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Planar Cubic Spirals

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Abstract

Spiral segments have several advantages of containing neither inflection points, singularities nor curvature extrema. The object of this note is to obtain sufficient conditions for cubic spiral segments with help of *Mathematica*.

1 Introduction

Spiral segments have several advantages of containing neither inflection points, singularities nor curvature extrema. They are used to join (i) a straight line to a circle, (ii) two circles with a broken back C, (iii) two circles with an S, (iv) two non-parallel straight lines and also (v) two circles with one circle inside the other ([4]). In this paper, all spirals have zero curvature at their beginning points, nonnegative curvature at their ending points, and are parameterized on the interval $[0,1]$. Walton & Meek have obtained a cubic spiral condition by choosing the second control point of a cubic Bézier midway between the first and third control points when the quintic numerator of the derivative of the curvature reduces to a special quartic polynomial composed of even powers, i.e., they successfully relaxed the requirement of a curvature extremum at the ending fourth control point([4]). In addition, the spiral region has been given for the first or fourth control point. Farouki has explored generalization by moving the second control point along the line segment joining the first and third control points([1]).

The object of this paper is to expand their regions for the cubic spiral with help of *Mathematica* (A system for Mathematics by Computer). In Section 3, we examine a

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nonnegativity condition for a quintic numerator of the derivative of the curvature on $[0,1]$ which is to be reduced to $[0,\infty)$. In Section 4, we determine the position of the end vertex of a cubic spiral with the other one fixed .

2 Preliminary results

For a parametric cubic curve segment $\boldsymbol{z}(t) = (x(t), y(t)), 0 \le t \le 1$ with control vertices $\boldsymbol{P}_i, 0 \le i \le 3$:

$$\boldsymbol{z}(t) = \boldsymbol{P}_0(1-t)^3 + 3\boldsymbol{P}_1(1-t)^2 t + 3\boldsymbol{P}_2(1-t)t^2 + \boldsymbol{P}_3 t^3, \tag{2.1}$$

its curvature $\kappa(t)$ is given by

$$\kappa(t) = (\boldsymbol{z}' \times \boldsymbol{z}'')(t) / \left\|\boldsymbol{z}'(t)\right\|^3, \quad 0 \le t \le 1 \tag{2.2}$$

where $\|\bullet\|$ means the Euclidean norm. Since the cubic curve segment has eight parameters, we require it to satisfy the following six conditions:

$$\begin{gathered} (i)\ \boldsymbol{z}(0) = (0,0), \quad (ii)\ \boldsymbol{z}'(0) \parallel (1,0), \quad (iii)\ \kappa(0) = 0 \\ (iv)\ \kappa(1) = 1, \quad (v)\ \boldsymbol{z}'(1) \parallel (\cos\theta, \sin\theta) \end{gathered} \tag{2.3}$$

The above four conditions (i)-(iii) mean that the segment is tangent to the the positive part of the x-axis at the origin (the starting point $\boldsymbol{P}_0$ of the segment), and the remaining three ones (iv)-(v) mean that the segment meets the circle centered $(x(1)-\sin\theta, y(1)+\cos\theta)$(the terminal point $\boldsymbol{P}_3$ of the segment) with a radius 1. For analysis, we rewrite $\boldsymbol{z}(t)$ of the form:

$$\begin{aligned} x(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ y(t) &= b_0 + b_1 t + b_2 t^2 + b_3 t^3 \end{aligned} \tag{2.4}$$

with

$$\begin{aligned} a_0 = 0, a_3 &= \frac{-a_1 - 2a_2 + \sqrt{2(a_1+a_2)\sin\theta\cos\theta}}{3} \\ b_0 = 0, b_1 = 0, b_2 = 0, b_3 &= \frac{(a_1 + 2a_2 + 3a_3)\tan\theta}{3} \end{aligned} \tag{2.5}$$

Introduce two parameters p, d as

$$a_1 = p, a_2 = d^2/2 - p \tag{2.6}$$

to obtain

$$\begin{aligned} x(t) &= pt + (\frac{d^2}{2} - p)t^2 + \frac{p - d^2 + d\cos\theta\sqrt{\sin\theta}}{3}t^3 \\ y(t) &= \frac{d\sqrt{\sin^3\theta}}{3}t^3 \end{aligned} \tag{2.7}$$

Then, we have

$$\kappa(t) = \frac{dt(2p + d^2t - 2pt)\sqrt{\sin^3\theta}}{\{(p + d^2t - 2pt - d^2t^2 + pt^2 + dt^2\cos\theta\sqrt{\sin\theta})^2 + d^2t^4\sin^3\theta\}^{3/2}} \tag{2.8}$$

First, differentiate $\kappa(t)$ with resperct to t and next let $t = 1/(1+s), 0 \leq s \leq \infty$ to obtain

$$\kappa'(t) = \{d(1+s)^5 \sin\theta\sqrt{\sin\theta}\sum_{i=0}^{5} u_i s^i\}/\{\sum_{i=0}^{4} v_i s^i\}^{5/2} \tag{2.9}$$

with

$$\begin{aligned} u_5 &= 2p^3, u_4 = 10p^3, u_3 = dp(-3d^3 + 20dp - 8p\cos\theta\sqrt{\sin\theta}) && (2.10) \\ u_2 &= d(-d^5 + 13d^3p - 16d^2p\cos\theta\sqrt{\sin\theta} + 8p^2\cos\theta\sqrt{\sin\theta}) \\ u_1 &= d^2(3d^4 - 5d^3\cos\theta\sqrt{\sin\theta} + 8dp\cos\theta\sqrt{\sin\theta} - 10p\sin\theta) \\ u_0 &= d^2(3d^3\cos\theta - 4d^2\sqrt{\sin\theta} - 2p\sqrt{\sin\theta})\sqrt{\sin\theta}; \\ v_4 &= p^2, v_3 = 2d^2p, v_2 = d^4 + 2dp\cos\theta\sqrt{\sin\theta} && (2.11) \\ v_1 &= 2d^3\cos\theta\sqrt{\sin\theta}, v_0 = d^2\sin\theta \end{aligned}$$

Being difficult to obtain a necessary and sufficient condition for the above quintic numerator of (2.9) to have no zero $\in (0, 1)$, we consider a sufficient condition, i.e., $u_i \geq 0, 0 \leq i \leq 5$ which gives the following Lemma. Note that functions $f_i(d, \theta), i = 1, 2, 3$ are obtained by solving $u_i = 0$ for p.

Lemma 2.1 *The cubic segment $\boldsymbol{z}(t)(= (x(t), y(t))$ of (2.7) is a spiral if*

$$\text{Max}\{f_1(d,\theta), f_2(d,\theta), f_3(d,\theta)\} \leq p \leq f_0(d,\theta) \tag{2.12}$$

where

$$f_0(d,\theta) = \frac{3d^3\cos\theta - 4d^2\sqrt{\sin\theta}}{2\sqrt{\sin\theta}}$$
$$f_1(d,\theta) = \frac{-(3d^4 - 5d^3\cos\theta\sqrt{\sin\theta})}{2(4d\cos\theta\sqrt{\sin\theta} - 5\sin\theta)}$$
$$\tag{2.13}$$
$$f_2(d,\theta) = \frac{d^2(-13d + 16\cos\theta\sqrt{\sin\theta} + \sqrt{169d^2 - 384d\cos\theta\sqrt{\sin\theta} + 256\cos^2\theta\sin\theta}}{16\cos\theta\sqrt{\sin\theta}}$$
$$f_3(d,\theta) = \frac{3d^3}{4(5d^2 - 2\cos\theta\sqrt{\sin\theta})}$$

In order to determine $f_4(d,\theta)(= \mathrm{Max}\{f_1(d,\theta), f_2(d,\theta), f_3(d,\theta)\})$, let $d_0 = 4\sqrt{\sin\theta}/(3\cos\theta)$ and $d_3 = 2(11+3\sqrt{5})\cos\theta\sqrt{\sin\theta}/19$. Then, we have

Case 1: $\cos 2\theta \le (152 - 57\sqrt{5})/57$ ($\Leftrightarrow 0.562821... \le \theta \le \pi/2$)

$$f_4(d,\theta) = f_3(d,\theta), d \in [d_0, \infty) \tag{2.14}$$

Case 2: $(152 - 57\sqrt{5})/57 \le \cos 2\theta \le (51 + \sqrt{105})/96$ ($\Leftrightarrow 0.439456... \le \theta \le 0.562820...$)

$$f_4(d,\theta) = \begin{cases} f_2(d,\theta), d \in [d_0, d_3] \\ f_3(d,\theta), d \in [d_3, \infty) \end{cases} \tag{2.15}$$

Note that $f_4(d,u)$ intersects $f_3(d,u)$ or $f_2(d,u)$ for $(152 - 57\sqrt{5})/57 \le \cos 2\theta \le (28 - 11\sqrt{5})/6$ or $(28 - 11\sqrt{5})/6 \le \cos 2\theta \le (51 + \sqrt{105})/96$.

Case 3: $(51 + \sqrt{105})/96 \le \cos 2\theta$ ($\Leftrightarrow 0 \le \theta \le 0.439456...$)

$$f_4(d,\theta) = \begin{cases} f_1(d,\theta), d \in [d_0, d_2] \\ f_2(d,\theta), d \in [d_2, d_3] \\ f_3(d,\theta), d \in [d_3, \infty) \end{cases} \tag{2.16}$$

where d_2 is a unique poisitive root of a cubic equation:

$$\begin{aligned} &48d^3\cos\theta - 6d^2(23 + 10\cos 2\theta)\sqrt{\sin\theta} + (119\sin 2\theta + 11\sin 4\theta)d \\ &-20\sqrt{\sin\theta}(\sin\theta + 2\sin 3\theta) = 0 \end{aligned} \tag{2.17}$$

Note that $f_4(d,\theta)$ intersects $f_2(d,\theta)$ or $f_1(d,\theta)$ for $(51 + \sqrt{105})/96 \le \cos 2\theta \le 13/14$ or $13/14 \le \cos 2\theta$. Consequently, we have a (sufficient) spiral condition:

Theorem 2.1 *The cubic segment $\boldsymbol{z}(t)(= (x(t), y(t))$ of (2.7) is a spiral if*

$$\begin{array}{ll} (i) & \textit{for } \cos 2\theta \le (28 - 11\sqrt{5})/6,\ f_3(d,\theta) \le p \le f_0(d,\theta)\ (d \ge d_4) \\ (ii) & \textit{for } (28 - 11\sqrt{5})/6 \le \cos 2\theta \le 13/14,\ f_2(d,\theta) \le p \le f_0(d,\theta)\ (d \ge d_5) \\ (iii) & \textit{for } 13/14 \le \cos 2\theta,\ f_1(d,\theta) \le p \le f_0(d,\theta)\ (d \ge d_6) \end{array} \tag{2.18}$$

Note that $f_i(d,\theta), i=3,2,1$ *intersect* $f_0(d,\theta)$ *at* $d=d_{7-i}$ *where*

$$
\begin{aligned}
d_4 &= \frac{(49+6\cos 2\theta)\sqrt{\sin\theta}+\sqrt{1645\sin\theta-195\sin 3\theta+9\sin 5\theta}}{60\cos\theta} \\
d_5 &= \frac{9(7+4\cos 2\theta)\sqrt{\sin\theta}+\sqrt{3(175\sin\theta-56\sin 3\theta+12\sin 5\theta)}}{3(22\cos\theta+3\cos 3\theta)} \\
d_6 &= \frac{18\cos\theta\sqrt{\sin\theta}+\sqrt{3(-13\sin\theta+7\sin 3\theta)}}{9+6\cos 2\theta}
\end{aligned}
\tag{2.19}
$$

Here we remark that the sufficient condition $a_2=0$ proposed by Walton & Meek [3] is equivalent to

$$p = d^2/2, \quad p \le f_0(d,\theta) \tag{2.20}$$

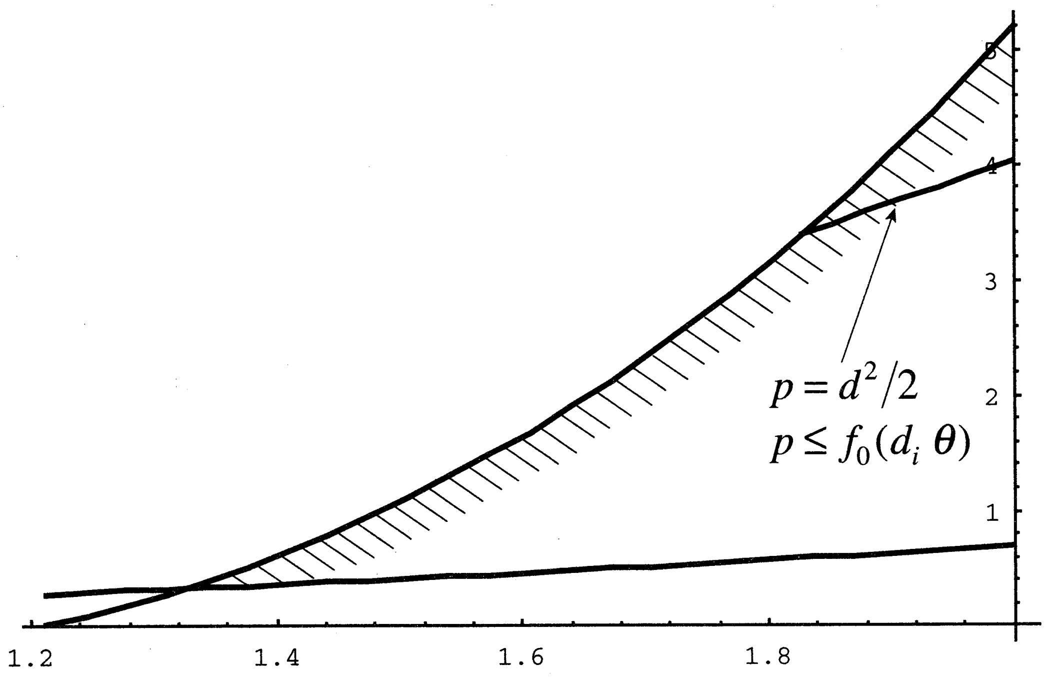

Figure 1. Spiral condition on p, d for $\theta = 0.6$

3 Main results

The above spiral condition (2.12) determines a spiral region for $\boldsymbol{P}_3$ (the end control point of the Bézier curve(2.1)).

For $(m,n)=\boldsymbol{P}_3$, we have

$$d=\frac{3n}{\sin\theta\sqrt{\sin\theta}}, \quad p=\frac{3(2m\sin^3\theta-2n\sin^2\theta\cos\theta-3n^2)}{2\sin^3\theta} \tag{3.1}$$

Sufficient condition 1 (Walton & Meek [4]):
Equation (2.20) gives

$$m = \frac{n\cos\theta}{\sin\theta} + \frac{3n^2}{\sin^3\theta}, \quad n \geq \frac{5\sin^2\theta}{9\cos\theta} \tag{3.2}$$

In Figure 2, the parametrized boundary of the above region is given by

$$m = \frac{5\sin\theta(2\cos^2\theta + 5)}{27\cos^2\theta}, \quad n = \frac{5\sin^2\theta}{9\cos\theta} \tag{3.3}$$

where its implicit form is given by

$$6561m^4 - 2187n^2(32 + 27n^2)m^2 - n^2(243n^2 + 320)^2 = 0 \tag{3.4}$$

Theorem 2.1 gives a larger spiral region:
Sufficient condition 2:
First, Theorem 2.1 requires $f_0(d,p) \geq p$ which is equivalent to

$$27n^3\cos\theta - 9n^2\sin^2\theta + 2n\cos\theta\sin^4\theta \geq 2m\sin^5\theta \tag{3.5}$$

(i) for $\cos 2\theta \leq (28 - 11\sqrt{5})/6$; in addition to (3.5)

$$m \geq \frac{n(117n^2 + 48n\cos\theta\sin^2\theta - 8\cos^2\theta\sin^4\theta)}{4(15n - 2\cos\theta\sin^2\theta)\sin^3\theta} \tag{3.6}$$

(ii) for $(28 - 11\sqrt{5})/6 \leq \cos 2\theta \leq 13/14$; in addition to (3.5)

$$16m\sin^5\theta\cos\theta \geq 3n^2\sqrt{1521n^2 - 1152n\sin^2\theta\cos\theta + 256\sin^4\theta\cos^2\theta} \tag{3.7}$$
$$-(117n^3 - 72n^2\sin^2\theta\cos\theta - 16n\cos^2\theta\sin^4\theta)$$

(iii) for $13/14 \leq \cos 2\theta$; in addition to (3.5)

$$m \geq \frac{-n\{81n^3 - 81n^2\cos\theta\sin^2\theta - 3n(8\cos^2\theta - 5)\sin^4\theta + 10\cos\theta\sin^6\theta\}}{2(12n\cos\theta - 5\sin^2\theta)\sin^5\theta} \tag{3.8}$$

The above result gives the spiral region for the fourth control points P_3 with P_0 fixed.

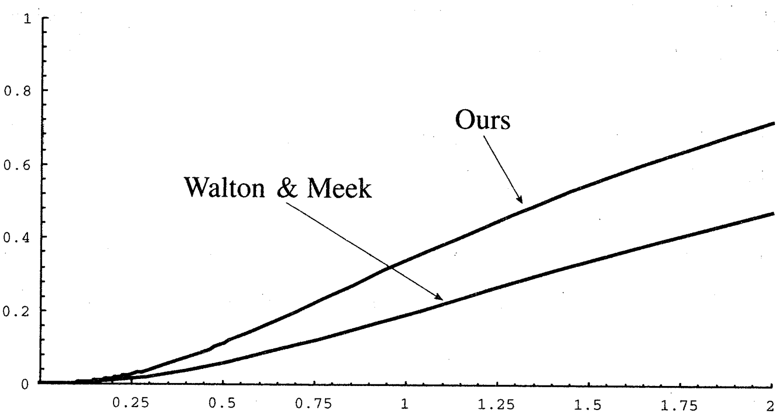

Figure 2. Spiral region for the fourth control point P_3 with P_0 fixed

A shift of $\boldsymbol{P}_3$ to the origin and $\boldsymbol{P}_2$ on the positive part of the x-axis gives

$$\boldsymbol{P}_0 = (x(1)\cos\theta + y(1)\sin\theta, x(1)\sin\theta - y(1)\cos\theta) \tag{3.9}$$

A combination of (3.2)-(3.9) yields the spiral region for the first control point P_0 with P_3 fixed.

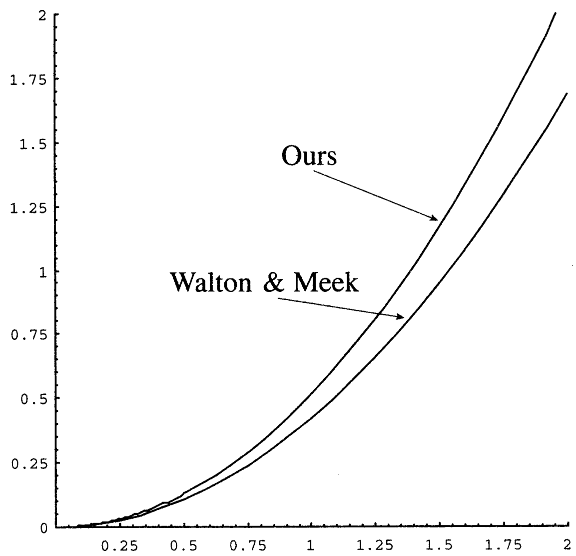

Figure 3. Spiral region for the fourth control point $\boldsymbol{P}_0$ with $\boldsymbol{P}_3$ fixed

References

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