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# 自己共役錐体に付随する作用素の不等式について

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ヒルベルト空間の中に自己共役錐体を設定し、そのヒルベルト空間上の２つの有界作用素の間に、差が自己共役錐体を保存するときに順序がつくと定義する。ここでは、そのような順序に関しての基本的な性質を考える。自己共役錐体として、初めは一般のものを考え、次に標準ノイマン環に付随するもの、さらには行列順序標準形に現れる自己共役錐体を扱う。なお、行列に関するこの順序の考察は文献 [IM] で行っている。

Let $\mathcal{H}$ be a separable complex Hilbert space with an inner product $(\ ,\ )$. A convex cone $\mathcal{H}^+$ in $\mathcal{H}$ is said to be selfdual if $\mathcal{H}^+ = \{\xi \in \mathcal{H} | (\xi, \eta) \geq 0\ \forall \eta \in \mathcal{H}^+\}$. The set of all bounded operators is denoted by $L(\mathcal{H})$. For $A, B \in L(\mathcal{H})$ we shall write

$$A \trianglelefteq B \quad \text{if} \quad (B - A)(\mathcal{H}^+) \subset \mathcal{H}^+.$$

Since $\mathcal{H}$ is algebraically spanned by $\mathcal{H}^+$, the relation "$\trianglelefteq$ " defines the partial order on $L(\mathcal{H})$. For example, let

$$\mathbb{C}^{n+} = \left\{ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \middle| \lambda_1, \cdots, \lambda_n \geq 0 \right\},$$

which is a selfdual cone in $\mathbb{C}^n$. Then $A = (\lambda_{ij}) \trianglerighteq O$ if and only if $\lambda_{ij} \geq 0$ for $i, j = 1, \cdots, n$. We have had many results of such positive matrices(see [**HJ**, Chapter 8]).

The next example is a set of all $n$-by-$n$ positive semi-definite matrices denoted by $M_n^+$, which is considered as a selfdual cone in $\mathbb{C}^{n^2}$. For an $n$-by-$n$ matrix $A$, let denote

$$\hat{A}:\ X \mapsto AXA^*, X \in M_n.$$

Then $\hat{A} \trianglerighteq O$ for all $A \in M_n$.

The following proposition is valid for a general selfdual cone.

**Proposition 1.** *Let $\mathcal{H}$ be a Hilbert space with a selfdual cone $\mathcal{H}^+$. Then for bounded operators on $\mathcal{H}$ we have the following properties:*

(1) *If $O \trianglelefteq A_1 \trianglelefteq B_1$ and $O \trianglelefteq A_2 \trianglelefteq B_2$, then $O \trianglelefteq A_1A_2 \trianglelefteq B_1B_2$. In particular, if $O \trianglelefteq A \trianglelefteq B$, then $A^n \trianglelefteq B^n$ for every natural number $n$.*

(2) *It is not true that $A, B \geq O$ and $O \trianglelefteq A \trianglelefteq B$ imply $A^{\frac{1}{2}} \trianglelefteq B^{\frac{1}{2}}$.*

(3) *If $O \trianglelefteq A \trianglelefteq B$, then $O \trianglelefteq A^* \trianglelefteq B^*$.*

(4) *If $A, A^{-1}, B, B^{-1} \trianglerighteq O$ and $A \trianglelefteq B$, then $B^{-1} \trianglelefteq A^{-1}$.*

(5) *If $O \trianglelefteq A \trianglelefteq B$, then $\| A \| \leq \| B \|$.*

*Proof.* (1) By assumption $A_i(\mathcal{H}^+) \subset \mathcal{H}^+$ and $(B_i - A_i)(\mathcal{H}^+) \subset \mathcal{H}^+$ hold for $i = 1, 2$. Since

$$B_1B_2 - A_1A_2 = B_1(B_2 - A_2) + (B_1 - A_1)A_2,$$

we obtain the desired inequality.

(2) Consider the case where $\mathcal{H} = \mathbf{C}^2, \mathcal{H}^+ = \mathbf{C}^{2+}$. Put for a sufficiently large number $\lambda$ and a sufficiently small positive number $\mu$

$$A^{\frac{1}{2}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B^{\frac{1}{2}} = \begin{pmatrix} 2+\lambda & 1-\mu \\ 1-\mu & 2+\lambda \end{pmatrix}.$$

Then $A^{\frac{1}{2}} \ntrianglelefteq B^{\frac{1}{2}}$ and

$$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \trianglelefteq B = \begin{pmatrix} (2+\lambda)^2 + (1-\mu)^2 & 2(2+\lambda)(1-\mu) \\ 2(2+\lambda)(1-\mu) & (2+\lambda)^2 + (1-\mu)^2 \end{pmatrix}.$$

(3) Let $A(\mathcal{H}^+) \subset \mathcal{H}^+$. Then we have $(A^*\xi, \eta) = (\xi, A\eta) \geq 0$ for all $\xi, \eta \in \mathcal{H}^+$. The selfduality of $\mathcal{H}^+$ shows that $A^* \trianglerighteq O$. By substituting $B - A$ for $A$, we obtain the desired property.

(4) If $A \trianglelefteq B$, then from (1)

$$B^{-1} = A^{-1}AB^{-1} \trianglelefteq A^{-1}BB^{-1} = A^{-1}.$$

(5) For $A \trianglerighteq O$, put

$$\| A \|_+ = \sup\{\| A\xi \|; \| \xi \| \leq 1, \xi \in \mathcal{H}^+\}.$$

Suppose $O \trianglelefteq A \trianglelefteq B$. Note that if $\eta - \xi \in \mathcal{H}^+$ for $\xi, \eta \in \mathcal{H}^+$, then $\| \xi \| \leq \| \eta \|$, because $\| \eta \|^2 - \| \xi \|^2 = (\eta - \xi, \eta + \xi) \geq 0$. Since $\| A \|_+ \leq \| B \|_+$, it suffices to show $\| \cdot \|_+ = \| \cdot \|$.

It is known that any element $\xi \in \mathcal{H}$ can be written as $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4), \xi_1 \perp \xi_2, \xi_3 \perp \xi_4$, for some $\xi_i \in \mathcal{H}^+$. Then $\| \xi \|^2 = \sum_{i=1}^4 \| \xi_i \|^2$. Noticing that $A \trianglerighteq O$, we see that

$$\begin{aligned} \| A\xi \|^2 &= \| A\xi_1 - A\xi_2 + i(A\xi_3 - A\xi_4) \|^2 = \sum_{i=1}^{4} \| A\xi_i \|^2 - 2(A\xi_1, A\xi_2) - 2(A\xi_3, A\xi_4) \\ &\leq \| A\xi_1 + A\xi_2 + i(A\xi_3 + A\xi_4) \|^2 = \| A(\xi_1 + \xi_2) \|^2 + \| A(\xi_3 + \xi_4) \|^2 \\ &\leq \| A \|_+^2 \| \xi_1 + \xi_2 \|^2 + \| A \|_+^2 \| \xi_3 + \xi_4 \|^2 = \| A \|_+^2 \| \xi \|^2 . \end{aligned}$$

It follows that $\| A \| \leq \| A \|_+$. The converse inequality is trivial. □

We shall next deal with a selfdual cone associated with a standard von Neumann algebra. Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ be a standard form of a von Neumann algebra in the sense of Haagerup [**H**]. Namely,

(i) $J\xi = \xi,\ \xi \in \mathcal{H}^+$,

(ii) $J\mathcal{M}J = \mathcal{M}'$,

(iii) $JXJ = X^*,\ X \in Z(\mathcal{M})$,

(iv) $XJXJ(\mathcal{H}^+) \subset \mathcal{H}^+,\ X \in \mathcal{M}$.

Every von Neumann algebra has a standard representation. In particular, suppose that $\mathcal{M}$ is a von Neumann algebra with a cyclic and separating vector $\xi_0 \in \mathcal{H}$, i.e. $\overline{\mathcal{M}\xi_0} = \overline{\mathcal{M}'\xi_0} = \mathcal{H}$. Put $S_{\xi_0}X\xi_0 = X^*\xi_0, \forall X \in \mathcal{M}$. Then $S_{\xi_0}$ is a closable conjugate linear operator on $\mathcal{H}$, and the closure of $S_{\xi_0}$ is also denoted by $S_{\xi_0}$. Let $S_{\xi_0} = J_{\xi_0}\Delta_{\xi_0}^{\frac{1}{2}}$ be a polar decomposition of $S_{\xi_0}$, where $J_{\xi_0}$ is an isometric involution on $\mathcal{H}$ and $\Delta_{\xi_0} = S_{\xi_0}^* S_{\xi_0}$. Put

$$\mathcal{H}_{\xi_0}^+ = \{XJ_{\xi_0}XJ_{\xi_0}\xi_0 | X \in \mathcal{M}\}^- = \{\Delta_{\xi_0}^{\frac{1}{4}}X^*X\xi_0 | X \in \mathcal{M}\}^-,$$

which is a selfdual cone in $\mathcal{H}$. Then $(\mathcal{M},\mathcal{H},J_{\xi_0},\mathcal{H}_{\xi_0}^+)$ is a standard form.

**Proposition 2.** *Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ be a standard form of a von Neumann algebra. Given an element $A \in \mathcal{M}$, the following conditions are equivalent:*

(1) $A \trianglerighteq O$.

(2) $A \in Z(\mathcal{M})$ *and* $A \geq O$.

*Proof.* Suppose $A \trianglerighteq O, A \in \mathcal{M}$. Choose an arbitrary element $\xi \in \mathcal{H}$. Then one can write as $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4), \xi_i \in \mathcal{H}^+$ such that $\mathcal{M}\xi_1 \perp \mathcal{M}\xi_2, \mathcal{M}\xi_3 \perp \mathcal{M}\xi_4$. We ten have

$$(A\xi, \xi) = \sum_{i=1}^{4} (A\xi_i, \xi_i) \trianglerighteq O.$$

Hence $A \geq O$. Moreover,since $J\xi = \xi_1 - \xi_2 - i(\xi_3 - \xi_4)$, we get

$$(JAJ\xi, \xi) = (J\xi, AJ\xi) = \sum_{i=1}^{4} (\xi_i, A\xi_i) = (\xi, A\xi).$$

It follows that $A = A^* = JAJ \in \mathcal{M}'$. The converse implication is immediate. □

Let $\mathcal{H}_n^+, n \in \mathbb{N}$, be a family of selfdual cones in $\mathcal{H}_n$, where $\mathcal{H}_n$ means $\mathcal{H} \otimes M_n (= M_n(\mathcal{H}))$. Put $\mathrm{st} : A \mapsto A^*, A \in M_{n,m}$, where $M_{n,m}$ means a set of all $n$-by-$m$ matrices. We write $J_{n,m} = J \otimes \mathrm{st}$. We call $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+, n \in \mathbb{N})$ a matrix ordered standard form, if for every $A \in \mathcal{M} \otimes M_{n,m}$

$$AJ_{n,m}AJ_m(\mathcal{H}_m^+) \subset \mathcal{H}_n^+$$

holds. Every von Neumann algebra can be represented as a matrix ordered standard form(see [**SW2**]). In the case where $\mathcal{M}$ has a cyclic and separating vector as above, put for each $n \in \mathbb{N}$

$$(\mathcal{H}_{\xi_0})_n^+ = \overline{\mathrm{co}}\{[X_i J_{\xi_0} X_j J_{\xi_0} \xi_0]_{i,j=1}^n | X_i \in \mathcal{M}\}.$$

Here $\overline{\mathrm{co}}$ denotes the closed convex hull. Then $(\mathcal{M}, \mathcal{H}, (\mathcal{H}_{\xi_0})_n^+)$ is a matrix ordered standard form. Such a von Neumann algebra associated with $(\mathcal{H}_{\xi_0})_n^+, n \in \mathbb{N}$, is uniquely determined. Given a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$, put, for $A \in \mathcal{M}$

$$\hat{A}\xi = AJAJ\xi \text{ for all } \xi \in \mathcal{H}.$$

If $A \in \mathcal{M}$, then $\hat{A}$ is completely positive, and we shall write $\hat{A} \trianglerighteq_{cp} O$. In fact, we obtain for $[\xi_{ij}] \in \mathcal{H}_n^+, n \in \mathbb{N}$

$$\begin{aligned}\hat{A} \otimes \mathrm{id}_n[\xi_{ij}] &= [\hat{A}\xi_{ij}] = [AJAJ\xi_{ij}] \\ &= (A \otimes \mathrm{id}_n)J_n(A \otimes \mathrm{id}_n)J_n[\xi_{ij}] \in \mathcal{H}_n^+.\end{aligned}$$

It is immediate that for $A \in L(\mathcal{H})$, $A \trianglerighteq_{cp} O$ implies $A \trianglerighteq O$. The sufficient and necessary condition that $A \trianglerighteq O$ is equivalent to $A \trianglerighteq_{cp} O$ for all $A \in L(\mathcal{H})$ is that $\mathcal{M}$ is abelian(see [**M**, Corollary 1.6]).

**Proposition 3.** *For a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$, suppose $A \in L(\mathcal{H})$, $A \geq O, A \trianglerighteq O$. If $A$ has a closed range and the support projection of $A$ is completely positive, then for all $\alpha \in \mathbb{R}$, $A^\alpha \trianglerighteq_{cp} O$.*

*Proof.* Let $P$ be a support projection of $A$. Put $\mathcal{N} = P\mathcal{M}|_{P\mathcal{H}}$. Since $P$ is completely positive, we see from [**MN**, Lemma 3] that $(\mathcal{N}, P\mathcal{H}, P_n\mathcal{H}_n^+)$ is a matrix ordered standard form. By assumption, $PA = AP \geq O$, $\trianglerighteq O$, and $PA$ maps a selfdual subcone $P\mathcal{H}^+$ in $P\mathcal{H}^+$) onto itself. It follows from [**C**, Theorem 3.3] that there exists an element $B \in \mathcal{N}^+$ such that $PA = BJBJP$. Hence

$$A^\alpha = B^\alpha J B^\alpha J P \trianglerighteq_{cp} O$$

for every real number $\alpha$. $\square$

**Theorem 4.** *With $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ as above, let $O \trianglelefteq A \trianglelefteq \hat{B}, A \in L(\mathcal{H}), B \in \mathcal{M}$. If $B$ is injective and has a dense range, then there exists an element $C \in Z(\mathcal{M})^+$ with $\| C \| \leq 1$ such that $A = C\hat{B}$. In particular, if $\mathcal{M}$ is factor, then one can choose a scalar $\lambda$ with $0 \leq \lambda \leq 1$ such that $A = \lambda\hat{B}$.*

*Proof.* Consider the polar decomposition $B = U|B|$ of $B$. By assumption $U$ is a unitary element of $\mathcal{M}$, and so $\hat{U} \trianglerighteq O$ and $\hat{U}^* \trianglerighteq O$ by Proposition 1 (3). Hence we may assume $B$ to be positive semi-definite. Let $B = \int_0^{\|B\|} \lambda dE_\lambda$ be a spectral decomposition of $B$. Put $P_n = \int_{\frac{1}{n}}^{\|B\|} dE_\lambda$ for $n \in \mathbb{N}$. Then one sees that $\hat{P}_n \nearrow I$ and $\hat{P}_n A \hat{P}_n \trianglelefteq \hat{P}_n \hat{B} \hat{P}_n$ by Proposition 1 (1). Since $\hat{P}_n \hat{B} \hat{P}_n$ is invertible on $\hat{P}_n\mathcal{H}$, where the inverse shall be denoted by $(\hat{P}_n \hat{B} \hat{P}_n)^{-1}$, we have

$$O \trianglelefteq \hat{P}_n A \hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \trianglelefteq \hat{P}_n.$$

There then exists an element $c_n$ in an order ideal $Z_{\hat{P}_n\mathcal{H}^+}$ of a selfdual cone $\hat{P}\mathcal{H}^+$ with $\| c_n \| \leq 1$ such that $\hat{P}_n A \hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1}\xi = c_n\xi$ for all $\xi \in \hat{P}_n\mathcal{H}$. By [**I**, Theorem VI.1,2 3)] we obtain that $c_n \in Z(\hat{P}_n\mathcal{M}|_{\hat{P}_n\mathcal{H}})^+$. Since $\hat{P}_n Z(\mathcal{M}) \hat{P}_n = Z(\hat{P}_n \mathcal{M} \hat{P}_n)$, we can find an element $C_n \in Z(\mathcal{M})$ such that $c_n\xi = \hat{P}_n C_n \hat{P}_n \xi$ for all $\xi \in \hat{P}_n\mathcal{H}$. Since $P_n B = B P_n, n \in \mathbb{N}$, we have

$$\begin{aligned} \hat{P}_{n+1} C_{n+1} \hat{P}_{n+1} \xi &= \hat{P}_{n+1} A \hat{P}_{n+1} (\hat{P}_{n+1} \hat{B} \hat{P}_{n+1})^{-1} \hat{P}_n \xi \\ &= \hat{P}_{n+1} A \hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \xi = \hat{P}_n C_n \hat{P}_n \xi \end{aligned}$$

for all $\xi \in \hat{P}_n\mathcal{H}$. Since $\{\hat{P}_nC_n\hat{P}_n\}$ is a bounded sequence, one can define

$$C\xi = \lim_{n\to\infty} \hat{P}_nC_n\hat{P}_n\xi,\ \xi \in \mathcal{H}.$$

Thus $C \in Z(\mathcal{M})^+$, $\| C \| \leq 1$ and we get

$$\begin{aligned} A &= \text{s-}\lim_{n\to\infty} \hat{P}_nA\hat{P}_n \\ &= \text{s-}\lim_{n\to\infty} \hat{P}_nC_n\hat{P}_nA\hat{P}_n \\ &= C\hat{B}. \end{aligned}$$

This completes the proof. $\square$

Now, consider two matrix ordered standard forms $(\mathcal{M}^{(1)}, \mathcal{H}^{(1)}, \mathcal{H}_n^{(1)+})$ and $(\mathcal{M}^{(2)}, \mathcal{H}^{(2)}, \mathcal{H}_n^{(2)+})$ with respective canonical involutions $J^{(1)}$ and $J^{(2)}$. Given an arbitrary element $\xi \in \mathcal{H}^{(1)}$, let $R_\xi$ be a right slice map of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ into $\mathcal{H}^{(2)}$ such that

$$R_\xi(\xi' \otimes \eta') = (\xi', \xi)\eta', \xi' \in \mathcal{H}^{(1)}, \eta' \in \mathcal{H}^{(2)}.$$

For any element $x \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, we put

$$\Phi(x)(\xi) = R_{J^{(1)}\xi}(x), \xi \in \mathcal{H}^{(1)}.$$

Then $\Phi(x)$ is a map of Hilbert-Schmidt class of $\mathcal{H}^{(1)}$ to $\mathcal{H}^{(2)}$. A set of all maps of Hilbert-Schmidt class of $\mathcal{H}^{(1)}$ to $\mathcal{H}^{(2)}$ is denoted by $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$. A set of all completely positive maps of $(\mathcal{H}^{(1)}, \mathcal{H}_n^{(1)+'})$ to $(\mathcal{H}^{(2)}, \mathcal{H}_n^{(2)+})$ in $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$ is denoted by $CPHS(\mathcal{H}^{(1)+'}, \mathcal{H}^{(2)+})$. Here $\mathcal{H}_n^{(1)+'} = \{{}^t[\xi_{ij}]_{i,j=1}^n | [\xi_{ij}]_{i,j=1}^n \in \mathcal{H}_n^{(1)+}\}$ is a selfdual cone corresponding to $\mathcal{M}^{(1)\prime}$. We shall here write $\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+}$ for a selfdual cone corresponding to $\mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)}$. It was shown in [**MT**, **SW1**] that

$$\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+} = \{x \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} | \Phi(x) \in CPHS(\mathcal{H}^{(1)+'}, \mathcal{H}^{(2)+})\}.$$

Thus

$$\Phi :\ \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \to HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$$

is an isometry mapping $\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+}$ onto $CPHS(\mathcal{H}^{(1)+'}, \mathcal{H}^{(2)+})$. In fact, $\Phi$ is isometric. Suppose that $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$ has an inner product

$$\langle A, B\rangle = \sum_{k=1}^{\infty}(Ae_k, Be_k),$$

where $\{e_k\}$ is a complete orthogonal basis of $\mathcal{H}^{(1)}$. Noticing that $\{J^{(1)}e_k\}$ is a complete orthogonal basis of $\mathcal{H}^{(1)}$, we obtain for a complete orthogonal basis $\{f_k\}$ of $\mathcal{H}^{(2)}$

$$\begin{aligned}
\langle\Phi(J^{(1)}e_i \otimes f_j),&\Phi(J^{(1)}e_{i'} \otimes f_{j'})\rangle \\
&= \sum_{k=1}^{\infty}(\Phi(J^{(1)}e_i \otimes f_j)(e_k), \Phi(J^{(1)}e_{i'} \otimes f_{j'})(e_k)) \\
&= \sum_{k=1}^{\infty}(R_{J^{(1)}e_k}(J^{(1)}e_i \otimes f_j), R_{J^{(1)}e_k}(J^{(1)}e_{i'} \otimes f_{j'})) \\
&= \sum_{k=1}^{\infty}((J^{(1)}e_i, J^{(1)}e_k)f_j, (J^{(1)}e_{i'}, J^{(1)}e_k)f_{j'}) \\
&= \delta_{ii'}\delta_{jj'},
\end{aligned}$$

for $i, j, i', j' = 1, 2, \cdots$. Therefore, $(\Phi(\mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)})\Phi^{-1}$, *HS*$(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$, $\Phi(J^{(1)} \otimes J^{(2)})\Phi^{-1}$, *CPHS*$(\mathcal{H}^{(1)+'}, \mathcal{H}^{(2)+})$ is a standard form. Using the Radon-Nikodym theorem for $L^2$-spaces [**S**, Theorem 1.2], we obtain the following proposition:

**Proposition 5.** *Let $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ be a matrix ordered standard form. Then $(\Phi(\mathcal{M}' \otimes \mathcal{M})\Phi^{-1}$,HS$(\mathcal{H}, \mathcal{H})$,$\Phi(J \otimes J)\Phi^{-1}$,CPHS$(\mathcal{H}^+, \mathcal{H}^+)$ is a standard form which is isomorphic to $(\mathcal{M}' \otimes \mathcal{M}, \mathcal{H} \otimes \mathcal{H}, J \otimes J, \mathcal{H}^+ \otimes \mathcal{H}^+)$ by the identification $\Phi : \mathcal{H} \otimes \mathcal{H} \mapsto$ HS$(\mathcal{H}, \mathcal{H})$defined as above. If $A, B \in$ HS$(\mathcal{H}, \mathcal{H})$such that $O \trianglelefteq_{cp} A \trianglelefteq_{cp} B$, then there exists an element $C \in (\mathcal{M}' \otimes \mathcal{M})^+$with $\| C \| \leq 1$ such that $A = \Phi\hat{C}\Phi^{-1}B$.*

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