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Principal 3-blocks of the Chevalley groups $G_2(q)$

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Abstract

The principal 3-block of a Chevalley group $G_2(q)$ with q a power of 2 satisfying $q \equiv 2$ or $5 \pmod 9$ and the principal 3-block of $G_2(2)$ are Morita equivalent.

§1 Introduction

1.1 In this paper we consider the Chevalley groups $G_2(q)$ over the finite field $GF(q)$. We show a Morita equivalence between the principal 3-block of $G_2(q)$ with q a power of 2 satisfying $q \equiv 2$ or $5 \pmod 9$ and the principal 3-block of $G_2(2)$. These groups have the same 3-local structure with common Sylow 3-subgroup which is isomorphic to $M(3)$ the extra-special group of order 27 and of exponent 3. To be accurate, here we state some notation and some definition. Let $(\mathcal{K}, \mathcal{O}, \kappa)$ be a splitting p-modular system for all subgroups of the considering groups, that is, $\mathcal{O}$ is a complete discrete valuation ring with unique maximal ideal $\mathcal{P}, \mathcal{K}$ is its quotient field of characteristic zero and κ is its residue field $\mathcal{O}/\mathcal{P}$ of prime characteristic p and we assume that $\mathcal{K}$ and κ are both big enough such that they are splitting fields for all subgroups of the considering groups. The principal p-block $B(G)$ of a group G is the indecomposable two-sided ideal of the group ring $\mathcal{O}G$ to which the trivial module belongs. According to Rickard's definition in [Ri], finite groups G and H have the same p-local structure if they have a common Sylow p-subgroup P such that whenever Q_1 and Q_2 are subgroups of P and $f : Q_1 \to Q_2$ is an isomorphism, then there is an element $g \in G$ such that $f(x) = x^g$ for all $x \in Q_1$ if and only if there is an element $h \in H$ such that $f(x) = x^h$ for all $x \in Q_1$.

1.2 First we explain our motivation. There is a famous conjecture:

Broué's conjecture: Let G and H be finite groups having the same p-local structure with common Sylow p-subgroup P. If P is abelian, is it true that their principal p-blocks $B(G)$ and $B(H)$ are derived equivalent?

It is known that if P is not abelian, this is not true. Nevertheless, it seems that there are not so many derived category equivalence classes among the principal p-blocks of groups having a fixed common p-local structure. Keeping this in mind, we investigate principal 3-blocks of an infinite series of the Chevalley groups $G_2(q)$ having the same 3-local structure with common non-abelian Sylow 3-subgroup P. If

$$q \equiv 2, 4, 5 \text{ or } 7 \pmod 9, \tag{1.1}$$

then any group G among the Chevalley groups $G_2(q)$ has Sylow 3-subgroup P which is isomorphic to $M(3)$ and $N_G(P)$ is isomorphic to the semi-direct product of $M(3)$ by the semidihedral group SD_{16} of order 16 with the faithful action. Furthermore G and $N_G(P)$ have the same 3-local structure. Now let H be a finite group satisfying the same 3-local structure as that of the semi-direct product of $M(3)$ by SD_{16} with the faithful action. Furthermore we assume that the maximal normal 3-subgroup of H is trivial, since in general, we may assume that the maximal normal p-subgroup is trivial, when we consider only principal p-blocks. If $Z(P)$ is not normal in H for a Sylow 3-subgroup P, then using the classification of finite simple groups we can conclude that H is either one of the Chevalley groups $G_2(q)$ satisfying (1.1) or its automorphism group or the automorphism group of J_2 (cf. [U]). Now our main theorem is as follows:

Theorem 1.3 *Assume that*

$$q \text{ is a power of } 2 \text{ and } q \equiv 2 \text{ or } 5 \pmod 9. \tag{1.2}$$

Then the principal 3*-block of* $G_2(q)$ *and the principal* 3*-block of* $G_2(2)$ *are Morita equivalent. Here a* $\Delta(P)$*-projective trivial source* $G_2(2) \times G_2(q)$*-module and its* $\mathcal{O}$*-dual induce this Morita equivalence as bimodules, where* P*is a common Sylow* 3*-subgroup of* $G_2(q)$ *and* $G_2(2)$ *and* $\Delta(P)$ *is the diagonal groups of* P *in* $G_2(2) \times G_2(q)$.

Remark 1.4 In fact, by Scott-Puig theorem in Marcus's paper (Theorem 1.6 [M]) this Morita equivalence is a so-called Puig equivalence (i.e. it implies the coincidence of their source algebras). Here we refer a by-product of this theorem. The decomposition matrices of the principal 3-blocks of the groups $G_2(q)$ were determined by Hiss and Shamash (3.3 [HS]) and that of $G_2(2)$ was determined completely, but in general they were incomplete. Table II in 3.3 in [HS] contains three unknown parameters. Now by this theorem the decomposition matrices of the principal 3-blocks of the groups $G_2(q)$ satisfying (1.2) are completely determined, since Morita equivalent blocks have the same decomposition matrix.

1.5 Theorem 1.3 is based on the following three theorems.

Let G be a group $G_2(q)$ satisfying (1.2), and P be a Sylow 3-subgroup of G. Set

$$N = N_G(Z(P)) \text{ and } H = N_G(P)$$

for short. Here N is an index two extension of $C_G(z)$ with $\langle z\rangle = Z(P)$ and $C_G(z)$ is isomorphic to $SU(3,q^2)$ by Appendix B in [H]. We would like to compare the principal 3-block $B(G)$ of G and the principal 3-block $B(G_2(2))$ of $G_2(2)$ via $B(H)$, since H is a common subgroup of G and $G_2(2)$.

Although H does not depend on q, H is so small. Hence first we compare $B(G)$ with $B(N)$, since Theorem 1.8 below guarantees that $B(N)$ and $B(H)$ $(= \mathcal{O}H)$ are Morita equivalent to each other. In order to prove Theorem 1.8 we need Theorem 1.7 which is based on Theorem 1.6.

Morita equivalences in Theorem 1.6, 1.7 and 1.8 are also Puig equivalences.

Theorem 1.6 (Koshitani and Kunugi [KK]) *The principal 3-block of $PSU(3,q^2)$ defined over the finite field $GF(q^2)$ satisfying $q \equiv 2,5 \pmod 9$ and the principal 3-block of $PSU(3,2^2)$ are Morita equivalent. If we set $G = PSU(3,q^2)$, with above q and $H = N_G(P)$ for a Sylow 3-subgroup P of G, then H is isomorphic to the semi-direct product of the elementary abelian group of order 9 by the quaternion group of order 8 with the faithful action. Furthermore H is isomorphic to $PSU(3,2^2)$ and $\mathcal{O}H = B(H)$, that is, the principal 3-block of H. Here let $B(G)$ be the principal 3-block of G. Then*

$$_{B(G)}B(G)_{\mathcal{O}H}$$

and its $\mathcal{O}$-dual induce a stable equivalence of Morita type between $B(H)$ and $B(G)$, and its unique indecomposable non-projective direct summand induces a Morita equivalence between them.

Theorem 1.7 *The principal 3-block of $SU(3, q^2)$ defined over the finite field $GF(q^2)$ satisfying $q \equiv 2, 5 \pmod 9$ and the principal 3-block of $SU(3, 2^2)$ are Morita equivalent. If we set $G = SU(3, q^2)$ with above q and $H = N_G(P)$ for a Sylow 3-subgroup P of G, then H is isomorphic to the semi-direst product of the extra-special group of order 27 and of exponent 3 by the quaternion group of order 8 with the faithful action. Furthermore, H is isomorphic to $SU(3, 2^2)$ and $\mathcal{O}H = B(H)$, that is, the principal 3-block of H. Let $B(G)$ be the principal 3-block of G. Then the unique indecomposable direct summand with vertex ΔP of*

$$_{B(G)}B(G)_{\mathcal{O}H}$$

and its $\mathcal{O}$-dual induce a Morita equivalence between $B(H)$ and $B(G)$.

Theorem 1.8 *Let G be one of the Chevalley groups $G_2(q)$ satisfying $q \equiv 2$ or $5 \pmod 9$. Let P be a Sylow 3-subgroup of G. Then the principal 3-blocks of*

$$N = N_G(Z(P)) \text{ and } H = N_G(P)$$

are Morita equivalent. Furthermore, this Morita equivalence is induced by an indecomposable $N \times H$-module which is a direct summand of the restriction of an $N \times N$-module $B(N)$ to $N \times H$ and has vertex $\Delta(P)$.

§2 Preliminaries

2.1 In this section we introduce some notation and explain the common frame of the proofs in Theorem 1.3, 1.7 and 1.8.

In this paper "modules" always mean finitely generated modules. They are left modules, unless stated otherwise. For a subgroup H of a group G, let U and V be $\mathcal{O}G$- and $\mathcal{O}H$-modules. We write $U_{\downarrow H}$ for the restriction of U to H, namely

$$U_{\downarrow H} = _{\mathcal{O}H} \mathcal{O}G \underset{\mathcal{O}G}{\otimes} U$$

and $V^{\uparrow G}$ for the induction of V to G, namely

$$V^{\uparrow G} =_{\mathcal{O}G} \mathcal{O}G \underset{\mathcal{O}H}{\otimes} V.$$

We use the similar notation for kG-modules and kH-modules and even for ordinary characters. Let $\mathcal{O}_G$ be the trivial $\mathcal{O}G$-module and k_G be the trivial kG-module. For an $\mathcal{O}$-algebra B we write

$$\overline{B} = k \underset{\mathcal{O}}{\otimes} B,$$

and for a $\overline{B}$-module U soc(U) means the socle of U.

For other notation and terminology we follow the books of Benson [Be], Landrock [La] and Nagao-Tsushima [NT]. Since the Brauer homomorphism plays an important role in this paper, we state its definition here.

Definition 2.2 (6.C. in [Br]) For an $\mathcal{O}G$-module V and a p-subgroup P of G, we set

$$Br_P(V) = V^P/(\sum_{Q \not\subseteq P} Tr_Q^P(V^Q) + \mathcal{P}V^P) \tag{1.1}$$

where V^P denotes the set of fixed points of V under P and Q runs over all proper subgroups of P and

$$Tr_Q^P(v) = \sum_{x \in P/Q} x(v) \tag{1.2}$$

for a p-subgroup Q of P and $v \in V^Q$.

Definition 2.3 (Definition 1.1 in [Li]) Let A and B be $\mathcal{O}$-algebras, $M(=_A M_B)$ an (A,B)-bimodule, $N(=_B N_A)$ a (B,A)-bimodule. We say M and N induce a stable equivalence of Morita type between B and A, if

(i) M is projective as a left A-module and as a right B-module,
(ii) N is projective as a left B-module and as a right A-module,
(iii) $M \underset{B}{\otimes} N = A \oplus X$ for a projective (A,A)-bimodule X and $N \underset{A}{\otimes} M = B \oplus Y$ for a projective (B,B)-bimodule Y.

For k-algebras we define a stable equivalence of Morita type similarly.

2.4 Let P be a Sylow 3-subgroup of any group in Theorem 1.3, 1.7 or 1.8. We would like to find a $\Delta(P)$-projective trivial source module such that it and its $\mathcal{O}$-dual induce a stable equivalence of Morita type as a bimodule in each case. For Theorem 1.7 we choose a suitable indecomposable summand of a $(B(G),\ B(N_G(P)))$-bimodule $B(G)$ where

$$G = SU(3, q^2) \text{ with } q \equiv 2 \text{ or } 5 \pmod 9.$$

For Theorem 1.8 we construct such a bimodule by an induction.

For Theorem 1.3 first we choose a $(B(G), B(N_G(Z(P))))$-bimodule $B(G)f$ as a bimodule which with its $\mathcal{O}$-dual induces a stable equivalence of Morita type between $B(G)$ and $B(N_G(Z(P)))$, where

$$G = G_2(q) \text{ with } q \text{ satisfying } (1.2),$$

and f is the central idempotent corresponding to $B(N_G(Z(P)))$. By Theorem 1.8 we already have a bimodule which induces stable equivalence of Morita type between $B(N_G(Z(P)))$ and $B(N_G(P))$. If we set $G_0 = G_2(q)$, then we already have chosen a bimodule which induces a stable equivalence of Morita type between $B(G_0)$ and $B(N_{G_0}(Z(P))) = B(N_{G_0}(P)) = B(N_G(P))$ as a special case. From these bimodules we will construct a required $(B(G), B(G_0))$-bimodule.

In each case we check local structure in order to guarantee a stable equivalence of Morita type by the following Broué's theorem.

Theorem 2.5 (cf. Broué, Theorem 6.3 in [Br]) *Let G be a finite group with a Sylow p-subgroup P and H be a subgroup of G containing $N_G(P)$. Assume that G and H have the same fusion on p-subgroups contained in P (i.e. the same p-local structure). Let b and b' be central primitive idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively such that there is a Brauer correspondence between*

$$A = \mathcal{O}Gb \text{ and } B = \mathcal{O}Hb'$$

having common defect group P. For a subgroup R of P, set

$$\overline{b}_R = Br_R(b)\ ,\ \overline{b'}_R = Br_R(b').$$

Let M be an (A, B)-bimodule and N be a (B, A)-bimodule. For each subgroup R of P set

$$\overline{M}_R = Br_{\Delta(R)}(M) \text{ and } \overline{N}_R = Br_{\Delta(P)}(N).$$

Assume that

(i) *M is a direct summand of the restriction of A from $G \times G$ to $G \times H$.*
(ii) *For each non-trivial subgroup R of P, $\overline{M}_R$ and $\overline{N}_R$ induce a Morita equivalence between $kC_H(R)\overline{b'}_R$ and $kC_G(R)\overline{b}_R$.*

Then M and its $\mathcal{O}$-dual induce a stable equivalence of Morita type between B and A.

2.6 Now we would like to apply the following Linckelmann's theorem and prove that the chosen bimodule for Theorem 1.7 (respectively, the unique non-projective direct summand of the composed $(B(G_2(q)), B(G_2(2))))$-bimodule for Theorem 1.3) induces a required Morita equivalence. In each case we get the bimodule over k-algebras from the above bimodule over $\mathcal{O}$-algebras by

$$k \underset{\mathcal{O}}{\otimes} -$$

and we have only to prove that it sends the simple modules to the simple modules, since it is also a trivial source module and it is liftable to the original bimodule (cf. §5 [Ri]).

In case of Theorem 1.3, instead of saying it directly, we prove that the composed $(kH, \overline{B}(G_2(q)))$-bimodule sends each simple $\overline{B}(G_2(q))$-module to a direct sum of a non-projective indecomposable kH-module and a projective kH-module and that this non-projective summand does not depend on q.

This is the main part of the proof of Theorem 1.3 and we use the same tools as those of the proof of Theorem 1.6, namely following Green-Landrack-Scott lemma and Robinson's lemma.

Theorem 2.7 (Linckelmann, Theorem 2.1 in [Li]) *Let G and H be two finite groups and b and b' be central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. Set*

$$A = \mathcal{O}Gb\,,\ B = \mathcal{O}Hb'\,,\ \overline{A} = k \underset{\mathcal{O}}{\otimes} A \text{ and } \overline{B} = k \underset{\mathcal{O}}{\otimes} B.$$

Let M be an (A, B)-bimodule which is projective as left and right module, such that the functor

$$M \underset{B}{\otimes} -$$

induces an $\mathcal{O}$-stable equivalence between B and A.

(i) *Up to isomorphism, M has the unique indecomposable non-projective direct summand M' as an (A, B)-bimodule and then $k \underset{\mathcal{O}}{\otimes} M'$ is, up to isomorphism, the unique indecomposable non-projective direct summand of $k \underset{\mathcal{O}}{\otimes} M$ as an $(\overline{A}, \overline{B})$-bimodule.*

(ii) *If M is indecomposable, for any simple B-module S, the A-module $M \underset{B}{\otimes} S$ is indecomposable and non-projective as an $\overline{A}$-module,*

(iii) *If for any simple B-module S, the A-module $M \underset{B}{\otimes} S$ is simple, then the functor $M \underset{B}{\otimes} -$ is a Morita equivalence.*

Lemma 2.8 (Green-Landrock-Scott) (see [La], II, Lemma 12.6) *Let M be a trivial source kG-module, so that M is liftable to a trivial source $\mathcal{O}G$-lattice $\widehat{M}$. Let $\chi_{\widehat{M}}$ be the ordinary character of G afforded by the $\widehat{M}$.*

(i) *Let Q be a p-subgroup of G. Then*

$$\dim_k[(\mathrm{soc}(M_{\downarrow Q})] = (\chi_{\widehat{M}}, 1_Q)_Q$$

where 1_Q is the trivial ordinary character of Q.

(ii) *Let x be a p-element in G. Then $\chi_{\widehat{M}}(x)$ equals to the number of indecomposable direct summands of the $k\langle x\rangle$-module $M_{\downarrow\langle x\rangle}$ which are isomorphic to the trivial $k\langle x\rangle$-module $k_{\langle x\rangle}$. In particular, $\chi_{\widehat{M}}(x)$ is a non-negative integer.*

(iii) *Let x be a p-element in G. Then, $\chi_{\widehat{M}}(x) \neq 0$ if and only if x belongs to some vertex of M.*

Lemma 2.9 (Robinson [Ro], Theorem 3) *Let H be a subgroup of G, and let S and T respectively be a simple kG-module and a simple kH-module. Then, the multiplicity of $P(S)$ as a direct summand of $T^{\uparrow G}$ is equal to the multiplicity of $P(T)$ as a direct summand of $S_{\downarrow H}$, where $P(S)$ is the projective cover of S.*

§3 Induction and restriction between G and $N_G(Z(P))$

3.1 From this section we assume that

$$q \text{ is a power of } 2 \text{ and } q \equiv 2 \text{ or } 5 \pmod 9 \tag{3.1}$$

and set

$$G = G_2(q)\,,\ N = N_G(Z(P)) \text{ and } H = N_G(P)$$

with a suitable Sylow 3-subgroup P of G. Then P, H do not depend on q and P is isomorphic to the extra-special group of order 27 and of exponent 3 and H is the semi-direct product of P by the semi dihedral group SD_{16} of order 16 with the faithful action. On the other hand, N depends on q and satisfies

$$N = N_G(Z(P)) \triangleright C_G(Z(P)) \cong SU(3, q^2).$$

Here

$$\{z, v\} \text{ with } z \in Z(P) - \{1\} \text{ and } v \in P - Z(P) \tag{3.2}$$

are representatives of the G-conjugacy classes (respectively, the N-conjugacy classes, the H-conjugacy classes) of the 3-elements.

Furthermore G, N and H have the same 3-local structure. Let $B(G)$, $B(N)$ and $B(H)(= \mathcal{O}H)$ be the principal 3-blocks of G, N and H respectively. Let e (respectively, f) be the central primitive idempotent of $\mathcal{O}G$ (respectively, $\mathcal{O}N$) corresponding to $B(G)$ (respectively, $B(N)$). Clearly for a p-subgroup Q of P such that $|Q| > 3$, we have

$$C_G(Q) = C_N(Q)$$

and $C_G(Z(P)) = C_N(Z(P))$. For $v \in P - Z(P)$

$$C_G(\langle v\rangle) \cong GU(2, q^2)$$

and $C_N(v)$ is the semi-direct product of $Z_{q+1} \times Z_{q+1}$ by Z_2 with the fixed-point-free action. By Proposition 2.6 in [KU] we already know that $\overline{B}(C_N(v))$ and $\overline{B}_G(v)$ are Morita equivalent by a bimodule

$${}_{\overline{B}(C_G(v))}\overline{B}(C_G(v))\overline{f'}_{\overline{B}(C_N(v))}$$

where $\overline{f'}$ is the central primitive idempotent corresponding to $\overline{B}(C_N(v))$. Hence by Theorem 2.5 what we have proved is the following proposition.

Proposition 3.2 The principal 3-block $B(G)$ of G and $B(N)$ of N are stable equivalent of Morita type by a bimodule

$$_{B(G)}B(G)f_{B(N)}$$

where f is the central primitive idempotent corresponding to $B(N)$.

3.3 Our main task is to determine

$$S_{\downarrow N}\overline{f} \quad \text{for each simple } \overline{B}(G)\text{-module } S, \tag{3.3}$$

where $\overline{f}$ is the central primitive idempotent corresponding to $\overline{B}(N)$. By Theorem 2.7 and Proposition 3.2 this is the direct sum of an indecomposable non-projective $\overline{B}(N)$-module and a projective $\overline{B}(N)$-module. Since there exists Lemma 2.9, it is useful to determine

$$s^{\uparrow G}\overline{e} \quad \text{for each simple } \overline{B}(N)\text{-module } s, \tag{3.4}$$

where $\overline{e}$ is the central primitive idempotent corresponding to $\overline{B}(G)$. Also by Theorem 2.7 and Proposition 3.2 this is a direct sum of an indecomposable non-projective $\overline{B}(G)$-module and a projective $\overline{B}(N)$-module. To do these tasks we need some information:

(i) the multiplicities of the irreducible characters in $B(G)$ as constituents of the induction of each irreducible characters in $B(N)$ from N to G.
(ii) the decomposition matrix of $B(G)$.
(iii) the decomposition matrix of $B(N)$ and Loewy series of each indecomposable projective $\overline{B}(N)$-modules.

In order to know (i) we need the character table of $B(G)$ ([EY]) and the character table of $B(N)$ which is recently determined by Enomoto. We also have to know which G-conjugacy class each element of N belongs to. This is also due to Enomoto. From these materials we can get information (i) by a usual character calculation. All simple $\overline{B}(G)$-modules are

$$S_{11}, S_{18}, S_{16}, S_{17}, S_{19}, S_{14}, S_{12}$$

and all degrees except for S_{12} are known by Hiss and Shamash (page 380 [HS]). According to Hiss-Shamash notation all irreducible ordinary characters in $B(G)$ are

$$X_{11}, X_{18}, X_{17}, X_{13}, X_{14}, X_{12}, X_{32}, X_{31}, X_{33}, X_{16}, X_{19}, \overline{X}_{19}, X'_{2a} \text{ and } X_{2a},$$

where "a" takes a specified one value in X'_{2a} and X_{2a}, which correspond to

$$\chi'_3\left(\frac{q+1}{3}\right) \text{ and } ,\chi'_4\left(\frac{q+1}{3}\right)$$

respectively according to Enomoto-Yamada notation [EY]. For information (ii) see Table II in Hiss-Shamash paper ([HS]) where α, β and γ are the unknown parameters.

3.4 For (iii) we have only to determine them for $B(H)$ by Theorem 1.8, since Morita equivalent blocks have the same decomposition matrix and the same Loewy series of the principal indecomposable modules (up to the one-to-one correspondence of the simple modules). Since P is normal in H and a 3-compliment of H is isomorphic to SD_{16}, the simple kH-modules $s_0, s'_3, s'_0, s_3, s_1, s'_1$ and s_4 are trivial source modules and the characters of their lifts correspond to the irreducible characters of SD_{16} whose kernels contain P. We can easily determine the decomposition matrix of H from its character table and we can also determine the Loewy series of the projective covers of all simple kH-modules (cf. Jenning's theorem).

3.5 Here we explain an outline of determining (3.3). Assume that

$$S \not\simeq S_{12}.$$

Then we can determine all composition factors of $S_{\downarrow N}\overline{f}$ completely from information (i), (ii) and (iii). The number of the composition factors are rather small, and it follows that it is indecomposable from information (iii). In particular, $S_{11\downarrow N}\overline{f}$ and $S_{18\downarrow N}\overline{f}$ are simple and

$$S_{11} \leftrightarrow s_0 \text{ and } S_{18} \leftrightarrow s'_3 \tag{3.5}$$

are Green correspondences and we may assume that

$$S_{19\downarrow N}\overline{f} = \begin{matrix} s_1 \\ s_4 \\ s'_1 \end{matrix}\ . \tag{3.6}$$

Furthermore by Lemma 2.9 we can conclude that any indecomposable projective direct summand of $s^{\uparrow G}\overline{e}$ for a simple $kN\overline{f}$-module s is $P(S_{12})$, that is the projective cover of S_{12}.

On the other hand, we can know the characters of the lifts of (3.4) from information (i). In particular, we know that only ${s'_3}^{\uparrow G}\overline{e}$, ${s_3}^{\uparrow G}\overline{e}$ and ${s_4}^{\uparrow G}\overline{e}$ can have projective summands and we can determine the projective summand of ${s'_3}^{\uparrow G}\overline{e}$ from (3.5). We can do the determination of heads and socles of ${s'_0}^{\uparrow G}\overline{e}$, ${s_1}^{\uparrow G}\overline{e}$ and ${s'_1}^{\uparrow G}\overline{e}$ and the determination of the Loewy series of $S_{\downarrow N}\overline{f}$ simultaneously by Frobenius reciprocity theorem. Now we know that these heads and socles are simple and if $S \not\sim S_{14}$, then $S_{\downarrow N}\overline{f}$ is uniserial. We determine the projective summands of ${s_3}^{\uparrow G}\overline{e}$ and ${s_4}^{\uparrow G}\overline{e}$, and then by Lemma 2.9 we know the projective summand $P_{(q)}$ of $S_{12\downarrow N}\overline{f}$. Set

$$S_{12\downarrow N}\overline{f} = W_{(q)} \oplus P_{(q)} \text{ and } S_{\downarrow N}\overline{f} = S_{(q)},$$

where the suffix (q) means that we are treating the case $G = G_2(q) = G(q)$.

3.6 The remaining task is the characterization of $S_{(q)}$ and $W_{(q)}$, and furthermore we have to guarantee that $S_{(q)}$ (respectively, $W_{(q)}$) corresponds $S_{(2)}$ (respectively, $W_{(2)}$) by the given Morita equivalence between the principal 3-blocks of $N_{G(q)}(Z(P))$ and $N_{G(2)}(Z(P)) = H$ in Theorem 1.8.

If $S \not\sim S_{14}$, then $S_{(q)}$ is uniserial and it is characterized as the unique submodule in its injective hull having the given Loewy series and then $S_{(q)}$ corresponds $S_{(2)}$. If $S \sim S_{14}$, then the $S_{(q)}$ is not uniserial but its Loewy series is slim and it is also characterized by its Loewy series and then $S_{(q)}$ corresponds $S_{(2)}$. Now we consider $W_{(q)}$. We may assume that the head of ${s_1}^{\uparrow G}\overline{e}$ is isomorphic to S_{12} and that its socle is isomorphic to S_{19}. We have

$$\left({s_1}^{\uparrow G}\overline{e}\right)_{\downarrow N}\overline{f} = s_1 \oplus P(s'_1) \oplus P_{(q)} \tag{3.7}$$

where $P(s'_1)$ is the projective cover of s'_1. Here we set

$$V = {s_1}^{\uparrow G}\overline{e} \,/\, \mathrm{soc}({s_1}^{\uparrow G}\overline{e}).$$

Then its head is isomorphic to S_{12} and its composition factors are $(\alpha - 1)S_{18}$, βS_{19}, γS_{17}, S_{16} and S_{12}, and $V_{\downarrow N}\overline{f}$ has $P_{(q)}$ as the projective direct summand. Now the non-projective direct summand of $V_{\downarrow N}\overline{f}$ is a factor module Q of $P(s'_1)$ (cf. (3.6) and (3.7)).

First we observe that the isomorphism classes of the composition factors of $\mathrm{soc}(V)$ are mutually distinct. If we remove a composition factor S from soc(V), then we get a smaller factor module of $P(s'_1)$ (actually it is a factor module of Q factored by a uniserial module $S_{\downarrow N}\overline{f}$), and since this uniserial module is a unique submodule having the same Loewy series as that of $S_{\downarrow N}\overline{f}$, the factor module is uniquely determined by this process.

Next we continue the above process for V/S and gain a smaller factor module of $P(s'_1)$. It is known that $\alpha = \beta = \gamma = 1$ when $q = 2$. When we remove composition factors S_{16}, S_{17} and S_{19} from V according to the above process, we get a factor module of $P(s'_1)$ whose socle is simple and it is not isomorphic to the socle of $S_{18\downarrow N}\overline{f}$ nor $S_{17\downarrow N}\overline{f}$ nor $S_{19\downarrow N}\overline{f}$. This implies that we have already obtained $W_{(q)}$ and $\alpha = \beta = \gamma = 1$ for any q. Since we can guarantee that the factor module which we obtain in each step is uniquely determined, the final $W_{(q)}$ is unique and characterized by this removing process. Then $W_{(q)}$ corresponds to $W_{(2)}$.

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