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Direction and curvature of 2-dimensional cracks

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- abstract -

We propose some formulas connecting the direction and curvature of 2-dimensional cracks in elastic solids with the coefficients of the eigenfunction expansion of the displacement at the crack tip. These formulas may be used as basic equations to derive a governing equation of cracks.

1 Introduction

There are at least four famous criteria for predicting the direction of the extension of a crack (see, for example,[1],[2],[5],[9]). Among them, two criteria are frequently used in the practical computation. The first is the one proposed by Erdogan and Sih [2] and the other is the one derived from the assumption of the local symmetry of stress distribution. The former which is usually called the maximum stress criterion conjectures that the crack will grow in a direction perpendicular to the maximum principal stress. This criterion is applicable not only to elastic bodies but also to plastic bodies, at least formally. On the other hand, the latter is based on the assumption that the ***stress intensity factor*** K_{II} will vanish along the crack path, if the path is smooth. Cotterell and Rice[1], Sumi et al.[10], for example, have developed effective ways of predicting the crack path based on this criterion.

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Different criteria lead different formulations of the crack problem and, to our knowledge, no final conclusion is obtained on the reality of these criteria, although it is known that these formulations can yield useful numerical results.

For a rigorous mathematical discussion, the problem has to be well formulated mathematically. In this context the work of Friedman and Liu[3] is important. They formulated a quasi-stational model of crack propagation based on the J-integrals and gave a proof on the existence and uniqueness of the crack propagation.
In the present paper we first give a breaf introduction to the author's recent paper [7] and then add some new results.

2 General framewok

Let C be a crack in a 2-dimensional elastic solid with one crack tip. The body must satisfy the elastic equlibrium equations:

$$\begin{aligned} \sum_{j=1}^{2} \sigma_{ij,j} &= 0 \quad in \quad \Omega \\ T_i(u) &= 0 \quad on \quad C \quad (i=1,2) \end{aligned}$$

Symbols:

$$\begin{aligned} E, \nu &: \ Young's\ modulus, Poisson's\ ratio \\ u = (u_1, u_2) &: \ displacements\ in\ x - coordinates \\ \sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12}) &: \ stresses \\ \epsilon = (\epsilon_{11}, \epsilon_{22}, \epsilon_{12}) &: \ strains \\ \nu_i &: \ directional\ cosines\ to\ C \\ T_i(u) &: \ surface\ tractions. \end{aligned}$$

$$\epsilon_{11} = u_{1,1} \quad \epsilon_{22} = u_{2,2} \quad \epsilon_{12} = u_{1,2} + u_{2,1}$$

We assume a linear elastic, plain-stress state.

3 Eigenfunction expansion of the displacements

Our discussion is based on an assumption that the displacements can be expanded in a certain series. To derive this series we first rewrite the equilibrium equations as follows.

$$
\begin{aligned}
U_{r,rr} + \frac{1}{r}U_r + \frac{1-\nu}{2r^2}U_{r,\theta\theta} - \frac{1}{r^2}U_r & \\
+\frac{1+\nu}{2r}U_{\theta,r\theta} - \frac{3-\nu}{2r^2}U_{\theta,\theta} &= 0 \\
\frac{1+\nu}{2r}U_{r,r\theta} + \frac{3-\nu}{2r^2}U_{r,\theta} + \frac{1}{r^2}U_{\theta,\theta\theta} & \\
+\frac{1-\nu}{2}U_{\theta,rr} + \frac{1-\nu}{2r}U_{\theta,r} - \frac{1-\nu}{2r^2}U_\theta &= 0.
\end{aligned}
$$

If the crack is straight, there is no special solution of the form $r^p(\log r)^m\phi(\theta)$ ($m \neq 0$) and of finite energy, satisfying the free boundary conditions on the crack.

Therefore we substituting the functions

$$
\begin{pmatrix} U_r^{(\alpha)} \\ U_\theta^{(\alpha)} \end{pmatrix} = \begin{pmatrix} r^\alpha\varphi(\theta) \\ r^\alpha\psi(\theta) \end{pmatrix}.
$$

into the equilibrium equations to get the special solutions of the form

$$
\begin{cases} \varphi = \cos(\alpha+1)\theta \\ \psi = -\sin(\alpha+1)\theta \end{cases} \qquad \begin{cases} \varphi = c_\alpha\cos(\alpha-1)\theta \\ \psi = -\tilde{c}_\alpha\sin(\alpha-1)\theta \end{cases},
$$

$$
\begin{cases} \varphi = \sin(\alpha+1)\theta \\ \psi = \cos(\alpha+1)\theta \end{cases} \qquad \begin{cases} \varphi = c_\alpha\sin(\alpha-1)\theta \\ \psi = \tilde{c}_\alpha\cos(\alpha-1)\theta \end{cases},
$$

where

$$
\begin{aligned}
c_\alpha &= (1+\nu)\alpha - 3 + \nu \\
\tilde{c}_\alpha &= (1+\nu)\alpha + 3 - \nu.
\end{aligned}
$$

We therefore assume that the solution is expressed by a linear combination of the functions

$$\begin{pmatrix} U_r^{(\alpha)} \\ U_\theta^{(\alpha)} \end{pmatrix} = r^\alpha \begin{pmatrix} A^{(\alpha)} c_\alpha \cos(\alpha-1)\theta + B^{(\alpha)} c_\alpha \sin(\alpha-1)\theta \\ +C^{(\alpha)} \cos(\alpha+1)\theta + D^{(\alpha)} \sin(\alpha+1)\theta \\ -A^{(\alpha)} \tilde{c}_\alpha \sin(\alpha-1)\theta + B^{(\alpha)} \tilde{c}_\alpha \cos(\alpha-1)\theta \\ -C^{(\alpha)} \sin(\alpha+1)\theta + D^{(\alpha)} \cos(\alpha+1)\theta \end{pmatrix}.$$

$$\begin{pmatrix} U_r \\ U_\theta \end{pmatrix} = \sum_{k=1}^{\infty} \begin{pmatrix} U_r^{(\frac{1}{2}k)}(r,\theta) \\ U_\theta^{(\frac{1}{2}k)}(r,\theta), \end{pmatrix}$$

4 Eigenfunction expansion of the tractions

We expand the tractions on the crack in r.

$$\begin{aligned} T_1 &= \sigma_{11}\nu_1 + \sigma_{12}\nu_2, \\ T_2 &= \sigma_{21}\nu_1 + \sigma_{22}\nu_2 \end{aligned}$$

Propositon 1. (I) $T_1 = 0$ implies :

$$\frac{1}{4}(1-\nu^2)A^{(\frac{1}{2})} - \frac{1}{2}(1-\nu)C^{(\frac{1}{2})} = 0,$$

$$\begin{aligned} &-\frac{15}{16}(1-\nu^2)\kappa B^{(\frac{1}{2})} + \frac{3}{8}(1-\nu)\kappa D^{(\frac{1}{2})} \\ &+ \frac{3}{4}(1-\nu^2)A^{(\frac{3}{2})} + \frac{3}{2}(1-\nu)C^{(\frac{3}{2})} = 0, \end{aligned}$$

$$\begin{aligned} &\frac{31}{128}(1-\nu^2)\kappa^2 A^{(\frac{1}{2})} + \frac{25}{48}(1-\nu^2)\kappa' B^{(\frac{1}{2})} \\ &+ \frac{9}{64}(1-\nu)\kappa^2 C^{(\frac{1}{2})} - \frac{5}{24}(1-\nu)\kappa' D^{(\frac{1}{2})} \\ &+ \frac{45}{16}(1-\nu^2)\kappa B^{(\frac{3}{2})} - \frac{15}{8}(1-\nu)\kappa D^{(\frac{3}{2})} \\ &- \frac{15}{4}(1-\nu^2)A^{(\frac{5}{2})} - \frac{5}{2}(1-\nu)C^{(\frac{5}{2})} = 0. \end{aligned}$$

(II) $T_2 = 0$ implies:

$$\begin{aligned}
-\frac{3}{4}(1-\nu^2)B^{(\frac{1}{2})} - \frac{1}{2}(1-\nu)D^{(\frac{1}{2})} &= 0,\\
\frac{3}{16}(1-\nu^2)\kappa A^{(\frac{1}{2})} - \frac{3}{8}(1-\nu)\kappa C^{(\frac{1}{2})} & \\
+\frac{15}{4}(1-\nu^2)B^{(\frac{3}{2})} + \frac{3}{2}(1-\nu)D^{(\frac{3}{2})} &= 0,\\
-\frac{5}{48}(1-\nu^2)\kappa' A^{(\frac{1}{2})} + \frac{67}{128}(1-\nu^2)\kappa^2 B^{(\frac{1}{2})} & \\
+\frac{5}{24}(1-\nu)\kappa' C^{(\frac{1}{2})} + \frac{9}{64}(1-\nu)\kappa^2 D^{(\frac{1}{2})} & \\
+\frac{15}{16}(1-\nu^2)\kappa A^{(\frac{3}{2})} + \frac{15}{8}(1-\nu)\kappa C^{(\frac{3}{2})} & \\
-\frac{35}{4}(1-\nu^2)B^{(\frac{5}{2})} - \frac{5}{2}(1-\nu)D^{(\frac{5}{2})} &= 0.
\end{aligned}$$

Remark. Equation (19) reduces to

$$\frac{15}{4}(1-\nu^2)B^{(\frac{3}{2})} + \frac{3}{2}(1-\nu)D^{(\frac{3}{2})} = 0$$

by the identity (15).

5 Direction and curvature

$A^{(\alpha)}$etc. are coordinate-dependent. We fix a coordinate system and introduce the following quantity. Let k be the angle of the crack at the tip.

$$\begin{aligned}
k_1 &= A^{(\frac{1}{2})}\cos\frac{k}{2} + B^{(\frac{1}{2})}\sin\frac{k}{2}\\
k_2 &= -A^{(\frac{1}{2})}\sin\frac{k}{2} + B^{(\frac{1}{2})}\cos\frac{k}{2}\\
k_3 &= C^{(\frac{1}{2})}\cos\frac{3k}{2} - D^{(\frac{1}{2})}\sin\frac{3k}{2}\\
k_4 &= C^{(\frac{1}{2})}\sin\frac{3k}{2} + D^{(\frac{1}{2})}\cos\frac{3k}{2}.
\end{aligned}$$

Substituting these relations into

$$\begin{aligned} C^{(\frac{1}{2})} &= \frac{1}{2}(1+\nu)A^{(\frac{1}{2})} \\ D^{(\frac{1}{2})} &= -\frac{3}{2}(1+\nu)B^{(\frac{1}{2})}. \end{aligned}$$

we have for $x = \tan\frac{k}{2}$,

$$\begin{aligned} \frac{1}{2}(1+\nu)k_1 - k_3 - (\frac{1}{2}(1+\nu)k_2 + 3k_4)x & \\ +(\frac{1}{2}(1+\nu)k_1 + 3k_3)x^2 - (\frac{1}{2}(1+\nu)k_2 - k_4)x^3 &= 0, \\ \frac{3}{2}(1+\nu)k_2 + k_4 + (\frac{3}{2}(1+\nu)k_1 - 3k_3)x & \\ +(\frac{3}{2}(1+\nu)k_2 - 3k_4)x^2 + (\frac{3}{2}(1+\nu)k_1 + k_3)x^3 &= 0. \end{aligned}$$

Proposition 2. The angle k and the coefficients of the lowest terms satisfy the following relations.

$$\begin{aligned} P(k) &= (1+\nu)(3k_1k_4 + k_2k_3) \pm \sqrt{D} \\ Q(k) &= \frac{3}{4}(1+\nu)^2(k_1^2 + k_2^2) \\ &\quad +(1+\nu)(5k_1k_3 - 3k_2k_4) \\ &\quad +3(k_3^2 + k_4^2) \\ \tan\frac{k}{2} &= \frac{P(k)}{Q(k)} \end{aligned}$$

$$\begin{aligned} D &= (1+\nu)^2(3k_1k_4 + k_2k_3)^2 \\ &\quad - \left(\frac{3}{4}(1+\nu)^2(k_1^2 + k_2^2) + (1+\nu)(5k_1k_3 - 3k_2k_4) + 3(k_3^2 + k_4^2)\right) \\ &\quad \times \left(\frac{3}{4}(1+\nu)^2(k_1^2 + k_2^2) - (1+\nu)(k_1k_3 + k_2k_4) - (k_3^2 + k_4^2)\right) \\ &= 16(B^{(\frac{1}{2})})^2(1+\nu)^4(A^{(\frac{1}{2})} + A^{(\frac{1}{2})}\cos k + 3B^{(\frac{1}{2})}\sin k)^2 \end{aligned}$$

Proposition 3. The crack tip curvature κ satisfies

$$8B^{(\frac{1}{2})}\kappa' + 3A^{(\frac{1}{2})}\kappa^2 + 72B^{(\frac{3}{2})}\kappa - 36A^{(\frac{5}{2})} - \frac{24}{1+\nu}C^{(\frac{5}{2})} = 0.$$

6 Determination of $k_i's$

$$\begin{aligned} \Gamma &= \{x; |r| = r_0\}, \quad \Gamma_\delta = \{x; |r| = \delta\} \\ \Omega &= \{x : \delta < |r| < r_0\} \cap \Omega \\ C_\delta^{\pm} &= C^{\pm} \cap \bar{\Omega}_\delta \end{aligned}$$

Starting from the identity

$$\int_{\partial\Omega_\delta} < T(u), v > ds = \int_{\partial\Omega_\delta} < u, T(v) > ds$$

we have

$$\begin{aligned} &\int_{\Gamma_\delta} (< T(u), v > - < u, T(v) >) ds \\ = &\int_{\Gamma} (< u, T(v) > - < T(u), v >) ds \\ &+ \int_{C_\delta^{\pm}} < u, T(v) > ds \end{aligned}$$

We express u in a fixed coordinate system and take the singular parts as v(4 independent particular solutions). Then we have a system of linear equations w.r.t k_i with the coefficient matrix

$$\begin{pmatrix} 0 & 0 & -p & 0 \\ 0 & 0 & 0 & p \\ -p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \end{pmatrix} \qquad p = 4\pi(1-\nu).$$

which is, essentially, an identity matrix !
The integration on $C_\delta^{\pm}$ converges as $\delta \to 0$, which is proved by considering the sign of the singular part of the solution on $C_\delta^{\pm}$. Let $u = u^{(S)} + u^{(R)}$. The signs of $u^{(S)}$ are different but those of $T(v)$ are the same, and so we have

$$\int_{C_\delta^{\pm}} < u, T(v) > ds = \int_{C_\delta^{\pm}} < u^{(R)}, T(v) > ds$$

7 Illustrative examples

(1). The direction.

We examined if the given direction of the curvature can be reproduced by the given formulas. We considered the following 4 cases.
(1) Criterion based on J-integrals [3].
(2) Criterion based on the presented where $+\sqrt{D}$ is taken.
(3) Criterion based on the presented where $-\sqrt{D}$ is taken.
(4) Criterion based on the presented where $\sqrt{D}$ is negracted.
All calculation are carried by a code using Mathematica. The result is as follows.
(a) If $K_{II} = 0$ all cases reproduce the correct direction.
(b) In any case (2) gives the correct direction.
(c) If K_{II} is small all cases give the similar result.
(d) (1) and (2) gives almost equal direction even for fairly large K_{II}.

(2). Semi-circular crack.

Let $z = x_1 + ix_2$ be the complex number.

$$(1) \qquad \sigma_{11} + \sigma_{22} = 4Re\phi'(z)$$
$$(2) \qquad \sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2(\bar{z}\phi''(z) + \psi'(z)).$$

If the stresses determined by these equations satisfy the traction free condition on the crack C, then the functions (u, v) determined by the following equation give the exact displacements (modulo constant) satisfying a certain boundary conditions at the outer boundary.

$$(3) \qquad 2G(u + iv) = \lambda\phi(z) - z\bar{\phi'}(z) - \bar{\psi}(z).$$

Here G and λ are defined as follows

$$G = \frac{E}{2(1+\nu)} \qquad \lambda = \frac{3-\nu}{1+\nu}.$$

We take the following function as $\phi(z)$.

(4) $$\phi(z) = \sqrt{z(z-2iR)}.$$

Then it is easily verified that if we choose the function ψ as

$$\psi'(z) = \frac{iR^3 z}{(z-iR)(z(z-2iR))^{3/2}} + \frac{R^2(z-iR-R)}{(z-iR)^2\sqrt{z(z-2iR)}},$$

then the stresses determined by the above equations satisfy the boundary condition on the crack.

We cite below the coefficients(multiplied by $2G$) of several terms in the expansion of displacements. The calculation of these coefficients is also carried by the use of Mathematica.

$$A^{(\frac{1}{2})} = \frac{2\lambda-1}{2c_{\frac{1}{2}}}\sqrt{R}, \qquad B^{(\frac{1}{2})} = A^{(\frac{1}{2})}, \qquad C^{(\frac{1}{2})} = -\frac{1}{2}\sqrt{R}, \qquad D^{(\frac{1}{2})} = \frac{3}{2}\sqrt{R},$$

$$A^{(\frac{3}{2})} = \frac{2\lambda-3}{8c_{\frac{3}{2}}}R^{-\frac{1}{2}}, \qquad B^{(\frac{3}{2})} = -A^{(\frac{3}{2})}, \qquad C^{(\frac{3}{2})} = -\frac{7}{8}R^{-\frac{1}{2}}, \qquad D^{(\frac{3}{2})} = -\frac{5}{8}R^{-\frac{1}{2}},$$

$$A^{(\frac{5}{2})} = \frac{2\lambda-5}{64c_{\frac{5}{2}}}R^{-\frac{3}{2}}, \qquad B^{(\frac{5}{2})} = A^{(\frac{5}{2})}, \qquad C^{(\frac{5}{2})} = \frac{43}{64}R^{-\frac{3}{2}}, \qquad D^{(\frac{5}{2})} = -\frac{49}{64}R^{-\frac{3}{2}}.$$

It is clear that these coefficients satisfy all the equations in Propositions 1 and 3.

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