On powers of class $A(k)$ operators including $p$-hyponormal and log-hyponormal operators

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This report is based on the following paper:

**Abstract**

In [15], we introduced class $A(k)$ as a class of operators including $p$-hyponormal and log-hyponormal operators. In this report, we shall show that “if $T$ is an invertible class $A(k)$ operator for $k \in (0, 1]$, then $T^n$ is a class $A(\frac{k}{n})$ operator for all positive integer $n$.”

Moreover, we shall show a similar result on powers of class $A_1(s, t)$ operators which were introduced in [7] as extensions of class $A(k)$ operators, that is, “if $T$ is a class $A_1(s, t)$ operator for $s, t \in (0, 1]$, then $T^n$ is a class $A_1(\frac{s}{n}, \frac{t}{n})$ operator for all positive integer $n$.”

**1. Introduction**

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$.

An operator $T$ is said to be $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$ for a positive number $p$, and $T$ is said to be log-hyponormal if $T$ is invertible and $\log T^*T \geq \log TT^*$. $p$-Hyponormal and log-hyponormal operators were defined as extensions of hyponormal operators, i.e., $T^*T \geq TT^*$. It is well known that “every $p$-hyponormal operator is a $q$-hyponormal operator for $p \geq q > 0$” by the celebrated Löwner-Heinz theorem “$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$,” and “every invertible $p$-hyponormal operator is a log-hyponormal operator” since $\log t$ is an operator monotone function. It is also well known that there exists a hyponormal operator $T$ such that $T^2$ is not a hyponormal operator [16, Problem 209]. Related to this fact, the following result on powers of $p$-hyponormal operators for $p > 0$ was shown by Aluthge-Wang [3] and Ito [19].
Theorem A.1 ([3, 19]). If $T$ is a $p$-hyponormal operator for $p > 0$, then $T^n$ is a \( \min\{1, \frac{p}{n}\} \)-hyponormal operator for all positive integer $n$.

We remark that Aluthge and Wang [3] showed Theorem A.1 in case $p \in (0, 1]$. Then Ito [19] showed Theorem A.1 in case $p > 0$. Moreover, we obtained the following result for log-hyponormal operators.

Theorem A.2 ([23]). If $T$ is a log-hyponormal operator, then $T^n$ is also a log-hyponormal operator for all positive integer $n$.

We remark that the best possibilities of Theorem A.1 and Theorem A.2 were shown in [14, 19]. Theorem A.1 and Theorem A.2 were shown as nice applications of the following Theorem F.

Theorem F (Furuta inequality [9]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) \( (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}} \)

and

(ii) \( (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \)

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

We remark that Theorem F yields the Löwner-Heinz theorem when we put $r = 0$. Alternative proofs of Theorem F are given in [6] and [20] and also an elementary one page proof in [10]. Tanahashi [22] showed that the domain drawn for $p$, $q$ and $r$ in the Figure 1 is the best possible one for Theorem F.

On the other hand, related to $p$-hyponormal, log-hyponormal and paranormal (i.e., $||T^2x|| \geq ||Tx||^2$ for every unit vector $x \in H$) operators, we introduced classes of operators in [15] as follows:

Definition 1 ([15]).

(i) An operator $T$ belongs to class A if $|T^2| \geq |T|^2$ where $|T| = (T^* T)^{\frac{1}{2}}$.

(ii) For each $k > 0$, an operator $T$ belongs to class A($k$) if

(1.1) \( (T^* |T|^{2k} T)^{\frac{1}{k+1}} \geq |T|^2 \)

(iii) For each $k > 0$, an operator $T$ is said to be absolute-$k$-paranormal if

\[ |||T||^k Tx|| \geq ||Tx||^{k+1} \]

holds for every unit vector $x \in H$. 

\begin{figure}[h]
\centering
\begin{tikzpicture}
    \fill[gray!20] (0,0) -- (6,0) -- (6,6) -- cycle;
    \draw[-stealth] (0,0) -- (7,0) node[anchor=north] {$p$};
    \draw[-stealth] (0,0) -- (0,7) node[anchor=east] {$q$};
    \draw (0,0) -- (6,6) node[anchor=north east] {\( (1+r)q = p+r \)};
    \draw (0,0) -- (0,6) node[anchor=east] {\( q = 1 \)};
    \draw (0,0) -- (6,0) node[anchor=north] {\( p = q \)};
    \draw (0,0) -- (6,6) node[anchor=south west] {\( (1,1) \)};
    \draw (0,0) -- (0,-r) node[anchor=east] {\( (0,-r) \)};
    \draw (0,0) -- (1,1) node[anchor=south west] {\( (1,0) \)};
\end{tikzpicture}
\caption{Figure 1}
\end{figure}
We remark that class $\mathrm{A}(1)$ equals class $\mathrm{A}$, and absolute-1-paranormal equals paranormal. Related to these classes, we obtained the following result on inclusion relations among them in [15].

**Theorem B** ([15]).

(i) Every log-hyponormal operator is a class $\mathrm{A}(k)$ operator for all $k > 0$.
(ii) Every invertible class $\mathrm{A}(k)$ operator is a class $\mathrm{A}(l)$ operator for $l \geq k > 0$.
(iii) Every absolute-$k$-paranormal operator is a absolute-$l$-paranormal operator for $l \geq k > 0$.
(iv) For each $k > 0$, every class $\mathrm{A}(k)$ operator is a absolute-$k$-paranormal operator.

Inclusion relations among the classes of operators mentioned above can be expressed as follows:

![Figure 2](image-url)
Related to Theorem A.1 and Theorem A.2 on powers of $p$-hyponormal and log-hyponormal operators, Ito [18] showed the following results on powers of class A operators as follows:

**Theorem C.1 ([18]).** If $T$ is an invertible class A operator, then $T^n$ is also a class A operator for all positive integer $n$.

**Theorem C.2 ([18]).** Let $T$ be an invertible class A operator. Then

(i) $|T^n|^\frac{2}{n} \geq \cdots \geq |T^2| \geq |T|^2$

and

(ii) $|T^\ast|^2 \geq |T^{2\ast}| \geq \cdots \geq |T^n\ast|^\frac{2}{n}$

hold for all positive integer $n$.

It is interesting to point out that these theorems are parallel results to the following theorems on paranormal operators.

**Theorem C.3 ([8]).** If $T$ is a paranormal operator, then $T^n$ is also a paranormal operator for all positive integer $n$.

**Theorem C.4 ([8, 18]).** Let $T$ be a paranormal operator. Then

$$||T^n x||^\frac{2}{n} \geq \cdots \geq ||T^2 x|| \geq ||T x||^2$$

hold for every unit vector $x \in H$ and all positive integer $n$.

In this report, firstly, we shall show a result on powers of invertible class $A(k)$ operators for $k \in (0, 1]$ in Theorem 1, which is more precise result than Theorem C.1. Secondly, we shall show similar results to Theorem 1 for related classes to class $A(k)$.

### 2. Powers of Class $A(k)$ Operators

**Theorem 1.** If $T$ is an invertible class $A(k)$ operator for $k \in (0, 1]$, then $T^n$ is a class $A(\frac{k}{n})$ operator for all positive integer $n$.

**Corollary 2.** If $T$ is an invertible class A operator, then $T^n$ is a class $A(\frac{1}{n})$ operator for all positive integer $n$.

By using (ii) of Theorem B, Corollary 2 yields Theorem C.1 since class $A(\frac{1}{n})$ is included in class A, so that Corollary 2 is a more precise result than Theorem C.1.

To prove Theorem 1, we prepare the following Proposition 3 and Lemma F.
Proposition 3. $T$ is a class $A(k)$ operator for $k > 0$ if and only if

\[(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2.\]

Lemma F ([11, 15]). Let $A$ and $B$ be invertible operators. Then

\[(BA^*B^*)^\lambda = BA(A^*B^*BA)^{\lambda-1}A^*B^*\]

holds for any real number $\lambda$.

Proof of Proposition 3. Let $T = U|T|$ be the polar decomposition of $T$. Then $T^* = U^*|T^*|$ is also the polar decomposition of $T^*$. Suppose that $T$ is a class $A(k)$ operator. Then

\[
(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} = UU^*(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}U^*
\]

\[
= U(U^*|T^*||T|^{2k}|U)\frac{1}{k+1}U^*
\]

\[
= U(T^*|T|^{2k}T)\frac{1}{k+1}U^*
\]

\[
\geq U|T|^{2}U^* \quad \text{since } T \text{ is a class } A(k) \text{ operator}
\]

\[
= |T^*|^2.
\]

Hence (2.1) holds.

Conversely, suppose that (2.1) holds. Then

\[
(T^*|T|^{2k}T)\frac{1}{k+1} = (U^*|T^*||T|^{2k}|T^*|U)\frac{1}{k+1}
\]

\[
= U^*(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}U
\]

\[
\geq U^*|T^*|^2U \quad \text{by (2.1)}
\]

\[
= |T|^2.
\]

Hence $T$ is a class $A(k)$ operator.

Whence the proof of Proposition 3 is complete. $\square$

Proof of Theorem 1. Suppose that $T$ is an invertible class $A(k)$ operator for $k \in (0, 1]$. Then we have

\[(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2\]

by Proposition 3. By (ii) of Theorem B, $T$ is a class A operator, and also we have the following inequalities by Theorem C.2:

\[
|T^n|^\frac{2}{n} \geq |T|^2,
\]

(2.3)

\[
|T^*|^2 \geq |T^n*|^\frac{2}{n}.
\]

Then we have

\[
(|T^*||T^n|^{\frac{2k}{n}}|T^*|)^{\frac{1}{k+1}} \geq (|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2
\]

\[
(\text{by (2.1))}
\]

\[
\geq |T^*|^{2k}T \quad \text{since } T \text{ is a class } A(k) \text{ operator}
\]

\[
= |T|^2.
\]
by (2.1), (2.2) and Löwner-Heinz theorem. (2.4) implies the following (2.5) by Lemma F:

\[
|T^*|T^n\geq\left(|T^n|^{\frac{k}{n}}|T^*|^2|T^n|^{\frac{k}{n}}\right)^{\frac{1}{k+1}-1}|T^n|^{\frac{k}{n}}|T^*|^2.
\]

(2.5) is equivalent to

\[
|T^n|^{2k/n} \geq \left(|T^n|^{\frac{k}{n}}|T^*|^2|T^n|^{\frac{k}{n}}\right)^{\frac{1}{k+1}}.
\]

By (2.3), (2.6) and Löwner-Heinz theorem, we have

\[
|T^n|^{2k/n} \geq \left(|T^n|^{\frac{k}{n}}|T^*|^2|T^n|^{\frac{k}{n}}\right)^{\frac{1}{k+1}} \geq \left(|T^n|^{\frac{k}{n}}|T^n*|^2|T^n|^{\frac{k}{n}}\right)^{\frac{1}{k+1}}.
\]

(2.7) implies the following by Lemma F:

\[
|T^n|^{2k/n} \geq |T^n|^{\frac{k}{n}}|T^n*|^{\frac{1}{n}}(|T^n*|^\frac{2k}{n}|T^n|^\frac{1}{n})^{\frac{k}{k+1}}|T^n*|^{\frac{1}{n}}|T^n|^{\frac{k}{n}}.
\]

Then we have

\[
\left(|T^n*|^{\frac{1}{n}}|T^n|^{\frac{2k}{n}}|T^n*|^{\frac{1}{n}}\right)^{\frac{1}{k+1}} \geq |T^n*|^{\frac{k}{n}}.
\]

Put \(A = \left(|T^n*|^{\frac{1}{n}}|T^n|^{\frac{2k}{n}}|T^n*|^{\frac{1}{n}}\right)^{\frac{1}{k+1}}\) and \(B = |T^n*|^{\frac{2}{n}}\), then \(A \geq B > 0\). By using (i) of Theorem F, we have

\[
(B^\frac{k}{2}A^pB^\frac{k}{2})^{\frac{1+r}{p+r}} \geq B^{1+r}
\]

for \(p \geq 1\) and \(r \geq 0\).

Put \(p = k+1 \geq 1\) and \(r = n-1 \geq 0\) in (2.8). Then we have

\[
(B^{\frac{k+1}{2}}A^{k+1}B^{\frac{k+1}{2}})^{\frac{n}{k+n}} \geq B^n.
\]

(2.9) is equivalent to

\[
\left\{|T^n*|^{\frac{2k}{n}}|T^n|^\frac{1}{n}\right\}^{\frac{1}{k+1}} \geq |T^n*|^2.
\]

Then we have

\[
(|T^n*||T^n|^{\frac{2k}{n}}|T^n*|\right)^{\frac{1}{k+1}} \geq |T^n*|^2.
\]

Hence \(T^n\) is a class \(A(\frac{k}{n})\) operator by Proposition 3.

\[\square\]

Proof of Corollary 2. Put \(k = 1\) in Theorem 1.

\[\square\]

3. Powers of Class AI\((s, t)\) Operators

Very recently, the following classes of operators were defined in [7] as extensions of class \(A(k)\).
Definition 2 ([7]).

(i) For each $s > 0$ and $t > 0$, an operator $T$ belongs to class $A(s, t)$ if
\[(3.1) \quad (|T^*|^t|T|^2|T^*|^t)^\frac{1}{2t} \geq |T^*|^2t.\]

(ii) For each $s > 0$ and $t > 0$, an operator $T$ belongs to class $AI(s, t)$ if $T$ is an invertible class $A(s, t)$ operator.

We remark that class $A(k)$ coincides with class $A(k, 1)$ by Proposition 3. Related to class $A(s, t)$ operators, the following theorem was obtained in [7] as a nice application of Theorem F.

Theorem D ([7]).

(i) For each $s > 0$ and $t > 0$, every class $A(s, t)$ operator is a class $A(s, r)$ operator for $r \geq t > 0$.

(ii) For each $s > 0$ and $t > 0$, every class $AI(s, t)$ operator is a class $AI(p, r)$ operator for $p \geq s > 0$ and $r \geq t > 0$.

On the other hand, Aluthge and Wang defined $w$-hyponormal operators in [4] which was related to hyponormal operators as follows: An operator $T$ is said to be $w$-hyponormal if $|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$ where $\tilde{T} = |T|^{\frac{1}{2}}U|T|^\frac{1}{2}$ and $T = U|T|$ is the polar decomposition of $T$. $\tilde{T} = |T|^{\frac{1}{2}}U|T|^\frac{1}{2}$ is called Aluthge transformation of $T$. Aluthge transformation was studied in [1, 2, 12, 13, 17, 21, 24]. Related to $w$-hyponormal operators, the following result was shown in [5].

Theorem E.1 ([5]). An operator $T$ is a $w$-hyponormal operator if and only if
\[ (|T^*|^\frac{1}{2}|T||T^*|^\frac{1}{2}) \geq |T^*| \quad \text{and} \quad |T| \geq (|T|^{\frac{1}{2}}|T^*||T|^\frac{1}{2})^{\frac{1}{2}}. \]

By Theorem E.1, an invertible $w$-hyponormal operator coincides with a class $AI(\frac{1}{2}, \frac{1}{2})$ operator since the first inequality in Theorem E.1 is equivalent to the second inequality in Theorem E.1 in case $T$ is invertible by Lemma F. Moreover, the following Theorem E.2 for $w$-hyponormal operators was shown by Aluthge and Wang [5].

Theorem E.2 ([5]). If $T$ is an invertible $w$-hyponormal operator, then $T^2$ is also a $w$-hyponormal operator.

In this section, we shall show the following results for class $AI(s, t)$ operators and $w$-hyponormal operators as parallel results to Theorem 1 for class $A(k)$ operators.

Theorem 4. If $T$ is a class $AI(s, t)$ operator for $s, t \in (0, 1]$, then $T^n$ is a class $AI(\frac{s}{n}, \frac{t}{n})$ operator for all positive integer $n$. 

**Corollary 5.** If $T$ is an invertible $w$-hyponormal operator, then $T^n$ is a class AI($\frac{1}{2n}, \frac{1}{2n}$) operator for all positive integer $n$.

By (ii) of Theorem D and Theorem E.1, Corollary 5 yields Theorem E.2 since class AI($\frac{1}{2n}, \frac{1}{2n}$) is included in $w$-hyponormal for all positive integer $n$.

**Proof of Theorem 4.** Suppose that $T$ is a class AI($s, t$) operator for $s, t \in (0, 1]$, i.e.,

$$(3.1) \quad (|T^*|^t |T^n|^2 |T^n|^t)\frac{t}{s+t} \geq |T^n|^2t.$$ 

By (ii) of Theorem D and Proposition 3, $T$ is a class A operator, and also we have the following inequalities by Theorem C.2:

$$(2.2) \quad |T^n|^{\frac{s}{n}} \geq |T|^2,$$

$$(2.3) \quad |T^*|^2 \geq |T^n|^\frac{s}{n}. $$

(2.2) and (2.3) imply the following (3.2) and (3.3), respectively, by Löwner-Heinz theorem for $s \in (0, 1]$ and $t \in (0, 1]$:

$$(3.2) \quad |T^n|^{\frac{2s}{n}} \leq |T|^{2s},$$

$$(3.3) \quad |T^*|^2t \geq |T^n|^\frac{2t}{n}. $$

Then we have

$$(3.4) \quad (|T^*|^t |T^n|^\frac{2s}{n} |T^n|^t)\frac{t}{s+t} \geq (|T^*|^t |T^n|^\frac{2s}{n} |T^n|^t)\frac{t}{s+t} \geq |T^*|^t.$$ 

by (3.1), (3.2) and Löwner-Heinz theorem. (3.4) implies the following (3.5) by Lemma F:

$$(3.5) \quad |T^*|^t |T^n|^{\frac{2s}{n}}(|T^n|^\frac{s}{n} |T^n|^2 |T^n|^\frac{s}{n})\frac{t}{s+t} \geq |T^n|^t.$$ 

(3.5) is equivalent to

$$(3.6) \quad |T^n|^{\frac{2s}{n}} \geq (|T^n|^\frac{s}{n} |T^n|^2 |T^n|^\frac{s}{n})\frac{t}{s+t}.$$ 

By (3.3), (3.6) and Löwner-Heinz theorem, we have

$$(3.7) \quad |T^n|^{\frac{2s}{n}} \geq (|T^n|^\frac{s}{n} |T^n|^2 |T^n|^\frac{s}{n})\frac{t}{s+t} \geq (|T^n|^\frac{s}{n} |T^n|^\frac{2s}{n} |T^n|^\frac{s}{n})\frac{t}{s+t}.$$ 

(3.7) implies the following by Lemma F:

$$(3.8) \quad |T^n|^{\frac{2s}{n}} \geq |T^n|^\frac{s}{n} |T^n|^\frac{s}{n} (|T^n|^\frac{s}{n} |T^n|^\frac{2s}{n} |T^n|^\frac{s}{n})\frac{t}{s+t} \geq |T^n|^\frac{s}{n} |T^n|^\frac{2s}{n}.$$ 

Then we have

$$(3.9) \quad (|T^n|^\frac{s}{n} |T^n|^\frac{2s}{n} |T^n|^\frac{s}{n})\frac{t}{s+t} \geq |T^n|^\frac{2s}{n}.$$ 

Hence $T^n$ is a class AI($\frac{s}{n}, \frac{s}{n}$) operator. \qed
Proof of Corollary 5. Put $s = \frac{1}{2}$ and $t = \frac{1}{2}$ in Theorem 4. Then we obtain Corollary 5 since the class of all invertible $w$-hyponormal operators equals class AI($\frac{1}{2}, \frac{1}{2}$). □

4. CONCLUDING REMARKS

Firstly, it is interesting to point out the contrast between the following two facts: Theorem A.1 asserts that if $T$ is a $p$-hyponormal operator for $p \in (0,1]$, then $T^n$ belongs to the class of $(\frac{p}{n})$-hyponormal operators which is a larger class of operators than the class of $p$-hyponormal operators to which $T$ belongs. Contrary to Theorem A.1, Theorem 1 asserts that if $T$ is an invertible class $A(k)$ operator for $k \in (0,1]$, then $T^n$ belongs to class $A(\frac{k}{n})$ which is a smaller class of operators than class $A(k)$ to which $T$ belongs.

Secondly, it is shown in (i) of Theorem B that every log-hyponormal operator is a class $A(k)$ operator for all $k > 0$. Here, we shall discuss a more precise relation between class $A(k)$ and the class of log-hyponormal operators than (i) of Theorem B. Assume that $T$ is an invertible class $A(k)$ operator. Then we have

\[(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2\]

by Proposition 3. Then by using Lemma F, (2.1) is equivalent to the following (4.1):

\[|T^*||T|^k(|T|^k|T^*|^2|T^*|)^{\frac{k}{k+1}} |T|^k|T^*| \geq |T^*|^2.\]

Hence an invertible class $A(k)$ operator satisfies the following inequality by (4.1):

\[|T|^{2k} \geq (|T|^k|T^*|^2|T^*|)^{\frac{k}{k+1}}.\]

Then

\[\log |T|^{2k} \geq \log((|T|^k|T^*|^2|T^*|)^{\frac{k}{k+1}}\]

holds since $\log t$ is an operator monotone function. (4.2) is equivalent to

\[\log |T|^{2(k+1)} \geq \log(|T|^k|T^*|^2|T^*|).\]

Let $k \rightarrow +0$ in (4.3). Then we have $\log T^*T \geq \log TT^*$. Briefly speaking, the class of log-hyponormal operators can be regarded as invertible class $A(0)$. And it is well known that log-hyponormal also can be regarded as 0-hyponormal. It is interesting to point out this contrast.

REFERENCES


[9] T.Furuta, *A ≥ B ≥ 0 assures \( (B\leftarrow A^p B^r)^{1/q} \geq B^{(p+2r)/q} \) for \( r \geq 0, p \geq 0, q \geq 1 \) with \( (1 + 2r)/q \geq p + 2r \)*, Proc. Amer. Math. Soc., 101 (1987), 85–88.


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