Let $\mathcal{H}$ be a separable complex Hilbert space with an inner product $(\cdot, \cdot)$. A convex cone $\mathcal{H}^+$ in $\mathcal{H}$ is said to be selfdual if $\mathcal{H}^+ = \{ \xi \in \mathcal{H} | (\xi, \eta) \geq 0 \forall \eta \in \mathcal{H}^+ \}$. The set of all bounded operators is denoted by $L(\mathcal{H})$. For $A, B \in L(\mathcal{H})$ we shall write

$$A \triangleleft B \quad \text{if} \quad (B - A)(\mathcal{H}^+) \subset \mathcal{H}^+.$$ 

Since $\mathcal{H}$ is algebraically spanned by $\mathcal{H}^+$, the relation "\(\triangleleft\)" defines the partial order on $L(\mathcal{H})$. For example, let

$$C_n^+ = \left\{ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \middle| \lambda_1, \cdots, \lambda_n \geq 0 \right\},$$

which is a selfdual cone in $C^n$. Then $A = (\lambda_{ij}) \geq O$ if and only if $\lambda_{ij} \geq 0$ for $i, j = 1, \cdots, n$. We have had many results of such positive matrices (see [HJ, Chapter 8]).

The next example is a set of all $n$-by-$n$ positive semi-definite matrices denoted by $M_n^+$, which is considered as a selfdual cone in $C^{n^2}$. For an $n$-by-$n$ matrix $A$, let denote

$$\hat{A} : X \mapsto AXA^*, \ X \in M_n.$$ 

Then $\hat{A} \geq O$ for all $A \in M_n$.

The following proposition is valid for a general selfdual cone.
Proposition 1. Let $\mathcal{H}$ be a Hilbert space with a selfdual cone $\mathcal{H}^+$. Then for bounded operators on $\mathcal{H}$ we have the following properties:

1. If $O \lneq A_1 \lneq B_1$ and $O \lneq A_2 \lneq B_2$, then $O \lneq A_1A_2 \lneq B_1B_2$. In particular, if $O \lneq A \lneq B$, then $A^n \lneq B^n$ for every natural number $n$.
2. It is not true that $A,B \geq O$ and $O \lneq A \lneq B$ imply $A^{\frac{1}{2}} \lneq B^{\frac{1}{2}}$.
3. If $O \lneq A \lneq B$, then $O \lneq A^* \lneq B^*$.
4. If $A,A^{-1},B,B^{-1} \geq O$ and $A \lneq B$, then $B^{-1} \lneq A^{-1}$.
5. If $O \lneq A \lneq B$, then $\|A\| \leq \|B\|$.

Proof. (1) By assumption $A_i(\mathcal{H}^+) \subset \mathcal{H}^+$ and $(B_i-A_i)(\mathcal{H}^+) \subset \mathcal{H}^+$ hold for $i = 1, 2$. Since
\begin{equation*}
B_1B_2 - A_1A_2 = B_1(B_2 - A_2) + (B_1 - A_1)A_2,
\end{equation*}
we obtain the desired inequality.

(2) Consider the case where $\mathcal{H} = \mathbb{C}^2$, $\mathcal{H}^+ = \mathbb{C}^{2+}$. Put for a sufficiently large number $\lambda$ and a sufficiently small positive number $\mu$
\begin{equation*}
A^{\frac{1}{2}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B^{\frac{1}{2}} = \begin{pmatrix} 2 + \lambda & 1 - \mu \\ 1 - \mu & 2 + \lambda \end{pmatrix}.
\end{equation*}
Then $A^{\frac{1}{2}} \not\simeq B^{\frac{1}{2}}$ and
\begin{equation*}
A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \lneq B = \begin{pmatrix} (2 + \lambda)^2 + (1 - \mu)^2 & 2(2 + \lambda)(1 - \mu) \\ 2(2 + \lambda)(1 - \mu) & (2 + \lambda)^2 + (1 - \mu)^2 \end{pmatrix}.
\end{equation*}

(3) Let $A(\mathcal{H}^+) \subset \mathcal{H}^+$. Then we have $(A^*\xi, \eta) = (\xi, A\eta) \geq 0$ for all $\xi, \eta \in \mathcal{H}^+$. The selfduality of $\mathcal{H}^+$ shows that $A^* \succeq O$. By substituting $B - A$ for $A$, we obtain the desired property.

(4) If $A \lneq B$, then from (1)
\begin{equation*}
B^{-1} = A^{-1}AB^{-1} \lneq A^{-1}BB^{-1} = A^{-1}.
\end{equation*}

(5) For $A \succeq O$, put
\begin{equation*}
\|A\|_+ = \sup \{ \|A\xi\|; \|\xi\| \leq 1, \xi \in \mathcal{H}^+ \}.
\end{equation*}
Suppose $O \lneq A \lneq B$. Note that if $\eta - \xi \in \mathcal{H}^+$ for $\xi, \eta \in \mathcal{H}^+$, then $\|\xi\| \leq \|\eta\|$, because $\|\eta\|^2 - \|\xi\|^2 = (\eta - \xi, \eta + \xi) \geq 0$. Since $\|A\|_+ \leq \|B\|_+$, it suffices to show $\|\cdot\|_+ = \|\cdot\|$.
It is known that any element $\xi \in \mathcal{H}$ can be written as $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4), \xi_1 \perp \xi_2, \xi_3 \perp \xi_4$, for some $\xi_i \in \mathcal{H}$. Then $\|\xi\|^2 = \sum_{i=1}^{4} \|\xi_i\|^2$. Noticing that $A \triangleright O$, we see that

$$
\|A\xi\|^2 = \|A\xi_1 - A\xi_2 + i(A\xi_3 - A\xi_4)\|^2 = \sum_{i=1}^{4} \|A\xi_i\|^2 - 2(A\xi_1, A\xi_2) - 2(A\xi_3, A\xi_4) \leq \|A\xi_1 + A\xi_2 + i(A\xi_3 + A\xi_4)\|^2 = \|A\xi_1 + A\xi_2\|^2 + \|A\xi_3 + A\xi_4\|^2 \leq \|A\xi_1 + A\xi_2\|^2 + \|A\xi_3 + A\xi_4\|^2 = \|A\|^2 \|\xi\|^2.
$$

It follows that $\|A\| \leq \|A\|_+$. The converse inequality is trivial. □

We shall next deal with a selfdual cone associated with a standard von Neumann algebra. Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ be a standard form of a von Neumann algebra in the sense of Haagerup [H]. Namely,

(i) $J\xi = \xi, \xi \in \mathcal{H}^+$,

(ii) $J\mathcal{M}J = \mathcal{M}'$,

(iii) $JXJ = X^*, X \in Z(\mathcal{M})$,

(iv) $XJX(\mathcal{H}^+) \subset \mathcal{H}^+, X \in \mathcal{M}$.

Every von Neumann algebra has a standard representation. In particular, suppose that $\mathcal{M}$ is a von Neumann algebra with a cyclic and separating vector $\xi_0 \in \mathcal{H}$, i.e. $\overline{\mathcal{M}\xi_0} = \overline{\mathcal{M}'\xi_0} = \mathcal{H}$. Put $S_{\xi_0}X\xi_0 = X^*\xi_0, \forall X \in \mathcal{M}$. Then $S_{\xi_0}$ is a closable conjugate linear operator on $\mathcal{H}$, and the closure of $S_{\xi_0}$ is also denoted by $S_{\xi_0}$. Let $S_{E_0} = J_{\xi_0}\Delta_{\xi_0}^{\frac{1}{2}}$ be a polar decomposition of $S_{\xi_0}$, where $J_{\xi_0}$ is an isometric involution on $\mathcal{H}$ and $\Delta_{\xi_0} = S_{\xi_0}^*S_{\xi_0}$. Put

$$
\mathcal{H}_{E_0}^+ = \{XJ_{\xi_0}XJ_{\xi_0}\xi_0|X \in \mathcal{M}\}^- = \{\Delta_{\xi_0}^{\frac{1}{2}}X^*X\xi_0|X \in \mathcal{M}\}^-,
$$

which is a selfdual cone in $\mathcal{H}$. Then $(\mathcal{M}, \mathcal{H}, J_{\xi_0}, \mathcal{H}_{E_0}^+)$ is a standard form.

**Proposition 2.** Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ be a standard form of a von Neumann algebra. Given an element $A \in \mathcal{M}$, the following conditions are equivalent:

1. $A \triangleright O$.
2. $A \in Z(\mathcal{M})$ and $A \geq O$.

**Proof.** Suppose $A \triangleright O, A \in \mathcal{M}$. Choose an arbitrary element $\xi \in \mathcal{H}$. Then one can write as $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4), \xi_i \in \mathcal{H}^+$ such that $\mathcal{M}\xi_1 \perp \mathcal{M}\xi_2, \mathcal{M}\xi_3 \perp \mathcal{M}\xi_4$. We ten
have
\[
(A \xi, \xi) = \sum_{i=1}^{4} (A \xi_i, \xi_i) \geq 0.
\]

Hence \( A \geq O \). Moreover, since \( J \xi = \xi_1 - \xi_2 - i(\xi_3 - \xi_4) \), we get
\[
(JAJ \xi, \xi) = (J \xi, AJ \xi) = \sum_{i=1}^{4} (\xi_i, A \xi_i) = (\xi, A \xi).
\]

It follows that \( A = A^* = JAJ \in \mathcal{M}' \). The converse implication is immediate. \( \square \)

Let \( \mathcal{H}_{n}^{+}, n \in \mathbb{N} \), be a family of selfdual cones in \( \mathcal{H}_n \), where \( \mathcal{H}_n \) means \( \mathcal{H} \otimes M_n(= M_n(\mathcal{H})) \). Put st : \( A \mapsto A^* \), \( A \in M_{n,m} \), where \( M_{n,m} \) means a set of all \( n \)-by-\( m \) matrices.

We write \( J_{n,m} = J \otimes \text{st} \). We call \( (\mathcal{M}, \mathcal{H}, \mathcal{H}_{n}^{+}, n \in \mathbb{N}) \) a matrix ordered standard form, if for every \( A \in \mathcal{M} \otimes M_{n,m} \)
\[
AJ_{n,m}AJ_{m}(\mathcal{H}_{m}^{+}) \subset \mathcal{H}_{n}^{+}
\]
holds. Every von Neumann algebra can be represented as a matrix ordered standard form (see [SW2]). In the case where \( \mathcal{M} \) has a cyclic and separating vector as above, put for each \( n \in \mathbb{N} \)
\[
(\mathcal{H}_{\xi_0})_{n}^{+} = \overline{\text{co}}\{[X_i J_{\xi_0} X_j J_{\xi_0} \xi_0]_{i,j=1}^{n} | X_i \in \mathcal{M}\}.
\]
Here \( \overline{\text{co}} \) denotes the closed convex hull. Then \( (\mathcal{M}, \mathcal{H}, (\mathcal{H}_{\xi_0})_{n}^{+}) \) is a matrix ordered standard form. Such a von Neumann algebra associated with \( (\mathcal{H}_{\xi_0})_{n}^{+}, n \in \mathbb{N} \), is uniquely determined. Given a matrix ordered standard form \( (\mathcal{M}, \mathcal{H}, \mathcal{H}_{n}^{+}) \), put, for \( A \in \mathcal{M} \)
\[
\hat{A} \xi = AJAJ \xi \text{ for all } \xi \in \mathcal{H}.
\]
If \( A \in \mathcal{M} \), then \( \hat{A} \) is completely positive, and we shall write \( \hat{A} \geq_{cp} O \). In fact, we obtain for \( [\xi_{ij}] \in \mathcal{H}_{n}^{+}, n \in \mathbb{N} \)
\[
\hat{A} \otimes \text{id}_n[\xi_{ij}] = [\hat{A} \xi_{ij}] = [AJAJ \xi_{ij}]
\]
\[
= (A \otimes \text{id}_n)J_{n}(A \otimes \text{id}_n)J_{n}[\xi_{ij}] \in \mathcal{H}_{n}^{+}.
\]
It is immediate that for \( A \in L(\mathcal{H}) \), \( A \geq_{cp} O \) implies \( A \geq O \). The sufficient and necessary condition that \( A \geq O \) is equivalent to \( A \geq_{cp} O \) for all \( A \in L(\mathcal{H}) \) is that \( \mathcal{M} \) is abelian (see [M, Corollary 1.6]).
**Proposition 3.** For a matrix ordered standard form \((\mathcal{M}, \mathcal{H}, \mathcal{H}_+^+)\), suppose \(A \in L(\mathcal{H})\), \(A \geq O\), \(A \succeq O\). If \(A\) has a closed range and the support projection of \(A\) is completely positive, then for all \(\alpha \in \mathbb{R}\), \(A^\alpha \succeq_{cp} O\).

**Proof.** Let \(P\) be a support projection of \(A\). Put \(N = PM|PH\). Since \(P\) is completely positive, we see from [MN, Lemma 3] that \((N, PH, P_n\mathcal{H}_+^+)\) is a matrix ordered standard form. By assumption, \(PA = AP \geq O\), \(A \succeq O\), and \(PA\) maps a selfdual subcone \(PH^+\) in \(P\mathcal{H}_+^+)\) onto itself. It follows from [C, Theorem 3.3] that there exists an element \(B \in N^+\) such that \(PA = BJBP\). Hence
\[
A^\alpha = B^\alpha JB^\alpha JP \succeq_{cp} O
\]
for every real number \(\alpha\). \(\square\)

**Theorem 4.** With \((\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)\) as above, let \(O \leq A \leq \hat{B}\), \(A \in L(\mathcal{H})\), \(B \in \mathcal{M}\). If \(B\) is injective and has a dense range, then there exists an element \(C \in Z(\mathcal{M})^+\) with \(\|C\| \leq 1\) such that \(A = C\hat{B}\). In particular, if \(\mathcal{M}\) is factor, then one can choose a scalar \(\lambda\) with \(0 \leq \lambda \leq 1\) such that \(A = \lambda\hat{B}\).

**Proof.** Consider the polar decomposition \(B = U|B|\) of \(B\). By assumption \(U\) is a unitary element of \(\mathcal{M}\), and so \(\hat{U} \geq O\) and \(\hat{U}^* \succeq O\) by Proposition 1 (3). Hence we may assume \(B\) to be positive semi-definite. Let \(B = \int_0^{\|B\|} \lambda dE_\lambda\) be a spectral decomposition of \(B\).

Put \(P_n = \int_{\frac{1}{n}}^{\|B\|} dE_\lambda\) for \(n \in \mathbb{N}\). Then one sees that \(\hat{P}_n \not\lesssim I\) and \(\hat{P}_n A\hat{P}_n \lesssim \hat{P}_n \hat{B} \hat{P}_n\) by Proposition 1 (1). Since \(\hat{P}_n \hat{B} \hat{P}_n\) is invertible on \(\hat{P}_n\mathcal{H}\), where the inverse shall be denoted by \((\hat{P}_n \hat{B} \hat{P}_n)^{-1}\), we have
\[
O \leq \hat{P}_n A\hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \leq \hat{P}_n.
\]

There then exists an element \(c_n\) in an order ideal \(Z_{\hat{P}_n\mathcal{H}^+}\) of a selfdual cone \(\hat{P}\mathcal{H}^+\) with \(\|c_n\| \leq 1\) such that \(\hat{P}_n A\hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \xi = c_n \xi\) for all \(\xi \in \hat{P}_n\mathcal{H}\). By [L, Theorem VI.1,2 3]) we obtain that \(c_n \in Z(\hat{P}_n\mathcal{M}|_{\hat{P}_n\mathcal{H}})^+\). Since \(\hat{P}_n Z(\mathcal{M})\hat{P}_n = Z(\hat{P}_n\mathcal{M}\hat{P}_n),\) we can find an element \(C_n \in Z(\mathcal{M})\) such that \(c_n \xi = \hat{P}_n C_n \hat{P}_n \xi\) for all \(\xi \in \hat{P}_n\mathcal{H}\). Since \(P_n B = BP_n, n \in \mathbb{N}\), we have
\[
\hat{P}_{n+1} C_{n+1} \hat{P}_{n+1} \xi = \hat{P}_{n+1} A\hat{P}_{n+1} (\hat{P}_{n+1} \hat{B} \hat{P}_{n+1})^{-1} \hat{P}_n \xi
= \hat{P}_{n+1} A\hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \xi = \hat{P}_n C_n \hat{P}_n \xi.
\]
for all $\xi \in \hat{P}_n \mathcal{H}$. Since $\{\hat{P}_n C_n \hat{P}_n\}$ is a bounded sequence, one can define
\[
C\xi = \lim_{n \to \infty} \hat{P}_n C_n \hat{P}_n \xi, \quad \xi \in \mathcal{H}.
\]
Thus $C \in Z(\mathcal{M})^+$, $||C|| \leq 1$ and we get
\[
A = \limsup_{n \to \infty} \hat{P}_n A \hat{P}_n = \limsup_{n \to \infty} \hat{P}_n C_n \hat{P}_n A \hat{P}_n = C \hat{B}.
\]
This completes the proof. □

Now, consider two matrix ordered standard forms $(\mathcal{M}^{(1)}, \mathcal{H}^{(1)}, \mathcal{H}^{(1)+})$ and $(\mathcal{M}^{(2)}, \mathcal{H}^{(2)}, \mathcal{H}^{(2)+})$ with respective canonical involutions $J^{(1)}$ and $J^{(2)}$. Given an arbitrary element $\xi \in \mathcal{H}^{(1)}$, let $R_\xi$ be a right slice map of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ into $\mathcal{H}^{(2)}$ such that
\[
R_\xi(\xi' \otimes \eta') = (\xi', \xi)\eta', \xi', \eta' \in \mathcal{H}^{(1)}, \eta' \in \mathcal{H}^{(2)}.
\]
For any element $x \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, we put
\[
\Phi(x)(\xi) = R_{J^{(1)}}(x), \xi \in \mathcal{H}^{(1)}.
\]
Then $\Phi(x)$ is a map of Hilbert-Schmidt class of $\mathcal{H}^{(1)}$ to $\mathcal{H}^{(2)}$. A set of all maps of Hilbert-Schmidt class of $\mathcal{H}^{(1)}$ to $\mathcal{H}^{(2)}$ is denoted by $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$. A set of all completely positive maps of $(\mathcal{H}^{(1)}, \mathcal{H}^{(1)+})$ to $(\mathcal{H}^{(2)}, \mathcal{H}^{(2)+})$ in $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$ is denoted by $CPHS(\mathcal{H}^{(1)+}, \mathcal{H}^{(2)+})$. Here $\mathcal{H}^{(1)+} = \{(\xi_{ij})_{i,j=1}^{n} \mid (\xi_{ij})_{i,j=1}^{n} \in \mathcal{H}^{(1)+}\}$ is a selfdual cone corresponding to $\mathcal{M}^{(1)}$. We shall here write $\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+}$ for a selfdual cone corresponding to $\mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)}$. It was shown in [MT, SW1] that
\[
\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+} = \{x \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \mid \Phi(x) \in CPHS(\mathcal{H}^{(1)+}, \mathcal{H}^{(2)+})\}.
\]
Thus
\[
\Phi : \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \to HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})
\]
is an isometry mapping $\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+}$ onto $CPHS(\mathcal{H}^{(1)+}, \mathcal{H}^{(2)+})$. In fact, $\Phi$ is isometric. Suppose that $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$ has an inner product
\[
\langle A, B \rangle = \sum_{k=1}^{\infty} \langle Ae_k, Be_k \rangle,
\]
where \( \{e_k\} \) is a complete orthogonal basis of \( \mathcal{H}^{(1)} \). Noticing that \( \{J^{(1)}e_k\} \) is a complete orthogonal basis of \( \mathcal{H}^{(1)} \), we obtain for a complete orthogonal basis \( \{f_k\} \) of \( \mathcal{H}^{(2)} \)

\[
(\Phi(J^{(1)}e_i \otimes f_j), \Phi(J^{(1)}e_{i'} \otimes f_{j'})) = \sum_{k=1}^{\infty} (\Phi(J^{(1)}e_i \otimes e_k), \Phi(J^{(1)}e_{i'} \otimes f_{j})) = \sum_{k=1}^{\infty} (R_{J^{(1)}}e_i \otimes f_{j'}, (J^{(1)}e_i \otimes J^{(1)}e_k)f_{j'}) = \delta_{ii'} \delta_{jj'},
\]

for \( i, j, i', j' = 1, 2, \cdots \). Therefore, \( (\Phi(M^{(1)} \otimes M^{(2)})\Phi^{-1}, HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)}), \Phi(J^{(1)} \otimes J^{(2)})\Phi^{-1}, CPHS(\mathcal{H}^{(1)+}, \mathcal{H}^{(2)+}) \) is a standard form. Using the Radon-Nikodym theorem for \( L^2 \)-spaces [S, Theorem 1.2], we obtain the following proposition:

**Proposition 5.** Let \( (\mathcal{M}, \mathcal{H}, \mathcal{H}^+) \) be a matrix ordered standard form. Then \( (\Phi(M' \otimes M)\Phi^{-1}, HS(\mathcal{H}, \mathcal{H}), \Phi(J \otimes J)\Phi^{-1}, CPHS(\mathcal{H}^+, \mathcal{H}^+) \) is a standard form which is isomorphic to \( (\mathcal{M}' \otimes M, \mathcal{H} \otimes \mathcal{H}, J \otimes J, \mathcal{H}^+ \otimes \mathcal{H}^+) \) by the identification \( \Phi : \mathcal{H} \otimes \mathcal{H} \mapsto HS(\mathcal{H}, \mathcal{H}) \) defined as above. If \( A, B \in HS(\mathcal{H}, \mathcal{H}) \) such that \( O \leq_{cp} A \leq_{cp} B \), then there exists an element \( C \in (\mathcal{M}' \otimes M)^+ \) with \( \|C\| \leq 1 \) such that \( A = \Phi C \Phi^{-1} B \).

**References**


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