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1145 巻 2000 年 211-219



# A Domain Decomposition Method for a Discretized Stokes Problem by P1/P1 Finite Element

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**abstract**

A parallel computation algorithm is studied for the Stokes problem subject to the slip boundary conditions in a spherical shell. Considering the computational amount of 3D problem, a P1/P1 finite element method with a GLS stabilizing technique is employed. A parallel iterative solver is presented with domain decomposition technique by congruent subdomains. This algorithm can drastically reduce required memory to store stiffness and mass matrices. Parallel efficiency of the iterative solver is reported on a shared memory-type parallel computer.

## 1 Introduction

We consider a parallel solver of a discretized Stokes problem by a finite element method, subject to the slip boundary conditions in a spherical shell. This concerns with an unsteady problem of Earth's mantle convection, where the Stokes problem and the convection-diffusion problem are solved repeatedly [4, 6]. Therefore a fast solver of the Stokes equations, which is suitable for parallel computation, is required. Here we present a domain decomposition algorithm by congruent subdomains. We also describe a method to treat the slip boundary conditions by projection operations. We omit proofs of Theorems, which are presented in [5].

## 2 The Stokes equations in a spherical shell

Let $\Omega$ be a spherical shell domain :

$$\Omega := \{x \in \mathbb{R}^3 \ ; \ R_1 < |x| < R_2\},$$

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where $|x|$ is the Euclidean norm of $x$, and $0 < R_1 < R_2$. Our purpose is to solve the Stokes equations : to find the velocity $u = (u_1, u_2, u_3)$ and the pressure $p$ satisfying in $\Omega$,

$$-\Delta u + \nabla p = f\,, \tag{1a}$$

$$\nabla \cdot u = 0\,, \tag{1b}$$

subject to the boundary conditions on $\Gamma(:= \partial\Omega)$,

$$u \cdot n = 0, \tag{2a}$$

$$t^{(k)} \cdot D(u)\,n = 0 \quad (k = 1, 2)\,. \tag{2b}$$

Here $D$ is the strain rate tensor defined by $D_{ij}(u) := \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$ $(1 \le i, j \le 3)$, $n$ and $t^{(k)}$ $(k = 1, 2)$ are unit outer normal and tangent vectors to the boundary, and $f = (f_1, f_2, f_3)$ is an external force. $f$ is assumed to be orthogonal to rigid body rotations :

$$\int_\Omega f \cdot v^{(i)}\, dx = 0 \quad (i = 1, 2, 3), \qquad v^{(i)}(x) := e^{(i)} \times x \quad (i = 1, 2, 3), \tag{3}$$

where $e^{(i)}$ is the unit vector to the $x_i$-direction. For the unique solvability we impose constraints to the velocity function space to eliminate the freedoms of rigid body rotations[6].

## 3 P1/P1 finite element approximation

Considering the cost of computation in 3D problem, we employ a cheap P1/P1 element, that is, both velocity and pressure are approximated by the piecewise linear elements in tetrahedra. The Galerkin least square (GLS) method [1, 2] is used for stabilizing the Stokes equations. Let $\Omega_h$ be a polyhedral approximation to $\Omega$, $\Gamma_h$ be the boundary of $\Omega_h$, and $\mathcal{T}_h$ be a partition of $\bar{\Omega}_h$ into tetrahedra, where $h$ is the maximum diameter of tetrahedral elements. Let $S_h(\Omega_h) \subset H^1(\Omega_h) \cap C^0(\bar{\Omega}_h)$ be the P1 finite element space whose degrees of freedom are on the vertices of tetrahedra. We introduce finite element spaces $X_h$, $V_h$, $M_h$, and $Q_h$,

$$\begin{aligned}
X_h &:= S_h(\Omega_h)^3, \\
V_h &:= \left\{v_h \in X_h\ ;\ (v_h \cdot n_\Omega)(P) = 0\ (\forall P)\,,\ (v_h,\, v^{(i)})_h = 0\ (i = 1, 2, 3)\right\}, \\
M_h &:= S_h(\Omega_h), \\
Q_h &:= \{q_h \in M_h\ ;\ (q_h,\, 1)_h = 0\}\,,
\end{aligned}$$

where $P$ stands for nodal point on $\Gamma_h$, and $n_\Omega$ is the unit outer normal to $\Gamma$. We use the same notation $(\cdot, \cdot)_h$ to represent the $L^2$-inner products in $X_h$ and $M_h$. We prepare the following bilinear forms for $u, v \in X_h$, and $p, q \in M_h$,

$$\begin{aligned}
a_h(u, v) &:= 2\int_{\Omega_h} D(u) : D(v)\, dx\,, \\
b_h(v, q) &:= -(\nabla \cdot v,\, q)_h\,, \\
d_h(p, q) &:= \textstyle\sum_{K \in \mathcal{T}_h} h_K^2 (\nabla p,\, \nabla q)_K\,,
\end{aligned}$$

where $(\cdot,\cdot)_K$ is the $L^2$-inner product on element $K$, and $D(u):D(v) := \sum_{1\le i,j\le 3} D_{ij}(u)D_{ij}(v)$.

A finite element approximation to the Stokes problem (1) and (2) by a stabilized technique [1, 2] is to find $(u_h, p_h) \in V_h \times Q_h$ satisfying

$$a_h(u_h, v_h) + b_h(v_h, p_h) = (f,\, v_h)_h\,, \tag{4a}$$

$$b_h(u_h, q_h) - \delta d_h(p_h, q_h) = -\delta \textstyle\sum_{K\in\mathcal{T}_h} h_K^2 (f,\, \nabla q_h)_K\,, \tag{4b}$$

for any $(v_h, q_h) \in V_h \times Q_h$. Here a positive constant $\delta$ is a stability parameter.

**Theorem 1** [6] *For every $f \in L^2(\Omega)^3$ there exists a unique solution of* (4).

## 4 Matrix formulation of the finite element scheme

Let $n_X$ and $n_M$ be dimensions of the spaces $X_h$ and $M_h$, respectively. We define index sets, $\Lambda_X := \{1, 2, \ldots, n_X\}$ and $\Lambda_M := \{1, 2, \ldots, n_M\}$. Let $\{\varphi_\alpha\}_{\alpha\in\Lambda_X}$ and $\{\psi_\mu\}_{\mu\in\Lambda_M}$ be finite element bases of $X_h$ and $M_h$, respectively,

$$X_h = \mathrm{span}[\varphi_1, \ldots, \varphi_{n_X}]\,,$$
$$M_h = \mathrm{span}[\psi_1, \ldots, \psi_{n_M}]\,.$$

Let $n_G$ be the number of vertices of tetrahedra in $\bar{\Omega}_h$. We denote the vertices ( nodal points ) by $P_j$, $j \in \Lambda_G := \{1, 2, \ldots, n_G\}$. Then, $n_X = 3\,n_G$ and $n_M = n_G$. We associate a pair $[\beta_0, \beta_1]$ of the nodal point number and the component number with index $\beta \in \Lambda_X$ and identify them,

$$\beta = [\beta_0, \beta_1] \qquad (\beta_0 \in \Lambda_G,\ \beta_1 \in \{1, 2, 3\})\,.$$

The finite element bases satisfy

$$[\varphi_\alpha(P_{\beta_0})]_{\beta_1} = \delta_{\alpha\beta} \qquad (\alpha,\, \beta = [\beta_0, \beta_1] \in \Lambda_X)\,,$$
$$\psi_\mu(P_\nu) = \delta_{\mu\nu} \qquad (\mu,\, \nu \in \Lambda_M)\,,$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. We define stiffness matrices, mass matrices and right-hand side vectors as follows,

$$\begin{aligned}
[A]_{\alpha\beta} &:= a_h(\varphi_\beta\,,\, \varphi_\alpha) && (\alpha,\, \beta \in \Lambda_X)\,,\\
[B]_{\mu\beta} &:= b_h(\varphi_\beta\,,\, \psi_\mu) && (\mu \in \Lambda_M,\ \beta \in \Lambda_X)\,,\\
[D]_{\mu\nu} &:= d_h(\psi_\nu\,,\, \psi_\mu) && (\mu,\, \nu \in \Lambda_M),\\
[M_X]_{\alpha\beta} &:= (\varphi_\beta\,,\, \varphi_\alpha)_h && (\alpha,\, \beta \in \Lambda_X),\\
[M_M]_{\mu\nu} &:= (\psi_\nu\,,\, \psi_\mu)_h && (\mu,\, \nu \in \Lambda_M),\\
[\vec{f}]_\alpha &:= (f\,,\, \varphi_\alpha)_h && (\alpha \in \Lambda_X),\\
[\vec{g}]_\mu &:= \textstyle\sum_{K\in\mathcal{T}_h} h_K^2 (f\,,\, \nabla\psi_\mu)_K && (\mu \in \Lambda_M)\,.
\end{aligned}$$

Let $\Lambda_\Gamma \subset \Lambda_G$ be an index set of every $j$ such that nodal point $P_j$ is on $\Gamma_h$. We define vectors $\{\vec{n}^{(j)}\}_{j\in\Lambda_\Gamma} \subset \mathbb{R}^{n_X}$ corresponding to the unit outer normals to the boundary, and $\{\vec{v}^{(j)}\}_{1\le j\le 3} \subset \mathbb{R}^{n_X}$ corresponding to the rigid body rotations by

$$[\vec{n}^{(j)}]_\alpha := \delta_{j\,\alpha_0} n_{\Omega\,\alpha_1}(P_j) \qquad (j \in \Lambda_\Gamma\,,\ \alpha = [\alpha_0, \alpha_1] \in \Lambda_X)\,,$$
$$[\vec{v}^{(j)}]_\alpha := v^{(j)}_{\alpha_1}(P_{\alpha_0}) \qquad (1 \le j \le 3\,, \alpha = [\alpha_0, \alpha_1] \in \Lambda_X)\,.$$

Then we have

$$\textstyle\sum_{\alpha\in\Lambda_X} [\vec{n}^{(j)}]_\alpha \varphi_\alpha(P_k) = \delta_{j\,k}\, n_\Omega(P_j) \qquad (j \in \Lambda_\Gamma,\ k \in \Lambda_G),$$
$$\textstyle\sum_{\alpha\in\Lambda_X} [\vec{v}^{(j)}]_\alpha \varphi_\alpha(x) = v^{(j)}(x) \qquad (1 \le j \le 3,\ x \in \bar{\Omega}_h).$$

We denote by $(\cdot,\cdot)$ the inner product in $\mathbb{R}^m$ for $m = n_X$ or $m = n_M$.

**Remark 1** $\{\vec{n}^{(i)}\}_{i\in\Lambda_\Gamma}$ are orthonormal, $(\vec{n}^{(i)}, \vec{n}^{(j)}) = \delta_{ij}$ $(i, j \in \Lambda_\Gamma)$.

We introduce the following spaces,

$$\begin{aligned} \vec{X} &:= \mathbb{R}^{n_X}\,,\\ \vec{V} &:= \{\vec{v} \in \vec{X}\ ;\ (\vec{v}, \vec{n}^{(i)}) = 0\ (i \in \Lambda_\Gamma),\ (M_X\vec{v}, \vec{v}^{(j)}) = 0\ (1 \le j \le 3)\}\,,\\ \vec{M} &:= \mathbb{R}^{n_M}\,,\\ \vec{Q} &:= \{\vec{q} \in \vec{M}\ ;\ (M_M\vec{q}, \vec{1}) = 0\}\,, \end{aligned}$$

where $\vec{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^{n_M}$. We define the following orthogonal projections,

$$P_{\vec{V}}\ :\ \vec{X} \to \vec{V}, \qquad (P_{\vec{V}}\vec{u}, \vec{v}) = (\vec{u}, \vec{v}) \qquad (\forall \vec{v} \in \vec{V})\,,$$
$$P_{\vec{Q}}\ :\ \vec{M} \to \vec{Q}, \qquad (P_{\vec{Q}}\vec{p}, \vec{q}) = (\vec{p}, \vec{q}) \qquad (\forall \vec{q} \in \vec{Q})\,.$$

The problem (4) is equivalent to the following: to find $(\vec{u}, \vec{p}) \in \vec{V} \times \vec{Q}$ satisfying

$$(A\vec{u} + B^T\vec{p},\ \vec{v}) = (\vec{f},\ \vec{v})\,,$$
$$(B\vec{u} - \delta D\vec{p},\ \vec{q}) = (-\delta\vec{g},\ \vec{q})\,,$$

for any $(\vec{v}, \vec{q}) \in \vec{V} \times \vec{Q}$. The matrix formulation of (4) is to find $\vec{u} \in \vec{V}$ and $\vec{p} \in \vec{Q}$ satisfying

$$\mathcal{PA}\begin{pmatrix}\vec{u}\\ \vec{p}\end{pmatrix} = \mathcal{P}\begin{pmatrix}\vec{f}\\ -\delta\vec{g}\end{pmatrix}\,, \tag{5}$$

where $\mathcal{P}$ and $\mathcal{A}$ are $(n_X + n_M) \times (n_X + n_M)$ matrices defined by

$$\mathcal{P} = \begin{pmatrix}P_{\vec{V}} & 0\\ 0 & P_{\vec{Q}}\end{pmatrix}\,, \quad \mathcal{A} = \begin{pmatrix}A & B^T\\ B & -\delta D\end{pmatrix}\,.$$

Theorem 1 implies

**Remark 2** $\mathcal{PA}$ is an isomorphism on $\vec{V} \times \vec{Q}$.

We solve the equation (5) by Krylov subspace method in $\vec{V} \times \vec{Q}$. Krylov subspaces are generated by multiplications of $\mathcal{PA}$. We use a preconditioned conjugate gradient method for the solution of (5).

**Remark 3** Let $\{\vec{w}^{(j)}\}_{1\le j\le 3}$ be an orthonormal set generated from $\{M_X\vec{v}^{(i)}\}_{1\le i\le 3}$ such that $(\vec{n}^{(i)},\, \vec{w}^{(j)}) = 0$ for any $i$ and $j$. Then, we have

$$P_{\vec{V}}\vec{v} = \vec{v} - \textstyle\sum_{i\in\Lambda_\Gamma}(\vec{v},\, \vec{n}^{(i)})\vec{n}^{(i)} - \sum_{1\le j\le 3}(\vec{v},\, \vec{w}^{(j)})\vec{w}^{(j)}\,.$$

We note that the operations concerning to $\{\vec{n}^{(i)}\}_{i\in\Gamma_\Lambda}$ are local, which are suitable for parallel computation.

# 5 Domain decomposition into congruent subdomains

We decompose the domain $\Omega_h$ into a union of non-overlapping $p$ subdomains,

$$\bar{\Omega}_h = \textstyle\bigcup_{0\le i<p}\bar{\Omega}_h^{(i)}, \quad \Omega_h^{(i)} \cap \Omega_h^{(j)} = \emptyset \quad (0 \le i < j < p)\,.$$

The interface introduced by this decomposition is denoted by $\mathcal{F} := \bigcup_{0\le i<j<p}\partial\Omega_h^{(i)} \cap \partial\Omega_h^{(j)}$. We call $\Omega_h^{(0)}$ a reference subdomain. We assume that all subdomains are obtained from the reference subdomain by transformations $\{R^{(i)}\}_{1\le i<p} \subset \mathbb{R}^{3\times 3}$, whose components consist of $-1$, 0, and 1,

$$\Omega_h^{(i)} = R^{(i)}\Omega_h^{(0)} \quad (1 \le i < p)\,.$$

We assume that nodal points in $\bar{\Omega}_h^{(0)}$ are numbered by an index set $\Lambda_G^{(0)} := \{1, 2, \dots, n_G^{(0)}\} \subset \Lambda_G$. Let $\Lambda_G^{(i)} \subset \Lambda_G$ ($\#\Lambda_G^{(i)} = n_G^{(0)}$) be an index set corresponding to nodal points in $\bar{\Omega}_h^{(i)}$ for $i = 1, \dots, p-1$. Let $\mathcal{T}_h^{(0)}$ be a partition of $\bar{\Omega}_h^{(0)}$, which produces the partition $\mathcal{T}_h^{(i)}$ of $\bar{\Omega}_h^{(i)}$ by

$$\mathcal{T}_h^{(i)} := \{K' \in \mathcal{T}_h\ ;\ K' = R^{(i)}K,\ \forall K \in \mathcal{T}_h^{(0)}\} \quad (1 \le i < p)\,.$$

We assume that the union of $\mathcal{T}_h^{(i)}$,

$$\mathcal{T}_h = \textstyle\bigcup_{0\le i<p}\mathcal{T}_h^{(i)}\,,$$

is a partition of the whole domain $\bar{\Omega}_h$.

**Lemma 1** *Available subdomain numbers by transformations described above are* 2, 3, 4, 6, 8, 12, 16, 24, *and* 48.

Figure 1 shows a domain decomposition when $p = 24$ and P1-mesh subdivision, where a 24th part of the spherical shell domain is cut off and every subdomain is shifted slightly to show the decomposition better.

### 5.1 Transformation of scalar valued FEM basis

We set $\Lambda_M^{(i)} := \Lambda_G^{(i)}$ for $i = 0, \dots, p-1$ and $n_M^{(0)} := n_G^{(0)}$, and define a bijection $r_M^{(i)}$ from $\Lambda_M^{(0)}$ onto $\Lambda_M^{(i)}$ by

$$r_M^{(i)}(\mu) = \nu \qquad (\mu \in \Lambda_M^{(0)},\ \nu \in \Lambda_M^{(i)}),$$

where $P_\nu = R^{(i)} P_\mu$ for $i = 1, \dots, p-1$.

**Lemma 2** *We have* $\psi_\nu(x) = \psi_\mu({R^{(i)}}^{-1} x)$, *where* $\mu = {r_M^{(i)}}^{-1}(\nu)$ *for* $\nu \in \Lambda_M^{(i)}$, $x \in \bar{\Omega}_h^{(i)}$, *and* $i = 1, \dots, p-1$.

Let $D^{(i)}$ and $M_M^{(i)}$ be sub-stiffness and sub-mass matrices defined by

$$[D^{(i)}]_{\mu\nu} := \textstyle\sum_{K \in \mathcal{T}_h^{(i)}} h_K^2 (\nabla\psi_\nu, \nabla\psi_\mu)_K \qquad (\mu, \nu \in \Lambda_M^{(i)},\ 0 \le i < p), \tag{6}$$

$$[M_M^{(i)}]_{\mu\nu} := \int_{\Omega_h^{(i)}} \psi_\nu\, \psi_\mu \, dx \qquad (\mu, \nu \in \Lambda_M^{(i)},\ 0 \le i < p). \tag{7}$$

**Remark 4** We note that the sizes of $D^{(i)}$ and $M_M^{(i)}$ are $n_M \times n_M$. Equations (6) and (7) define only $n_M^{(0)} \times n_M^{(0)}$ entries of these matrices. We may set all the other entries to be zero, which have no contribution in the following. The effective sizes of the matrices are $n_M^{(0)} \times n_M^{(0)}$. In advance of the subsection 5.2 we give similar remarks for $\vec{s}^{(i)}$, $A^{(i)}$, $B^{(i)}$, and $M_X^{(i)}$, whose sizes are $n_X \times 1$, $n_X \times n_X$, $n_M \times n_X$, and $n_X \times n_X$, respectively.

**Theorem 2** *It holds that, for* $i = 1, \dots, p-1$,

$$[D^{(i)}]_{\mu'\nu'} = [D^{(0)}]_{\mu\nu} \qquad (\mu, \nu \in \Lambda_M^{(0)}),$$

$$[M_M^{(i)}]_{\mu'\nu'} = [M_M^{(0)}]_{\mu\nu} \qquad (\mu, \nu \in \Lambda_M^{(0)}),$$

*where* $\mu' = r_M^{(i)}(\mu)$ *and* $\nu' = r_M^{(i)}(\nu)$.

### 5.2 Transformation of vector valued FEM basis

We assume that the association of $[\alpha_0, \alpha_1]$ with $\alpha \in \Lambda_X$ satisfies

$$\alpha_0 = [(\alpha - 1)/3] + 1, \qquad \alpha_1 = ((\alpha - 1) \bmod 3) + 1,$$

where $[\,\cdot\,]$ denotes the greatest integer less than or equal to the argument. We define

$$\Lambda_X^{(i)} := \{\alpha \in \Lambda_X\ ;\ \alpha = [\alpha_0, \alpha_1],\ \alpha_0 \in \Lambda_G^{(i)}\} \quad (0 \le i < p).$$

We note that $\Lambda_X^{(0)} = \{1, 2, \dots, 3n_G^{(0)}\}$. We define a bijection $r_X^{(i)}$ from $\Lambda_X^{(0)}$ onto $\Lambda_X^{(i)}$ by

$$r_X^{(i)}(\alpha) = \beta \qquad (\alpha = [\alpha_0, \alpha_1] \in \Lambda_X^{(0)},\ \beta = [\beta_0, \beta_1] \in \Lambda_X^{(i)}),$$

where $\beta_0 = r_M^{(i)}(\alpha_0)$ and $R_{\beta_1 \alpha_1}^{(i)} \ne 0$ for $i = 1, \dots, p-1$. We define sign vectors $\{\vec{s}^{(i)}\}_{1 \le i < p} \subset \mathbb{R}^{n_X}$ corresponding to the transformations $R^{(i)}$ by

$$[\vec{s}^{(i)}]_\beta := \textstyle\sum_{1 \le l \le 3} R_{\beta_1 l}^{(i)} \quad (\beta = [\beta_0, \beta_1] \in \Lambda_X^{(i)},\ 1 \le i < p).$$

We note that $[\vec{s}^{(i)}]_\beta = 1$ or $-1$ for $\beta \in \Lambda_X^{(i)}$.

**Lemma 3** *We have* $\varphi_\beta(x) = [\vec{s}^{(i)}]_\beta R^{(i)} \varphi_\alpha({R^{(i)}}^{-1}x)$, *where* $\alpha = {r_X^{(i)}}^{-1}(\beta)$ *for* $\beta \in \Lambda_X^{(i)}$, $x \in \bar{\Omega}_h^{(i)}$, *and* $i = 1, \dots, p-1$.

Let $A^{(i)}$, $B^{(i)}$, and $M_X^{(i)}$ be sub-stiffness and sub-mass matrices defined by

$$[A^{(i)}]_{\alpha\beta} := 2\int_{\Omega_h^{(i)}} D(\varphi_\beta) : D(\varphi_\alpha)\, dx \qquad (\alpha, \beta \in \Lambda_X^{(i)},\ 0 \le i < p), \tag{8}$$

$$[B^{(i)}]_{\gamma\beta} := -\int_{\Omega_h^{(i)}} \nabla \cdot \varphi_\beta \psi_\gamma\, dx \qquad (\gamma \in \Lambda_M^{(i)},\ \beta \in \Lambda_X^{(i)},\ 0 \le i < p), \tag{9}$$

$$[M_X^{(i)}]_{\alpha\beta} := \int_{\Omega_h^{(i)}} \varphi_\beta \cdot \varphi_\alpha\, dx \qquad (\alpha, \beta \in \Lambda_X^{(i)},\ 0 \le i < p). \tag{10}$$

**Theorem 3** *It holds that, for* $i = 1, \dots, p-1$,

$$\begin{aligned}
[A^{(i)}]_{\alpha'\beta'} &= [\vec{s}^{(i)}]_{\alpha'}\, [A^{(0)}]_{\alpha\beta}\, [\vec{s}^{(i)}]_{\beta'} && (\alpha,\ \beta \in \Lambda_X^{(0)}), \\
[B^{(i)}]_{\mu'\beta'} &= [B^{(0)}]_{\mu\beta}\, [\vec{s}^{(i)}]_{\beta'} && (\mu \in \Lambda_M^{(0)},\ \beta \in \Lambda_X^{(0)}), \\
[M_X^{(i)}]_{\alpha'\beta'} &= [M_X^{(0)}]_{\alpha\beta} && (\alpha,\ \beta \in \Lambda_X^{(0)}),
\end{aligned}$$

*where* $\alpha' = r_X^{(i)}(\alpha)$, $\beta' = r_X^{(i)}(\beta)$, *and* $\mu' = r_M^{(i)}(\mu)$.

**Remark 5** By virtue of Theorems 2 and 3 we do not need to store the total stiffness and mass matrices in the whole domain. It is sufficient to construct and store these matrices only in the reference subdomain $\bar{\Omega}_h^{(0)}$, which reduces required memory drastically.

## 6 Numerical result

We employ a preconditioned conjugate gradient (CG) method with projections. The preconditioner is a combination of an incomplete Cholesky decomposition of the discretized Stokes matrix corresponding to the nodes in $\bar{\Omega}_h \setminus \mathcal{F}$, and a diagonal scaling of the matrix corresponding to the nodes on the interface $\mathcal{F}$. The preconditioning operation of the former can be performed in parallel completely by $p$ processors (cf. Remark 5), and that of the latter can be also done by a suitable decomposition of the nodes on the interface $\mathcal{F}$.

We observe the efficiency of memory reduction and parallel computation by the decomposition into congruent subdomains. In a test problem, the solution is given by

$$\begin{aligned}
u_1 &= \sin x_1 - x_1 \cos x_2, \\
u_2 &= 2(\sin x_2 - x_2 \cos x_3), \\
u_3 &= 2 \sin x_3 - x_3(\cos x_2 + \cos x_1), \\
p &= \sin x_1 + \sin x_2 + \sin x_3 .
\end{aligned}$$

We impose inhomogeneous boundary conditions instead of (2), and seek the velocity in an affine space of $V_h$ subject to the inhomogeneous normal component. The radii of the spherical shell are set to be $R_1 = 0.5$ and $R_2 = 1.0$. Discretization parameters are listed in Table 1.

The stability parameter $\delta$ is set to be 0.1. The convergence criterion of the CG solver is set to be $10^{-6}$ in the relative residual. When the subdomain number $p = 24$, 373 iterations were done to get a solution, whose relative errors were $||u - u_h||_{H^1(\Omega_h)^3}/||u||_{H^1(\Omega_h)^3} = 2.826 \times 10^{-2}$ and $||p - p_h||_{L^2(\Omega_h)}/||p||_{L^2(\Omega_h)} = 1.933 \times 10^{-3}$. Table 2 shows total required memory storage of a program code. Although some arrays to store data of nodal points and some work vectors for the CG solver are invariant, this result shows the algorithm can reduce memory drastically using many subdomains. Table 3 shows the elapsed time of CG iterations, which does not include the time to construct the preconditioner, and the parallel efficiency when the subdomain number $p = 24$. We used Fujitsu GP7000F, parallel computer of shared memory type at the Computer Center of Kyushu University, and a thread library developed by RWCP OpenMP compiler project[3].

## 7 Conclusion

A technique of domain decomposition into a union of congruent subdomains can drastically reduce required memory to store stiffness and mass matrices. In combination with projection operations to treat the slip boundary conditions, a preconditioned conjugate gradient solver is easily implemented using the domain decomposition. It has high parallel efficiency on a shared memory-type parallel computer.

Acknowledgments

This study was supported by the Ministry of Education, Science, Sports and Culture of Japan under Grant-in-Aid for Scientific Research (B), No. 11554003.

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Table 1: Discretization parameters ($n_G$ and $n_E$ : numbers of vertices and elements, $h$ : maximum element diameter, $n_X$ and $n_M$ : degree of freedoms of velocity and pressure).

| $n_G$ | $n_E$ | $h$ | $n_X$ | $n_M$ |
|---|---|---|---|---|
| 324,532 | 1,868,544 | $5.558 \times 10^{-2}$ | 973,596 | 324,532 |

Table 2: Required memory ($p$ : number of subdomains).

| $p$ | M bytes |
|---|---|
| 1 | 1,773.8 |
| 8 | 513.5 |
| 24 | 456.7 |

Table 3: Elapsed time and parallel efficiency when $p = 24$ ($n$ : number of CPUs, $r$ : speed-up ratio, $e$ : parallel efficiency).

| $n$ | seconds | $r$ | $e$ |
|---|---|---|---|
| 1 | 5,961.89 | 1.00 | — |
| 2 | 3,096.23 | 1.93 | 96.28 |
| 4 | 1,561.35 | 3.82 | 95.46 |
| 8 | 789.76 | 7.55 | 94.36 |
| 24 | 352.22 | 16.93 | 70.53 |

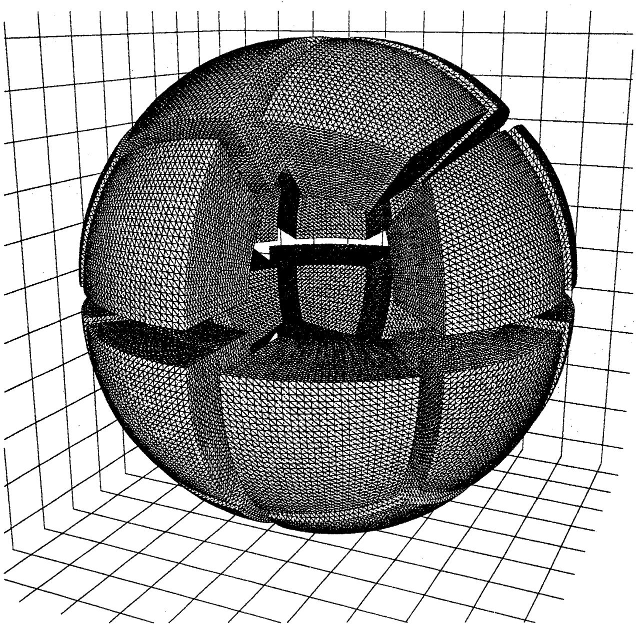

Figure 1: Domain decomposition of a spherical shell into 24 subdomains.