数理解析研究所講究録
1144 巻 2000 年 70-76

ON 2–HYPONORMAL OPERATORS

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The purpose of this talk is to make a brief survey of recent research related to 2–hyponormal operators.

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}):=\mathcal{L}(\mathcal{H},\mathcal{H})$. An operator $T\in\mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T=TT^*$, hyponormal if $T^*T\geq TT^*$, and subnormal if $T=N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K}\supseteq\mathcal{H}$. Thus the operator T is subnormal if and only if there exist operators A and B such that $\widehat{T}:=\left(\begin{smallmatrix} T & A\\ 0 & B\end{smallmatrix}\right)$ is normal, i.e.,

$$\begin{cases} [T^*,T]:=T^*T-TT^*=AA^* \\ A^*T=BA^* \\ [B^*,B]+A^*A=0. \end{cases} \tag{0.1}$$

An alternative description of subnormality is given by the Bram-Halmos criterion, which states that an operator T is subnormal if and only if

$$\sum_{i,j}(T^ix_j,T^jx_i)\geq 0$$

for all finite collections $x_0,x_1,\cdots,x_k\in\mathcal{H}$ ([Br],[Con, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix}\geq 0 \qquad (\text{all } k\geq 1). \tag{0.2}$$

Condition (0.2) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (0.2) for $k=1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (0.2) for all k. Let $[A,B]:=AB-BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k\times k$ operator matrix

$$M_k(T):=([T^{*j},T^i])_{i,j=1}^k \tag{0.3}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (0.3) is equivalent to the positivity of the $(k+1)\times(k+1)$ operator matrix in (0.2); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k\geq 1$ ([CMX]). Now it is natural to ask whether k-hyponormal operators admit an extension

1991 *Mathematics Subject Classification.* Primary 47B20, 47B35, 47B37;Secondary 47A20.

$\widehat{T}$ with one or more of the properties listed in (0.1). Recall ([At],[CMX],[CoS]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS((T, T^2, \cdots, T^k)) := \left\{ \sum_{j=1}^{k} \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e., ([CMX])

$$(0.4) \qquad (M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}) \geq 0 \qquad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbb{C}.$$

If $k = 2$ then T is said to be *quadratically hyponormal*. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general. The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([Cu1],[Cu2],[CF1], [CF2],[CF3],[CLL],[CL1],[CL2],[CL3],[CL4],[CMX], [DPY],[McCP]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle: in fact, even subnormality for Toeplitz operators has not yet been characterized (cf. [Hal], [Cow1], [CoL]). For weighted shifts, positive results appear in [Cu1] and [CF3], although no concrete example of a weighted shift which is polynomially hyponormal but not subnormal has yet been found (the existence of such weighted shifts was established in [CP1] and [CP2]).

The following notion was introduced in [CL4].

1. Definition. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly subnormal* if there exist operators $A \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}')$ such that the first two conditions in (0.1) hold: $[T^*, T] = AA^*$ and $A^*T = BA^*$. The operator $\widehat{T}$ is said to be a *partially normal extension* of T.

Clearly,

$$(1.1) \qquad \text{subnormal} \Longrightarrow \text{weakly subnormal} \Longrightarrow \text{hyponormal}.$$

However the converses of both implications in (1.1) are not true in general (see [CL4]).

The following theorem provides a clue for a model of 2–hyponormal operators.

2. Theorem ([CL4, Lemma 2.1]). *If $T \in \mathcal{L}(\mathcal{H})$ is 2-hyponormal then T has a linear (not necessarily bounded) extension $\widehat{T}$ on $\mathcal{H} \oplus \mathcal{H}$ satisfying the equality $\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f$ for all $f \in \mathcal{H}$. More precisely,*

$$(2.1) \qquad \widehat{T} := \begin{pmatrix} T & [T^*, T]^{\frac{1}{2}} \\ 0 & \widetilde{S} \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H},$$

where $\widetilde{S} : ker[T^*, T] \oplus ran[T^*, T] \longrightarrow \mathcal{H}$ is defined by

$$(2.2) \qquad \widetilde{S}f := \begin{cases} [T^*, T]^{\frac{1}{2}} T g & \text{if } f = [T^*, T]^{\frac{1}{2}} g \text{ with } g \in ran[T^*, T] \\ 0 & \text{if } f \in ker[T^*, T]. \end{cases}$$

Moreover if $[T^*, T]$ has closed range (e.g., if $[T^*, T]$ is finite rank) then T is weakly subnormal.

Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the *(unilateral) weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. If W_α is a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$, then the *moments* of W_α are usually defined by $\beta_0 := 1$, $\beta_{n+1} := \alpha_n \beta_n$ $(n \geq 0)$ [Shi]; however, we prefer to reserve this term for the sequence $\gamma_n := \beta_n^2$ $(n \geq 0)$. A criterion for k-hyponormality can be given in terms of these moments ([Cu1, Theorem 4]): if we build a $(k+1) \times (k+1)$ Hankel matrix $A(n;k)$ by

$$A(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \quad (n \geq 0), \tag{2.3}$$

then

$$W_\alpha \text{ is } k\text{-hyponormal} \iff A(n;k) \geq 0 \quad (n \geq 0). \tag{2.4}$$

In particular, for α strictly increasing, W_α is 2-hyponormal if and only if

$$\det \begin{pmatrix} \gamma_n & \gamma_{n+1} & \gamma_{n+2} \\ \gamma_{n+1} & \gamma_{n+2} & \gamma_{n+3} \\ \gamma_{n+2} & \gamma_{n+3} & \gamma_{n+4} \end{pmatrix} \geq 0 \quad (n \geq 0). \tag{2.5}$$

In [CL4] it was shown that 2-hyponormal weighted shifts are weakly subnormal operators possessing partially normal extensions which are hyponormal.

3. Theorem ([CL4, Theorem 1.2]). *Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ be a weight sequence. If W_α is a 2-hyponormal weighted shift on $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+)$, then W_α is weakly subnormal. Moreover, there exists a partially normal extension $\widehat{W}_\alpha$ on $\mathcal{K} := \mathcal{H} \oplus \mathcal{H}$ such that*

(i) *$\widehat{W}_\alpha$ is hyponormal;*
(ii) *$\sigma(\widehat{W}_\alpha) = \sigma(W_\alpha)$; and*
(iii) *$||\widehat{W}_\alpha|| = ||W_\alpha||$.*

In particular, if α is strictly increasing then $\widehat{W}_\alpha$ can be obtained as

$$\widehat{W}_\alpha = \begin{pmatrix} W_\alpha & [W_\alpha^*, W_\alpha]^{\frac{1}{2}} \\ 0 & W_\beta \end{pmatrix} \qquad \text{on } \mathcal{K} := \mathcal{H} \oplus \mathcal{H}, \tag{3.1}$$

where W_β is a weighted shift whose weight sequence $\{\beta_n\}_{n=0}^\infty$ given by

$$\beta_n = \alpha_n \sqrt{\frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2}} \quad (n = 0, 1, \cdots; \ \alpha_{-1} := 0).$$

In addition, if W_α is 3-hyponormal then $\widehat{W}_\alpha$ can be chosen to be weakly subnormal.

We now meet a natural question:

Question A. *Is every* 2*-hyponormal operator weakly subnormal ?*

Towards an affirmative answer we must find a partially normal extension $\widehat{T}$. As a candidate one might suggest, in view of (2.1), that

$$\widehat{T} = \begin{pmatrix} T & [T^*,T]^{\frac{1}{2}} \\ 0 & S \end{pmatrix},$$

where S is a continuous linear extension of $\widetilde{S}$ in Theorem 2. The key missing step is then to show that S is bounded.

On the other hand, do there exist hyponormal weighted shifts which are not weakly subnormal? To answer this question, we first give an outer propagation property of weakly subnormal weighted shifts.

4. Theorem ([CL4, Theorem 4.3]). *Let $T \equiv W_\alpha$ be a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$. Assume that T is weakly subnormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$ then $\alpha_{n+k} = \alpha_n$ for all $k \geq 1$.*

5. Example ([CL4, Example 4.4]). With the aid of Theorem 4 we can find examples of operators which are hyponormal (even quadratically hyponormal) but not weakly subnormal: for example, if

$$\alpha_0 = \alpha_1 = \sqrt{\frac{2}{3}}, \quad \alpha_n = \sqrt{\frac{n+1}{n+2}}\ (n \geq 2) \quad \text{(cf. [Cu1, Proposition 7])},$$

then W_α is quadratically hyponormal but not weakly subnormal. □

Now one might expect an inner propagation (and hence full propagation) for weakly subnormal weighted shifts. But we don't know if this is the case. In fact we were unable to decide if every weakly subnormal weighted shift is 2-hyponormal. If a weighted shift T has a partially normal extension $\widehat{T}$ of the form (3.1) then we can see ([CL4, the proof of Theorem 1.2]) that T should be 2-hyponormal. But it is not evident that whenever a hyponormal weighted shift T has a partially normal extension $\widehat{T}$, $\widehat{T}$ should be of the form (3.1) up to unitary equivalence. Thus we would like to pose the following:

Question B. *Is every weakly subnormal weighted shift* 2*-hyponormal ?*

Let's turn our attention to 2–hyponormality of Toeplitz operators. Recall that the Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \ldots\}$. An element $f \in L^2(\mathbb{T})$ is said to be analytic if $f \in H^2(\mathbb{T})$, and co-analytic if $f \in L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$. If P denotes the orthogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$ and J is the unitary operator from $L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$ onto $H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$, the operator T_φ on $H^2(\mathbb{T})$ defined by

$$T_\varphi g := P(\varphi g) \qquad (g \in H^2(\mathbb{T}))$$

is called the *Toeplitz operator* with symbol φ. It is well known that analytic Toeplitz operators are subnormal.

The study of subnormal Toeplitz operators was originated from P.R. Halmos's Problem 5 (cf. [Ha1],[Ha2]): *Is every subnormal Toeplitz operator either normal or analytic ?* As we know, this question was answered in the negative by C. Cowen and J. Long [CoL]. But it is still open *which Toeplitz operators are subnormal ?* The hyponormality of Toeplitz operators has been studied by M. Abrahamse [Ab], C. Cowen [Cow1],[Cow2], P. Fan [Fa], C. Gu [Gu], T. Ito and T. Wong [ItW], T. Nakazi and K. Takahashi [NT], D. Yu [Yu], K. Zhu [Zh], D. Farenick, the author, and his collaborators (cf. [FL1],[FL2],[CL1],[HKL],[KL]). An elegant theorem of C. Cowen [Cow3] characterizes the hyponormality of a Toeplitz operator T_φ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. K. Zhu [Zh] reformulated Cowen's criterion and then showed that the hyponormality of T_φ with polynomial symbols φ can be decided by a method based on the classical interpolation theorem of I. Schur [Sch].

Now it seems to be interesting to understand the gap between k-hyponormality and subnormality for Toeplitz operators. As a candidate for the first question in this line we posed the following ([CL1],[CL3]):

Question C. *Is every 2-hyponormal Toeplitz operator subnormal ?*

In [CL3], the following was shown:

6. Theorem ([CL3, Corollary 6]). *If T_φ is 2-hyponormal and if φ or $\bar{\varphi}$ is of bounded type (i.e., φ or $\bar{\varphi}$ is a quotient of two analytic functions) then T_φ is normal or analytic.*

In view of Theorem 6, it would be interesting to consider which 2–hyponormal Toeplitz operators are subnormal. The first inquiry involves the self–commutator. Subnormal operators with finite rank self-commutators have been studied by many authors ([Al], [McCYa], [Mo], [OTT], [Xi1], [Xi2]). In 1975, I. Amemiya, T. Ito and T. Wong [AIW] showed that if T_φ is a subnormal Toeplitz operator with rank–one self–commutator then φ is a linear function of a Blaschke product of degree 1. More generally, B. Morrel [Mo] showed that a pure subnormal operator with rank-one self-commutator is unitarily equivalent to a linear function of the unilateral shift. Very recently, in [CL4], it was shown that every pure 2–hyponormal operator with rank-one self–commutator is a linear function of the unilateral shift. On the other hand, J. McCarthy and L. Yang [McCYa] have classified all rationally cyclic subnormal operators with finite rank self–commutators. However it is still open which are the pure subnormal operators with finite rank self–commutator. Related to this, in [CL3] we formulated the following:

Question D. *If T_φ is a 2–hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that T_φ is analytic ?*

On the other hand, there exists a gap between 2–hyponormality and quadratic hyponormality for weighted shifts (cf. [Cu1]). However we have not been able to decide whether there exists a gap between 2–hyponormality and quadratic hyponormality for *Toeplitz operators.*

Question E. *Does there exist a quadratically hyponormal Toeplitz operator which is not 2–hyponormal ?*

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