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Unified Variational Approach for Numerical Solutions of the Laplace Equation

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1 Introduction

Let $\Omega \subset \boldsymbol{R}^2$ be a domain bounded by a smooth curve Γ with the Cartesian coordinates $\boldsymbol{x} = (x_1, x_2)$. We consider the Laplace equation

$$-\Delta u(\boldsymbol{x}) = 0 \qquad \text{in } \Omega. \tag{1}$$

Let $\boldsymbol{n}$ denote the outward unit normal to the boundary Γ, and let $q = \dfrac{\partial u}{\partial n}$ on the boundary. We assume that the Dirichlet data $\overline{u}$ are prescribed on a non-zero measure part Γ_u of the boundary Γ, and the Neumann data $\overline{q}$ are prescribed on a non-zero measure part Γ_q of the boundary Γ. We note here that Γ_u and Γ_q are taken arbitrarily. In this case, the problems can be classified into four groups according to the bounday conditions:

1. $\Gamma_u \cap \Gamma_q \neq \phi$ and $\Gamma_u = \Gamma_q$ —— Cauchy problem;
2. $\Gamma_u \cap \Gamma_q \neq \phi$ and $\Gamma_u \neq \Gamma_q$ —— Over-determined problem;
3. $\Gamma_u \cap \Gamma_q = \phi$ and $\Gamma_u \cup \Gamma_q = \Gamma$ —— Boundary value problem;
4. $\Gamma_u \cap \Gamma_q = \phi$ and $\Gamma_u \cup \Gamma_q \neq \Gamma$ —— Under-determined problem.

As to these four problems, we call them *boundary inverse problems*, because they essentially fall into identification of suitable boundary values. Amaya et al.[1] and

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Kuwayama et al.[2] investigated a numerical aspect of the propagation of errors for the Laplace equation. Kubo et al.[3] considered the ill-posedness of the Cauchy problem for the Laplace equation through a theoretical point of view and discussed a conditional stability estimate of the problem with respect to the L^2–norm[4].

The purpose of this paper is to present a unified numerical treatment of those four problems. We mainly consider the Cauchy problem and under-determined problem. To solve such problems, the basic idea is to identify a proper boundary value $u = \omega$ for the rest of the boundary $\Gamma_u^c = \Gamma \backslash \Gamma_u$ so that the solution $u(\boldsymbol{x})$ of the Laplace equation also satisfies the boundary condition given on Γ_q. The treatment is based on the direct variational method.

2 Direct Variational Method

We write $u(\boldsymbol{x})$ and $q(\boldsymbol{x})$ in the form of $u(\boldsymbol{x};\tau)$ and $q(\boldsymbol{x};\tau)$ to explicitly stress the dependence of the solution on the boundary value τ to be identified on Γ_q^c. We write $u(\boldsymbol{x})$ and $q(\boldsymbol{x})$ in the form of $u(\boldsymbol{x};\omega)$ and $q(\boldsymbol{x};\omega)$ to explicitly stress the dependence of the solution on the boundary value ω to be identified on Γ_u^c. Using the steepest decent method, we minimize either the object functional:

$$K(\tau) = \int_{\Gamma_u} |u(\boldsymbol{x};\tau) - \bar{u}(\boldsymbol{x})|^2 d\Gamma + \eta \int_\Omega |\nabla u(\boldsymbol{x};\tau)|^2 d\Omega \quad \text{subject to } \left.\frac{\partial u}{\partial n}\right|_{\Gamma_q} = \bar{q}, \quad (2)$$

or

$$J(\omega) = \int_{\Gamma_q} |q(\boldsymbol{x};\omega) - \bar{q}(\boldsymbol{x})|^2 d\Gamma + \eta \int_\Omega |\nabla u(\boldsymbol{x};\omega)|^2 d\Omega \quad \text{subject to } u|_{\Gamma_u} = \bar{u}. \quad (3)$$

Here we assume K: $H^{1/2}(\Gamma_q^c) \ni \tau \longmapsto \boldsymbol{R}_+ = [0, +\infty)$ and J: $H^{1/2}(\Gamma_u^c) \ni \omega \longmapsto \boldsymbol{R}_+ = [0, +\infty)$. The Dirichlet integrals added in Eqs.(2) and (3) are regularizers with the regularization parameter $\eta > 0$ to guarantee unique existence of the local minimum (as you will see on the next page) of the functional $K(\tau)$ or $J(\omega)$.

With a suitable choice of positive real numbers α_k for $k = 0, 1, 2, \cdots$, we consider the minimizing process (steepest decent method):

$$\tau_{k+1}(\boldsymbol{x}) = \tau_k(\boldsymbol{x}) - \alpha_k K'(\tau_k), \quad (4)$$

where the gradient $K'(\tau) \in H^{-1/2}(\Gamma_q^c)$ is defined by the first variation:

$$K(\tau + \delta\tau) - K(\tau) =< K'(\tau), \delta\tau > +o(\|\delta\tau\|) \quad (5)$$

with real-valued function $o(\|\delta\tau\|)$ of higher order than $\|\delta\tau\|$, as it tends to zero with respect to the $L^2(\Gamma_q^c)$-norm:

$$\|\varphi\| := \left\{ \int_{\Gamma_q^c} |\varphi|^2 d\Gamma \right\}^{1/2}.$$

We also assume $K'(\tau) \in H^{1/2}(\Gamma_q^c)$.

In fact, let the variation $\delta\tau$ in τ be of the form $\tau(\boldsymbol{x}) = \tau^{(0)}(\boldsymbol{x}) + \varepsilon\tau^{(1)}(\boldsymbol{x})$ with arbitrary real number ε, where $\dfrac{\partial\tau^{(0)}(\boldsymbol{x})}{\partial n} = \bar{q}(\boldsymbol{x})$ and $\dfrac{\partial\tau^{(1)}(\boldsymbol{x})}{\partial n} = 0$ on Γ_q. The solution $u(\boldsymbol{x})$ of Eq.(1) must satisty the Green's identity

$$u(\boldsymbol{\xi}) = -\int_\Gamma u(\boldsymbol{x}) \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}) + \int_\Gamma \frac{\partial u}{\partial n}(\boldsymbol{x}) G(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}), \quad \boldsymbol{\xi} \in \Omega \tag{6}$$

and the boundary integral equation

$$\frac{1}{2}u(\boldsymbol{\xi}) + \int_\Gamma u(\boldsymbol{x}) \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}) = \int_\Gamma \frac{\partial u}{\partial n}(\boldsymbol{x}) G(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}), \quad \boldsymbol{\xi} \in \Gamma \tag{7}$$

where $G(\boldsymbol{x};\boldsymbol{\xi})$ is the fundamental solution of the Laplacian $-\Delta$ so that $-\Delta G(\boldsymbol{x};\boldsymbol{\xi}) = \delta(\boldsymbol{x}-\boldsymbol{\xi})$, where $\delta(\boldsymbol{x}-\xi)$ is the Dirac measure at the point $\boldsymbol{\xi}$. Due to the variation $\delta\tau$ on Γ, we may assume that $u(\boldsymbol{x}) = u^{(0)}(\boldsymbol{x}) + \varepsilon u^{(1)}(\boldsymbol{x})$ on $\bar{\Omega}$ because of the linear dependence of the solution $u(\boldsymbol{x})$ on its boundary values. Substituting this expression into Eqs. (6) and (7), and noticing the arbitrariness of ε, we obtain

$$u^{(j)}(\boldsymbol{\xi}) = -\int_\Gamma u^{(j)}(\boldsymbol{x}) \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}) + \int_\Gamma \frac{\partial u^{(j)}}{\partial n}(\boldsymbol{x}) G(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}), \quad \boldsymbol{\xi} \in \Omega$$

and

$$\frac{1}{2}u^{(j)}(\boldsymbol{\xi}) + \int_\Gamma u^{(j)}(\boldsymbol{x}) \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}) = \int_\Gamma \frac{\partial u^{(j)}}{\partial n}(\boldsymbol{x}) G(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}), \quad \boldsymbol{\xi} \in \Gamma$$

for $j = 0, 1$. This implies that $q(\boldsymbol{x}) = \dfrac{\partial u}{\partial n}(\boldsymbol{x})$ on Γ has the form $q^{(0)}(\boldsymbol{x}) + \varepsilon q^{(1)}(\boldsymbol{x})$. This is a consequence of the fact that the Dirichlet-Neumann map $H^{1/2}(\Gamma) \longrightarrow$

$H^{-1/2}(\Gamma)$ is linear. We obtain (taking Eq. (2) as an example):

$$\begin{aligned}
K(\tau+\delta\tau) &= \int_{\Gamma_u} |u^{(0)}(\boldsymbol{x})+\varepsilon u^{(1)}(\boldsymbol{x})-\bar{u}(\boldsymbol{x})|^2 d\Gamma + \eta\int_\Omega |\nabla(u^{(0)}(\boldsymbol{x})+\varepsilon u^{(1)}(\boldsymbol{x}))|^2 d\Omega \\
&= \int_{\Gamma_u} |u^{(0)}(\boldsymbol{x})-\bar{u}(\boldsymbol{x})|^2 d\Gamma + \eta\int_\Omega |\nabla u^{(0)}(\boldsymbol{x})|^2 d\Omega \\
&\quad + 2\varepsilon\left[\int_{\Gamma_u}\{u^{(0)}(\boldsymbol{x})-\bar{u}(\boldsymbol{x})\}u^{(1)}(\boldsymbol{x})d\Gamma + \eta\int_\Omega \nabla u^{(0)}(\boldsymbol{x})\cdot\nabla u^{(1)}(\boldsymbol{x})d\Omega\right] \\
&\quad + \varepsilon^2\left[\int_{\Gamma_u}|u^{(1)}(\boldsymbol{x})|^2 d\Gamma + \eta\int_\Omega |\nabla u^{(1)}(\boldsymbol{x})|^2 d\Omega\right] \\
&= K(\tau) + < K'(\tau), \delta\tau > + \varepsilon^2\left[\int_{\Gamma_u}|u^{(1)}(\boldsymbol{x})|^2 d\Gamma + \eta\int_\Omega |\nabla u^{(1)}(\boldsymbol{x})|^2 d\Omega\right].
\end{aligned}$$

For $\eta > 0$, the term involving ε^2 is always positive. This guarantees that, when $K'(\tau) = 0$, $K(\tau)$ attains a local minimum.

To seek a concrete expression of $K'(\tau)$, we notice

$$\begin{aligned}
K(\tau+\delta\tau) - K(\tau) &= \int_{\Gamma_u}\{|u(\boldsymbol{x};\tau+\delta\tau)-\bar{u}(\boldsymbol{x})|^2 - |u(\boldsymbol{x};\tau)-\bar{u}(\boldsymbol{x})|^2\}d\Gamma \\
&\quad + \eta\int_\Omega\{|\nabla u(\boldsymbol{x};\tau+\delta\tau)|^2 - |\nabla u(\boldsymbol{x};\tau)|^2\}d\Omega \\
&= \int_{\Gamma_u} 2\{u(\boldsymbol{x};\tau)-\bar{u}(\boldsymbol{x})\}\delta u(\boldsymbol{x};\tau)d\Gamma + \int_{\Gamma_u}|\delta u(\boldsymbol{x};\tau)|^2 d\Gamma \\
&\quad + 2\eta\int_{\Gamma_q^c} u(\boldsymbol{x};\tau)\delta\tau(\boldsymbol{x};\tau)d\Gamma \\
&\quad + \eta\int_{\Gamma_q^c}\delta u(\boldsymbol{x};\tau)\delta\tau(\boldsymbol{x};\tau)d\Gamma. \qquad (8)
\end{aligned}$$

Here we put $\delta u(\boldsymbol{x};\tau) = u(\boldsymbol{x};\tau+\delta\tau) - u(\boldsymbol{x};\tau)$, and accordingly $\delta q(\boldsymbol{x};\tau) = q(\boldsymbol{x};\tau+\delta\tau) - q(\boldsymbol{x};\tau)$. We notice that $\Delta(\delta u) = 0$ in $\Omega, \delta q = 0$ on Γ_q, and $\delta q = \delta\tau$ on Γ_q^c.

We now consider $v(\boldsymbol{x}) \in H^2(\Omega)$ as a solution of the Laplace equation

$$-\Delta v(\boldsymbol{x}) = 0 \qquad (9)$$

subject to the boundary conditions

$$\left.\frac{\partial v}{\partial n}\right|_{\Gamma_u} = 2\{u(\boldsymbol{x};\tau) - \bar{u}(\boldsymbol{x})\}, \quad \left.\frac{\partial v}{\partial n}\right|_{\Gamma_u^c} = 0.$$

By Green's integral theorem:

$$\int_\Omega (\Delta v)\delta u d\Omega = \int_\Gamma \frac{\partial v}{\partial n}\delta u d\Gamma - \int_\Gamma v\frac{\partial \delta u}{\partial n}d\Gamma + \int_\Omega v\Delta(\delta u)d\Omega, \qquad (10)$$

we have

$$0=\int_{\Gamma_q^c} v(\boldsymbol{x})\delta\tau(\boldsymbol{x})d\Gamma-\int_{\Gamma_u} 2\{u(\boldsymbol{x};\tau)-\bar{u}(\boldsymbol{x})\}\delta u(\boldsymbol{x};\tau)d\Gamma. \tag{11}$$

Moreover, we note that

$$\int_{\Gamma_u} |\delta u(\boldsymbol{x};\tau)|^2 d\Gamma = o(\|\delta\tau\|),$$

and

$$\int_{\Gamma_u^c} \delta u(\boldsymbol{x};\tau)\delta\tau(\boldsymbol{x};\tau)d\Gamma = o(\|\delta\tau\|).$$

Consequently, from Eqs. (8) and (11) we have

$$\begin{aligned} K(\tau+\delta\tau)-K(\tau) &= \int_{\Gamma_q^c} v(\boldsymbol{x})\delta\tau(\boldsymbol{x})d\Gamma+\eta\int_{\Gamma_q^c} 2u(\boldsymbol{x};\tau)\delta\tau(\boldsymbol{x})d\Gamma+o(\|\delta\tau\|) \\ &= (v+2\eta u,\delta\tau)_{L^2(\Gamma_q^c)}+o(\|\delta\tau\|_{L^2(\Gamma_q^c)}). \end{aligned} \tag{12}$$

Therefore, we obtain the explicit form

$$K'(\tau)=v(\boldsymbol{x};\tau)+2\eta u(\boldsymbol{x};\tau) \quad \text{on} \quad \Gamma_u^c. \tag{13}$$

Through the direct variational method, and functional minimization method, these problems can be regarded as boundary inverse problems, in which the proper boundary conditions are to be identified for the rest of the boundary. Then the minimization problems (either (2) or (3)) can be recast into iterative primary and dual boundary value problems of the Laplace equation[5]. For example, a prototype of our algorithm for the under-determined problem is[6]:

<u>*Algorithm*</u>

Given $\tau_0|_{\Gamma_q^c}$.

For $k=0,1,2,\cdots$, *until convergence, do:*

- *Solve the primary problem:* $-\Delta u_k(\boldsymbol{x})=0$ *in* Ω, *with* $q_k|_{\Gamma_q}=\bar{q}$, $q_k|_{\Gamma_q^c}=\tau_k$ *to find* $u_k(\boldsymbol{x};\tau_k)$ *on* Γ_u.
- *Solve the dual problem:* $-\Delta v_k(\boldsymbol{x})=0$ *in* Ω, *with* $\left.\frac{\partial v_k}{\partial n}\right|_{\Gamma_u}=2\{u_k(\boldsymbol{x};\tau_k)-\bar{u}(\boldsymbol{x})\}$, $\left.\frac{\partial v_k}{\partial n}\right|_{\Gamma_u^c}=0$ *to find* $K'(\tau_k)=v_k+2\eta u_k$ *on* Γ_q^c.
- *Update* $\tau_{k+1}=\tau_k-\alpha_k K'(\tau_k)$ *on* Γ_q^c.

We shall discuss a suitable choice of the sequence of positive real numbers $\{\alpha_k\}$. To this end, we employ the Armijo criterion in mathematical programming. This guarantees that the sequence $\{\tau_k\}$ satisfies

$$K(\tau_k - \alpha_k K'(\tau_k)) \ \leq \ K(\tau_k) - \xi\alpha_k \parallel K'(\tau_k) \parallel^2$$

with a constant ξ $(0 < \xi < 1/2)$.

<u>*Controlling the step size α_k*</u>

Given parameters $0 < \xi < \frac{1}{2}$, $0 < \tau < 1$, *and* $0 < \varepsilon \ll 1$.

If $\parallel K'(\tau_k) \parallel < \varepsilon$, *then stop,*

else $\beta_0 := 1$.

For $m = 0, 1, 2, \cdots$, *do:*

If $K(\tau_k - \beta_m K'(\tau_k))$
$\leq K(\tau_k) - \xi\beta_m \parallel K'(\tau_k) \parallel^2$, *then* $\alpha_k := \beta_m$,
else $\beta_{m+1} := \tau\beta_m$.

3 Numerical Examples

The collocation boundary element method with C^0 linear finite elements is used for the numerical solutions of the primary and dual boundary value problems of the Laplace equation.

We note that the function

$$u^*(x_1, x_2) = x_1^2 - x_2^2 = r^2 \cos(2\vartheta) \tag{14}$$

is harmonic in two dimensions.

3.1 An Example of Boundary Value Problem

Suppose that the function u^* is unknown, and consider the Laplace equation in the unit circle

$$\Omega = \{(r, \vartheta) \mid 0 \leq r = \sqrt{x_1^2 + x_2^2} < 1, \quad 0 \leq \vartheta < 2\pi\}$$

with the Dirichlet data: $\bar{u} = \cos(2\vartheta)$ on $\Gamma_u = \{(1, \vartheta) \mid 0 \leq \vartheta \leq \pi\}$ and the Neumann data: $\bar{q} = 2\cos(2\vartheta)$ on $\Gamma_q = \{(1, \vartheta) \mid \pi \leq \vartheta \leq 2\pi\}$ as shown in Figure 1.

The boundary of Ω is uniformly divided into 24 boundary elements as shown in Figure 2. The black points are nodes on the boundary Γ_u and triangles are nodes on the boundary Γ_q. At points $\theta = 0$ and $\theta = \pi$ on the boundary Γ, we use double nodes.

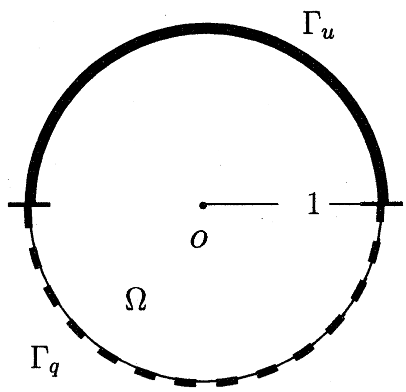

Figure 1. Boundaries Γ_u and Γ_q

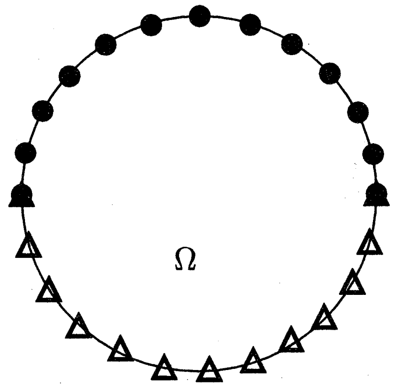

Figure 2. Boundary elements

We take $\eta = 0.1$, $\varepsilon = 10^{-4}$, and the initial guess: $\tau_0 = 1$ on Γ_q^c. Then $\| K'(\tau_k) \| < \varepsilon$ when $k = 18$. That means the functional $K(\tau_k)$ reached a local minimum at τ_{18}. Then we can regard u_{18} as our numerical solution. The convergent process and difference between the exact solution u^* and numerical solution u_{18} are shown in Figure 3. The result is fairly good.

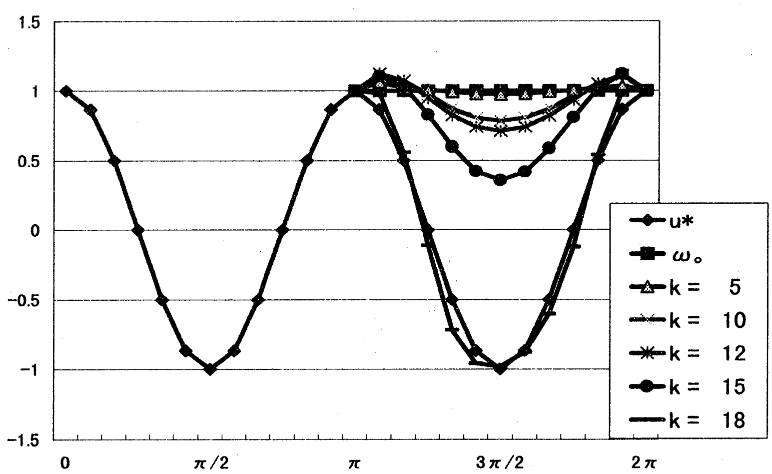

Figure 3. u_k and u^* on the boundary

3.2 An Example of Under-determined Problem

Consider the Laplace equation in the unit circle Ω with the Dirichlet data: $\bar{u} = \cos(2\vartheta)$ on $\Gamma_u = \{(1, \vartheta) \mid 0 \leq \vartheta \leq \frac{\pi}{2}\}$ and the Neumann data: $\bar{q} = 2\cos(2\vartheta)$

on $\Gamma_q = \{(1,\vartheta) \mid \pi \leq \vartheta \leq \frac{3}{2}\pi\}$ as shown in Figure 4. The boundary is uniformly divided into 24 boundary elements as shown in Figure 5. The black points are nodes on the boundary Γ.

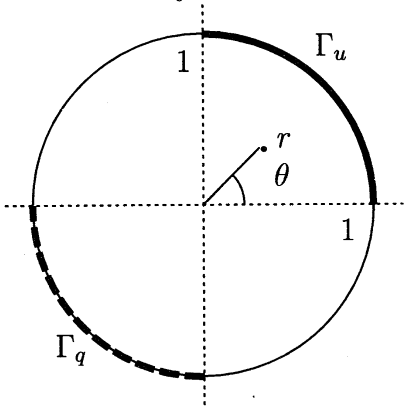

Figure 4. Boundaries Γ_u and Γ_q

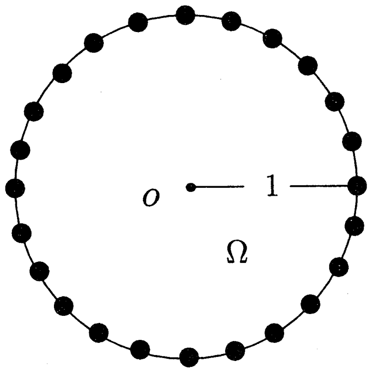

Figure 5. Boundary elements

We take the initial guess: $\tau_0 = -\cos(\frac{2}{3}(\theta - \frac{\pi}{2}))$ because it is smooth on Γ_q^c and periodic on the boundary as u^*. Besides, it has the same slope at points $(\theta, r) = (\pi/2, 1)$ and $(\theta, r) = (2\pi, 1)$ as u^*. We tried various values of η and found that among them our numerical solution is the best when $\eta = 0.15$. In this case we got $\| K'(\tau_k) \| < \varepsilon$ when $k = 20$. That means the functional $K(\tau_k)$ reached a local minimum at τ_{20}. Then we can regard u_{20} as our numerical solution. That does not coincide with u^*. The convergent process and difference between the exact solution u^* and numerical solution u_{20} are shown in Figure 6.

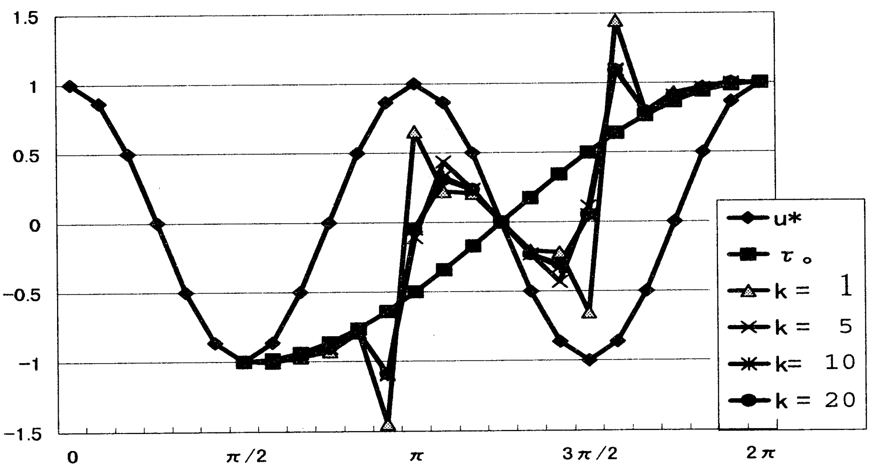

Figure 6. u_k and u^* on the boundary

4 Conclusions

We present a unified new treatment of under-determined boundary value problem, conventional boundary value problem, the Cauchy problem, and over- determined boundary value problem. We minimize functionals using the steepest decent method. The boundary element method is applied for numerical solution of the primary problem and dual problem. Two simple numerical examples from boundary value problem and under-determined problem are given seperately . We conclude that our numerical methods are efficient in dealing with these four kinds of inverse problems.

References

[1] Amaya, K. and Aoki, S., Inverse analysis of galvanic corrosion problems using *a priori* information, *Proceedings of the 43rd National Congress on Theoretical and Applied Mechanics*, Japan Academy Council, Tokyo, January (1994), 481-482.

[2] Kuwayama, S., Kubo, S., Ohji, K., and Takahashi, T., Analysis of boundary value inverse problems by using the relationship between magnification of errors and singular values, *Proceedings of the 43rd National Congress on Theoretical and Applied Mechanics*, Japan Academy Council, Tokyo, January (1994), 515-518.

[3] Kubo, M., Iso, Y., and Tanaka, O., Numerical analysis for the initial value problem for the Laplace equation, *Boundary Element Methods*, M.Tanaka, Q. Du, and T. Honma (eds.), Elsevier Science Publisher (1993), 337-344.

[4] Kubo, M., L^2–conditional stability estimate for the Cauchy problem of the Laplace equation. *Journal of Inverse and Ill-posed Problems*, Vol.2, 3 (1994), 252-261.

[5] Kobayashi, K., Onishi, K. and Ohura, Y., On identifying Dirichlet condition for 2D Laplace equation by BEM, *Engineering Analysis with Boundary Elements*, 17 (1996), 223-230.

[6] Ohura, Y., Kobayashi, K. and Onishi, K., Numerical solution of an under-determined problem of the Laplace equation, *Journal of Applied Mechanics*, JSCE, 2 (1999), 185-189.