ON SPECTRAL PROPERTIES OF LOG-HYAPONORMAL OPERATORS

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Abstract

In this paper we consider spectral mapping theorem about two kinds of functional transformations for log-hyponormal operators and the continuity of the spectrum for log-hyponormal operators.

Introduction.

Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. For $A \in B(\mathcal{H})$, we denote the spectrum, the point spectrum, the residual spectrum and the approximate point spectrum of $A$ by $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_a(A)$, respectively. For the study of spectral theory of operators, spectral mapping theorems are important. In this paper we consider spectral mapping theorems about two kinds of functional transformations for log-hyponormal operators. It is familiar that if $A$ is normal then for every polynomial $f(\lambda, \lambda^*)$ one has $\sigma(f(A)) = f(\sigma(A)) = \{f(\lambda, \lambda^*); \lambda \in \sigma(A)\}$. In particular, we called the equality $\sigma(\text{Re}(A)) = \text{Re}(\sigma(A))$ with the polynomial $f(\lambda + \lambda^*) := \frac{1}{2}(\lambda + \lambda^*) = \text{Re}(\lambda)$ for any operator $A$ the "projective" property.

The projective property for semi-normal operators was shown by C. Putnam [11] and the projective property for Toeplitz operators was shown by S. Berberian [2]. We will show the subprojective property for $p$-hyponormal or log-hyponormal operators. On the other hand, in [14], D Xia studied the following functional transformation $\varphi_{\{\xi, \psi\}}(T) = \xi(U)\psi(|T|)$ for a semi-hyponormal operator $T = U|T|$. And in [6], M. Itoh extended this result to $p$-hyponormal operators. Recently, M. Chô and B. P. Duggal [4] gave an elementary proof of
Itoh's result for invertible operator cases and generalized this result. We will extend this result for log-hyponormal operator.

On the other hand, in [8] it was shown that the spectrum $\sigma$ is continuous on the set of $p$-hyponormal operators. We also show that this is still true for log-hyponormal operators.

An operator $A$ is called $p$-hyponormal if $(A^*A)^p - (AA^*)^p \geq 0$ for some $p \in (0, \infty)$. If $p = 1$, $A$ is hyponormal and if $p = \frac{1}{2}$, $A$ is semi-hyponormal. By the consequence of Löwner's inequality [10] if $A$ is $p$-hyponormal for some $p \in (0, \infty)$, then $A$ is also $q$-hyponormal for every $q \in (0, p]$. Thus we assume, without loss of generality, that $p \in (0, \frac{1}{2})$. Let $H(p)$ denote the class of $p$-hyponormal operators.

An operator $T$ is called log-hyponormal if $T$ is invertible and satisfies $\log (T^*T) \geq \log (TT^*)$. Since $\log : (0, \infty) \rightarrow (-\infty, \infty)$ is monotone function, every invertible $p$-hyponormal operator is log-hyponormal. But there exists a log-hyponormal operator which is not $p$-hyponormal (cf. [12, Example 12]).

An operator $A \in B(H)$ has a unique polar decomposition $A = U|A|$, where $|A| = (AA^*)^{\frac{1}{2}}$ and $U$ is a partial isometry with the initial space the closure of the range of $|A|$ and the final space the closure of the range of $A$. In particular, if $A = U|A|$ is log-hyponormal, then the operator $U$ is unitary. Associated with $A$ there is a related operator $\tilde{A} = |A|^\frac{1}{2}U|A|^\frac{1}{2}$, we call it the Aluthge transform of $A$. Aluthge transform has been used as a useful tool for study of $p$-hyponormal operators.

The followings are basic properties for $\tilde{A}$.

(i) If $A = U|A|$ be $p$-hyponormal $(0 < p < \frac{1}{2})$, then the operator $\tilde{A} = |A|^\frac{1}{2}U|A|^\frac{1}{2}$ is $(p + \frac{1}{2})$ hyponormal (cf. [1, Theorem 2]).

(ii) If $A \in B(H)$ be a log-hyponormal operator with a polar decomposition $A = U|A|$, then $\tilde{A} = |A|^\frac{1}{2}U|A|^\frac{1}{2}$ is semi-hyponormal (cf. [12, Theorem 4]).

Form the fact above, the second Aluthge transform of a $p$-hyponormal operator or log-hyponormal operator is hyponormal.

**Theorem A** For every $A \in B(H)$ and its Aluthge transform $\tilde{T} = |A|^\frac{1}{2}U|A|^\frac{1}{2}$, it holds that

$$\omega(A) = \omega(\tilde{A})$$

where $\omega = \sigma, \sigma_a$ or $\sigma_p$.

**Proof.** It is known from [9, Theorem 1.3].
1. Functional transformations for log-hyponormal operators.

First, we will show the “subprojective” property for the spectra of $p$-hyponormal operators and log-hyponormal operators. For a operator $T$, a point $z$ is in the normal approximate point spectrum $\sigma_{na}(T)$ of $T$ if there exists a sequence $\{x_n\}$ of unit vectors such that

$$(T - z)x_n \to 0 \quad \text{and} \quad (T - z)^*x_n \to 0 \quad \text{as} \quad n \to \infty.$$ 

We begin with the following lemma. Proof is easy. So we omit it.

**Lemma 1.1.** If $T \in B(\mathcal{H})$ and $\sigma_{a}(T) = \sigma_{na}(T)$, then

$${\rm Re}(\sigma(T)) \subset \sigma({\rm Re} T) \quad \text{and} \quad {\rm Im}(\sigma(T)) \subset \sigma({\rm Im} T). \quad (1.1.1)$$

**Corollary 1.2.** Let $T$ be $p$-hyponormal or log-hyponormal. Then (1.1.1) holds.

**Proof.** Since $\sigma_{a}(T) = \sigma_{na}(T)$ for a $p$-hyponormal or a log-hyponormal operator $T$. This follows from Lemma 1.1 \hfill \Box

**Theorem 1.3.** Let $T = U|T| = H + iK$ be $p$-hyponormal or log-hyponormal and $\hat{T}$ be the second Aluthge transform of $T$. Let $\hat{T} = \hat{H} + i\hat{K}$ be the Cartesian decomposition of $\hat{T}$. Then

$$\sigma(\hat{H}) \subset \sigma(H) \quad \text{and} \quad \sigma(\hat{K}) \subset \sigma(K).$$

**Proof.** By Theorem A,

$$\sigma(T) = \sigma(\hat{T}) \implies {\rm Re}(\sigma(T)) = {\rm Re}(\sigma(\hat{T})), \quad {\rm Im}(\sigma(T)) = {\rm Im}(\sigma(\hat{T})).$$

Since $\hat{T}$ is hyponormal, $\sigma(\hat{T}) = \sigma(\Re \hat{T})$ and $\sigma(\hat{T}) = \sigma(\Im \hat{T})$. Thus

$$\sigma(\Re \hat{T}) \subset \sigma(\Re T) \quad \text{and} \quad \sigma(\Im \hat{T}) \subset \sigma(\Im T). \hfill \Box$$

**Corollary 1.4.** Let $T$ be log-hyponormal. If $T$ has a compact real (imaginary) part, then $T$ is normal.
Proof. Since, by Theorem 1.3, \( \text{meas}(\sigma(\hat{H})) = 0 \), \( \hat{T} \) is normal. And since \( T \) is normal if and only if \( \hat{T} \) is normal. Thus \( T \) is normal.

\[ \square \]

Let \( E \) be a bounded closed subset of all real numbers \( \mathbb{R} \), and \( \mathcal{M}(E) = \{ \psi : \psi \text{ is a bounded real Baire function on } E \} \). Let \( \mathcal{M}_0(E) = \{ \psi \in \mathcal{M}(E) : \psi(x) \geq 0 \text{ for all } x \in E \text{ and } \psi(0) = 0 \} \). Let \( \mathcal{J}(E) = \{ \psi : \psi \text{ is a strictly monotone increasing continuous function on } E \} \) and \( \mathcal{J}_0(E) = \mathcal{M}_0(E) \cap \mathcal{J}(E) \). Let \( \mathcal{S}(E) = \{ \psi \in \mathcal{M}(E) : K_\psi \geq 0 \} \), where \( K_\psi \) is the singular integral operator defined on \( L^2(E) \) by

\[
(K_\psi f)(x) = s - \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_E \frac{\psi(x) - \psi(y)}{x - (y + i\epsilon)} f(y) dy.
\]

If \( E \) is a closed subset of the unit circle \( T \), let \( \mathcal{M}_0(E) = \{ \xi : \xi \text{ is a complex Baire function on } E \to T \} \), \( \mathcal{J}_0(E) = \{ \xi : \xi \text{ is a direction preserving homomorphism on } E \} \) and \( \mathcal{S}_0(E) = \{ \xi : \xi \in \mathcal{M}_0(E) \text{ and } K_\xi \geq 0 \} \), where \( K_\xi \) is the singular integral operator defined on \( L^2(E) \) by

\[
(K_\xi f)(e^{i\theta}) = s - \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_E \frac{1 - \xi(e^{i\theta})\overline{\xi(e^{i\eta})}}{1 - e^{i\theta e^{-i\epsilon}}} f(e^{i\eta}) d\eta.
\]

For functions \( f \) and \( g \), we denote the functional transformation \( F_{[f,g]}(T) = f(U)\exp(g(\log|T|)) \) for a log-hyponormal operator \( T = U|T| \) and \( F_{[f,g]}(re^{i\theta}) = f(e^{i\theta})\exp(g(\log r)) \) in the complex plane.

**Lemma 1.5.** Let \( T \in B(H) \) be a semi-hyponormal operator with operator decomposition \( T = U|T| \). Then \( Ue^{|T|} \) is log-hyponormal and

\[
\sigma_a(Ue^{|T|}) = \{ e^r e^{i\theta} : re^{i\theta} \in \sigma_a(T) \};
\]

\[
\sigma_r(Ue^{|T|}) = \{ e^r e^{i\theta} : re^{i\theta} \in \sigma_r(T) \};
\]

\[
\sigma(Ue^{|T|}) = \{ e^r e^{i\theta} : re^{i\theta} \in \sigma(T) \}.
\]

**Proof.** Proof is from [13, Lemmas 5 and 6].

\[ \square \]

**Theorem 1.6.** Let \( T = U|T| \) be log-hyponormal and \( \log|T| \geq 0 \). Suppose that \( f \in \mathcal{J}_0(\sigma(U)) \cap \mathcal{S}_0(\sigma(U)) \) and \( g \in \mathcal{J}_0(\sigma(\log|T|)) \cap \mathcal{S}_0(\sigma(\log|T|)) \) if \( \sigma(U) \neq T \) and \( g \in \mathcal{J}_0([0, \|\log|T||]) \cap \mathcal{S}_0([0, \|\log|T||]) \) if \( \sigma(U) = T \). Then \( F_{[f,g]}(T) \) is log-hyponormal and \( F_{[f,g]}(\sigma_w(T)) = \sigma_w(F_{[f,g]}(T)) \), where \( \sigma_w = \sigma, \sigma_a \) or \( \sigma_r \).
Proof. Let $T = U|T|$ be log-hyponormal, then $S = U \log |T|$ is semi-hyponormal and $\sigma_w(S) = \{(\log r)e^{i\theta} : re^{i\theta} \in \sigma_w(T)\}$. From Theorem VI, 3.1 of [14], $f(U)g(\log |T|)$ is also semi-hyponormal. Thus $\sigma_w(f(U)g(\log |T|)) = \{f(e^{i\theta})g(\log r) : (\log r)e^{i\theta} \in \sigma_w(U \log |T|)\}$. Moreover, from Lemma 1.5 we can see that $F_{[f,g]}(T) = f(U)\exp(g(\log |T|))$ is log-hyponormal. Thus $\sigma_w(F_{[f,g]}(T)) = \{f(e^{i\theta})g(\log r) : (\log r)e^{i\theta} \in \sigma_w(U \log |T|)\}$.

2. Continuity of $\sigma$ on the set of all log-hyponormal operators.

In [8], it was shown that the spectrum $\sigma$ is continuous on the set of all $p$-hyponormal operators. In this section we show that this is still true for log-hyponormal operators. To do this we recall that $T \in B(\mathcal{H})$ is said to be bounded below if there exists $k > 0$ for which $\|x\| \leq k\|Tx\|$ for each $x \in \mathcal{H}$. For $A \in B(\mathcal{H})$, $\gamma(A)$ denote the reduced minimum modulus, $\gamma(A) = \inf_{x \in \mathcal{H}, \|x\|=1} \|Ax\|$. If $T$ is bounded below and $T_n$ converges to $T$, then $T_n$ converges to $T$.

Proof. Since $T$ is bounded below, we have that if $\gamma(\cdot)$ denote the reduced minimum modulus, then $\gamma(T) = \alpha > 0$ and $T$ is a continuity point of $\gamma$ (cf. [7, Theorem 4.3]). Hence, without loss of generality, we may assume that $\gamma(T_n) > \varepsilon/2$ for all $n$. Since the set of bounded below operators is an open set, it follows that for sufficiently large $n$, $T_n$'s are bounded below and hence $|T|$ and $|T_n|$ are invertible (cf. [5, Theorem 8.6.4]). Let $y \in \mathcal{H}$ and $\|y\| = 1$. Then there exist $x$ and $x_n$ in $\mathcal{H}$ ($n \in Z^+$) such that $y = |T|x$ and $y = |T_n|x_n$. Since $\gamma(S)$ is the supremum of all real number $\gamma$ such that $\gamma \|x\| \leq \|Sx\|$, we have

$$\|x\| \leq \frac{1}{\gamma(|T|)} \|T|x\| = \frac{1}{\gamma(T)}\|y\| = \frac{1}{\gamma(T)} < 2/\alpha.$$
Similarly, \( \|x_n\| < 2/\alpha \) for all \( n \in \mathbb{Z}^+ \). Therefore

\[
\|U_n y - U y\| = \|U_n|T_n|x_n - U|T|x\| \leq \|U_n|T_n|x_n - U_n|T_n|x\| + \|U_n|T_n|x - U|T|x\|.
\]

But

\[
\|U_n|T_n|x - U|T|x\| \leq \|T_n - T\| \|x\| < \frac{2\|T_n - T\|}{\alpha} \quad \text{as} \quad n \to \infty.
\]

We now claim that \( \|x_n - x\| \to 0 \) as \( n \to \infty \). If it is not so, then there exist \( \delta > 0 \) and a sequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \|x_{n_k} - x\| > \delta \) for all \( k \). Hence

\[
\|T| (x_{n_k} - x)\| = \|T|x_{n_k} - |T_n|x_{n_k}\| \leq \|T| - |T_n|\|\|x_{n_k}\| < \frac{2}{\alpha}\|T| - |T_n|\| \to 0
\]

as \( n \to \infty \). This implies that \( |T| \) is not bounded below. It is a contradiction. Therefore, we have

\[
\|U_n|T_n|x_n - U_n|T_n|x\| \leq \|T_n\|\|x_n - x\| \to 0 \quad \text{as} \quad n \to \infty.
\]

Now we have:

**Theorem 2.2.** The spectrum \( \sigma \) is continuous on the set of all log-hyponormal operators.

**Proof.** Suppose that \( T = U|T| \) and \( T_n = U_n|T_n| \) for \( n \in \mathbb{Z}^+ \) are log-hyponormal operators such that \( T_n \) converges to \( T \). Since \( T \) is invertible it follows from Lemma 2.1 that \( U_n \) converges to \( U \), so that

\[
\tilde{T}_n = |T_n|^{\frac{1}{2}} U_n |T_n|^{\frac{1}{2}} \to \tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} \quad \text{as} \quad n \to \infty.
\]

Since \( \tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} \) is semi-hyponormal and the spectrum is continuous on the set of all \( p \)-hyponormal operators, we have

\[
\sigma(T_n) = \sigma(\tilde{T}_n) \to \sigma(\tilde{T}) = \sigma(T).
\]

For an operator \( A \in B(\mathcal{H}) \), \( z \) is in the approximate defect spectrum \( \sigma_\delta(A) \) if there exists a sequence \( \{x_n\} \) of unit vectors in \( \mathcal{H} \) such that \( \lim_{n \to \infty} \|(A - z)^* x_n\| = 0 \). Then we have
Theorem 2.3. Let $T$ be a log-hyponormal operator. Then

$$\sigma(T) = \sigma_\delta(T).$$

Proof. By Lemma 3 of [13], we have

$$\sigma_a(T) \subset \sigma_\delta(T).$$

Therefore,

$$\sigma(T) = \sigma_\delta(T).$$

We conclude with:

Corollary 2.4. The approximate defect spectrum $\sigma_\delta$ is continuous on the set of all log-hyponormal operators.

References


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