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VIETORIS CONTINUOUS SELECTIONS ON SCATTERED SPACES

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Throughout this paper, all spaces are Hausdorff.

Let X be a topological space, and let $\mathcal{F}(X)$ be the set of all non-empty closed subsets of X. Let us recall the *Vietoris topology* τ_V on $\mathcal{F}(X)$. The base for it is defined by all collections of the following form

$$\langle\mathcal{V}\rangle = \{F \in \mathcal{F}(X) : F \cap V \neq \emptyset, V \in \mathcal{V}, F \subset \cup\mathcal{V}\}$$

where $\mathcal{V}$ runs over all finite families of open subsets of X. If $\mathcal{V} = \{V_0, V_1, \dots, V_n\}$ is a finite family of open subsets of X, then in some cases, we shall write $\langle V_0, V_1, \dots, V_n\rangle$ instead of $\langle\mathcal{V}\rangle$. A map $\sigma : \mathcal{F}(X) \to X$ is a *selection* for $\mathcal{F}(X)$ if $\sigma(F) \in F$ for every $F \in \mathcal{F}(X)$. A selection $\sigma : \mathcal{F}(X) \to X$ is a *continuous selection* for $\mathcal{F}(X)$ if it is continuous with respect to the Vietoris topology τ_V on $\mathcal{F}(X)$.

Fifty years ago Ernest Michael [8] has discovered a simple sufficient condition for the existence of a continuous selection on a Hausdorff space X.

Theorem 1. [8] *If there exists a linear order $<$ on X such that induced order topology is weaker than the original topology and every non-empty closed subspace of X has $<$-minimal element, then the space X has a continuous selection.*

The selection in this case is constructed by assigning to each non-empty closed subset of X its $<$-minimal element. In fact Michael has proved

[1] This paper is based on a joint work with S. Fuji and K. Miyazaki.

that this condition is not only sufficient but also necessary for connected Hausdorff spaces. Later on, van Mill and Wattel have proved the same for compact Hausdorff spaces[10]. It is still unknown if the condition is necessary for all regular spaces, that is all presently known regular spaces with continuous selections satisfy it as well. While this shows that the existence of special linear order on a space with continuous selection plays an important role, mere existence of some linear order does not suffice to imply the existence of a continuous selection: the real line $\mathbb{R}$ is a linearly ordered (metric) space without any continuous selection [3].

A space is *scattered* if and only if every its non-empty closed subset has an isolated point. First we have a sufficient condition for the existense of continuous selection.

Theorem 2. *Let X be a paracompact scattered and every point $x \in X$ is G_δ. Then $\mathcal{F}(X)$ has a continuous selection.*

A space have *Baire property* if the intersection of countably many open dense subsets is dense.

We also have a necessary condition as follows:

Theorem 3. [5] *Let X be a regular space. If $\mathcal{F}(X)$ has a continuous selection, then every closed subset of X has Baire property.*

Combining two theorems above, we completely characterize countable regular spaces which admit a continuous selection by proving that:

Theorem 4. *A countable regular space X has a continuous selection if and only if it is scattered.*

The following example demonstrate that the assumption of regularity is essential in the above characterization:

Example 5. Let $X = \mathbb{Q} \times \{0,1\}$, and let $\mathbb{Q}_i = \mathbb{Q} \times \{i\}$ for $i \in \{0,1\}$ where $\mathbb{Q}$ denotes the rational numbers. For $x \in \mathbb{Q}$ we use $x^0 = \{x\} \times \{0\}$ and $x^1 = \{x\} \times \{1\}$. Let the topology τ on X be generated by the singletons of $\mathbb{Q}_0$ together with all sets of the form $V_\epsilon(x^1) = \{x^1\} \cup \{y^0 \in \mathbb{Q}_0 : x - \epsilon < y < x + \epsilon\} - \{x^0\}$, where $<$ is the usual order of the real line, $\epsilon > 0$ and $x \in \mathbb{Q}$. Then the space is a countable, first countable scattered Hausdorff space which has no continuous selection.

Unfortunately, scatteredness is no longer a sufficient condition for the existence of a continuous selection outside of the class of countable spaces.

Example 6. Let $X = \omega_1 \times (\omega + 1)$ which is the product of the space of countable ordinals and a convergence sequence. This space is a scattered (collectionwise) normal, countably compact, first countable space which does not have a continuous selection.

Remark. In the above example if we do not require scatteredness, such an example is easily constructed. In fact if we glue the first points 0 of two copies of the long line, then it is a linearly orderable, collectionwise normal, countably compact, first countable space which does not have a continuous selection [6].

The first countability is a novel feature of the above example, the one point compactification of an uncountable discrete set provides an example of a Hausdorff compact scattered space without any continuous selection. Further, the next example shows that scatteredness and linear orderability even combined together do not guarantee the existence of a continuous selection.

A space is a *GO-space* if it is homeomorphic to a subspace of a linearly ordered space.

A subset $A \subset \omega_1$ is ***stationary*** if it has the non-empty intersection with any closed unbounded set in ω_1.

Example 7. Let $S \subset \omega_1$ be a stationary set such that $\omega_1 - S$ is also stationary (such a set exists; see [7]). Let M be the quotient space obtained from the product space $S \cup \{\omega_1\} \times \{0,1\}$ by identifying the points $\langle \omega_1, 0\rangle$ and $\langle \omega_1, 1\rangle$ to the singleton ∞, where we introduce the discrete topology on $\{0,1\}$. Then M is a regular Lindelöf scattered GO space which has no continuous selection.

There is a standard way (see for instance [11])to embed a GO-space X as a closed subspace in a linearly ordered space X^* which is a subset of the linearly ordered space $X \times \mathbb{Z}$ equipped with the lexicographical order of X and $\mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. In our case the resulting linearly ordered space X^* is automatically Lindelöf and scattered. Therefore there exists a Lindelöf scattered linearly ordered space which has no continuous selection.

It should be pointed out that both our examples have size ω_1, which is the smallest possible one.

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