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TOPOLOGICAL PROPERTIES OF PRODUCTS OF ORDINAL NUMBERS

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1. INTRODUCTION

The greek letters $\alpha, \beta, \gamma, \ldots$ denote ordinal numbers with the usual order topologies. A space means a T_1 (every one point set is closed) toplogical space. A space X is *regular* if every point $x \in X$ and every closed set F with $x \notin F$ are separated by disjoint open sets. It is easy to verify that:

(1) If X and Y are regular, then so is the product space $X \times Y$.
(2) If X is regular and $Y \subset X$, then Y is regular.

A space X is *normal* if every pair of disjoint closed sets are separated by disjoint open sets. All compact spaces are normal and all subspaces of ordinal numbers are also normal. On the other hand, $X = (\omega_1 + 1) \times \omega_1$ is not normal. Indeed, using the Pressing Down Lemma, we can show that the diagonal $\{\langle \alpha, \alpha \rangle \in X : \alpha < \omega_1\}$ and the set $\{\omega_1\} \times \omega_1$ cannot be separated by disjoint open sets. Therefore:

(1) $X = (\omega_1 + 1)$ and $Y = \omega_1$ are normal but $X \times Y$ is not normal.
(2) $X = (\omega_1 + 1)^2$ is compact so normal, but the subspace $Y = (\omega_1 + 1) \times \omega_1$ of X is not normal.

Thus the notion of normality is completely different from that of regularity.

The simplest non-trivial space is $\omega + 1$, that is, the convergent sequence with its unique limit point. The following famous result was proved by Dowker [Do]:

Dowker's Theorem. *If X is normal, then $X \times (\omega + 1)$ is normal iff X is countably paracompact.*

Here a space X is said to be *countably paracompact*(*countably metacompact*) if for every countable open cover $\mathcal{U} = \{U_n : n \in \omega\}$, there is a locally finite (point finite, respectively) open refinement $\mathcal{V}$ of $\mathcal{U}$, where $\mathcal{V}$ is locally finite (point finite) if for every $x \in X$, there is a neighborhood U of x such that $\{V \in \mathcal{V} : V \cap U \neq \emptyset\}$ is finite (if for every $x \in X$, $\{V \in \mathcal{V} : x \in V\}$ is finite, respectively), moreover an open cover $\mathcal{V}$ is said to be an open refinement of $\mathcal{U}$ if for every $V \in \mathcal{V}$, there is $U \in \mathcal{U}$ with $V \subset U$.

Dowker asked in [Do] whether there exists a normal space which is not countably paracompact. More than twenty years later, M. E. Rudin constructed in [Ru] such a space in ZFC.

In these connections, we present more definitions. A space X is *CollectionWise Normal*(abbreviated as CWN) if for every discrerte collection $\mathcal{F}$ of closed sets of X, there exists a disjoint (equivalently, discrete) collection $\mathcal{U} = \{U(F) : F \in \mathcal{F}\}$

of open sets with $F \subset U(F)$, where a collection $\mathcal{F}$ is discrete if for every $x \in X$, there is a neighborhood U of x with $|\{F \in \mathcal{F} : U \cap F \neq \emptyset\}| \leq 1$. A space X is *expandable* if for every locally finite collection $\mathcal{F}$ of closed sets, there is a locally finite collection $\mathcal{U} = \{U(F) : F \in \mathcal{F}\}$ of open sets with $F \subset U(F)$. For every space, it is not difficult to verify:

(1) CWN $\to$ normal.
(2) expandable $\to$ countably paracompact $\to$ countably metacompact.
(3) normal + countably metacompact $\to$ countably paracompact.
(4) normal + expandable $\leftrightarrow$ CWN + countably paracompact.
(5) $(\omega_1 + 1) \times \omega_1$ is expandable but not normal.

2. Results

In the past 10 years, we have investigated such topological properties described in section 1 of product spaces of ordinal numbers. In this section, α denotes an arbitrary large ordinal number. First we proved in [KOT]:

Theorem 1. *For every pair of subspaces A and B of α,*

(1) *$A \times B$ is normal iff it is CWN.*
(2) *$A \times B$ is countably paracompact iff it is expandable.*
(3) *If $A \times B$ is normal, then it is countably paracompact. Note that $(\omega_1+1)\times\omega_1$ is countably paracompact but not normal.*
(4) *For every pair of subspaces A and B of ω_1, $A\times B$ is normal iff it is countably paracompact iff A or B are non-stationary or $A \cap B$ is stationary. Thus, if A and B are disjoint stationary sets of ω_1, then $A \times B$ is neither normal nor countably paracompact.*

We asked in [KOT]:

(a) Is $A \times B$ countably metacompact for every pair of subspaces A and B of α?
(b) Are normality and CWN equivalent for *all* subspaces of α^2?
(c) Are countable paracompactness and expandability equivalent for *all* subspaces of α^2?

On (a), we got in [KS1] and [KS2]:

Theorem 2.

(1) *All subspaces of α^2 are countably metacompact.*
(2) *All subspaces of ω_1^n are countably metacompact for every $n \in \omega$.*
(3) *There is a subspace of ω_1^ω which is not countably metacompact.*

After then we got an affirmative answer of (b) in [KNSY]:

Theorem 3. *Normality and CWN are equivalent for all subspaces of α^2.*

However, the question (c) still remains open.
In connection with (4) of Theorem 1, we asked in [KNSY]:

(d) Are normality and countable paracompactness equivalent for all subspaces of ω_1^2?

On (d), we proved in [KSS]:

Theorem 4. *For every subspace X of ω_1^2,*

(1) *X is normal iff X is expandable iff X is countably paracompact and strongly collectionwise Hausdorff, where a space is strongly collectionwise Hausdorff (collectionwise Hausdorff) if for every subset F of X with the collection $\{\{x\} : x \in F\}$ discrete, there is a discrete (disjoint, respectively) collection $\mathcal{U} = \{U(x) : x \in F\}$ of open sets with $x \in U(x)$.*

(2) *If V=L or the Product Measure Extension Axiom are assumed, then X is normal iff X is countably paracompact.*

(3) *X is collectionwise Hausdorff.*

This theorem also says that the question (c) is closedly related to (d).

3. On the question (d)

Now we conjecture that there is a model in which (d) is not true, that is, there is a countably paracompact but not normal subspace of ω_1^2. American young mathematicians Eisworth, Just, Pavlov, Smith, Szeptycki are working on this problem. In discussion with them, we have had a candidate of such a subspace. The remaining is an unpublished work with them.

Let Lim $= \{\alpha < \omega_1 : \alpha \text{ is limit }\}$ and Succ $= \omega_1 \setminus$ Lim. For each $\alpha \in$ Lim, fix a strictly increasing ω-sequence L_α cofinal in α, moreover for simplicity of our discussion we assume $L_\alpha \subset$ Succ. Then we call $\mathcal{L} = \{L_\alpha : \alpha \in \mathrm{Lim}\}$ a ladder system. Set $L(\mathcal{L}) = \bigcup_{\alpha \in \mathrm{Lim}} L_\alpha$, then $L(\mathcal{L}) \subset$ Succ. The ladder space $X(\mathcal{L})$ determined by $\mathcal{L}$ is defined as follows:

$$X(\mathcal{L}) = [\bigcup_{\alpha \in L(\mathcal{L})} \{\alpha\} \times \{\beta \in \mathrm{Lim} : \alpha < \beta\}] \cup [\bigcup_{\alpha \in \mathrm{Lim}} (\{\alpha\} \cup L_\alpha) \times \{\alpha + 1\}].$$

This is our candidate. The following are proved in our discussion:

(1) In ZFC, $X(\mathcal{L})$ is not normal for every ladder system $\mathcal{L}$.

(2) If MA(ω_1) is assumed, then for every ladder system $\mathcal{L}$, $X(\mathcal{L})$ is not countably paracompact. In fact, MA(ω_1) destroys the property (+) below.

(3) In ZFC, $X(\mathcal{L})$ is not countably paracompact for some ladder system $\mathcal{L}$.

So our conjecture is:

(d') In some model, there is a ladder system $\mathcal{L}$ such that $X(\mathcal{L})$ is countably paracompact.

Finally we present a combinatorial equivalent property due to Pavlov and Szeptycki, independently.

(4) Let $\mathcal{L}$ be a ladder system. Then $X(\mathcal{L})$ is countably paracompact iff $\mathcal{L}$ satisfies the following two properties (WU) and (+):

(WU) $\forall f : \mathrm{Lim} \to \omega \exists g : L(\mathcal{L}) \to [\omega]^{<\omega} \forall \alpha \in \mathrm{Lim}(|\{\beta \in L_\alpha : f(\alpha) \notin g(\beta)\}| < \omega)$.

(+) $\forall f : L(\mathcal{L}) \to \omega(\{\alpha \in \mathrm{Lim} : |f'' L_\alpha| = \omega\}$ is not stationary).

So the conjecture (d') can be written as:

(d") In some model, there is a ladder system $\mathcal{L}$ satisfying both (WU) and (+).

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