARI-GP version 2.0.4

表 19: The machine environment

CPU	Intel Celeron 300A PPGA (Dual) 450MHz
Memory	128MB
OS	LASER5 Linux 6.0 (Kernel 2.2.5-rh60_L5_2smp)

The test function we used is:

$$\log a = \log \frac{a}{2^n} + n \log 2.$$

In the following table 20, $Asir_1$ shows the calculation time of the first term in the above right-hand side. $Asir_2$ shows the calculation time of the second term. The unit time is seconds. In the first colum a, F_7 shows $2^{2^7}+1$, which is a fermat prime.

表 20: The calculation time of Log_1 and Log_2

a	$Asir_1(Log_1)$[sec]			$Asir_2(Log_2)$[sec]		
	time	gc	total	time	gc	total
$2^{10}-1$	1.19	0.51	1.699	0.63	0.25	0.8898
$2^{20}-1$	2.03	0.96	2.997	0.65	0.2	0.8531
$2^{30}-1$	3.08	1.18	4.258	0.69	0.18	0.8663
$2^{40}-1$	4.2	1.48	5.708	0.67	0.18	0.8453
$2^{50}-1$	5.42	2.29	7.739	0.67	0.19	0.8518
$2^{60}-1$	6.77	2.49	9.283	0.65	0.19	0.8405
$2^{70}-1$	8.55	2.89	11.46	0.66	0.16	0.8202
$2^{80}-1$	10.16	3	13.15	0.66	0.17	0.8281
$2^{90}-1$	12.02	2.76	14.8	0.68	0.13	0.8074
$2^{100}-1$	14.03	3.42	17.46	0.68	0.13	0.8097
$2^{100}+1$	0.05		0.05013	0.65	0.15	0.8104
F_7	0.05	0.01	0.05853	0.65	0.14	0.8132
859433	1.56	0.39	1.956	0.64	0.16	0.8045

In the table 21, *PariTime* shows the calculation time of logarithms using PARI-GP, and *AsirTime* shows the calculation time of those using Asir. The unit time is seconds for each.

表 21: *PariTime* and *AsirTime*

a	*PariTime*[sec]		*AsirTime*[sec]		
	time	total	time	gc	total
$2^{10}-1$	6.85	6.848	1.9	0.67	2.575
$2^{20}-1$	6.91	6.91	2.68	1.16	3.841
$2^{30}-1$	6.99	6.989	3.73	1.37	5.096
$2^{40}-1$	6.97	6.975	4.84	1.6	6.456
$2^{50}-1$	6.95	6.951	6.08	2.5	8.578
$2^{60}-1$	7.07	7.066	7.55	2.58	10.13
$2^{70}-1$	7.04	7.042	9.18	3.07	12.26
$2^{80}-1$	7.04	7.047	10.69	3.28	13.97
$2^{90}-1$	7.03	7.041	12.63	3.19	15.83
$2^{100}-1$	7.02	7.02	14.92	3.34	18.28
$2^{100}+1$	7.02	7.02	0.73	0.14	0.8704
F_7	7.12	7.123	0.7	0.15	0.8563
859433	6.91	6.913	2.22	0.53	2.761

References

[1] Isao Makino, Research on Algorithms of Number Theory, Abstracts of Research Project Grant-in-aid for Scientific Research (c)(2) 09640061, Mar 1999

[2] Peter Henrici, Applied and Computational Complex Analysis Volume 2,Interscience,1974

[3] Alexey,Nikolaevitch,Khovanskii, The Application of Continued Fraction and Their Generalizations to Problems in Approximation Theory,Noordhoff 1963

[4] Isao Makino, Takeshi Aoyama, The calculation of logarithmic function with continued fraction expansions, Journal of Japan Society for Symbolic and Algebraic Computation Vol.7 No.3, pp.15-16,1999.

[5] Isao Makino, Takeshi Aoyama, The calculation of logarithmic function with continued fraction expansions, to appere in NLA99: Computer Albebra(Sakado,1999),Josai Math.Monogr.,2, Josai Univ. Graduate School of Science Sakado, to appere