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Bilinear estimates and critical Sobolev inequality in BMO, with applications to the Navier-Stokes and the Euler equations

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Introduction.

In this paper we prove that the BMO norm of the velocity and the vorticity controls the blow-up phenomena of smooth solutions to the Navier-Stokes and the Euler equations. Our result is applied to the criterion on regularity of weak solutions to the Navier-Stokes equations.

We consider the Navier-Stokes and the Euler equations in $\boldsymbol{R}^n$, $n \geq 3$:

$$\text{(N-S)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } x \in \boldsymbol{R}^n,\ t > 0, \\ u\,|_{t=0} = a, \end{cases}$$

$$\text{(E)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } x \in \boldsymbol{R}^n,\ t > 0, \\ u\,|_{t=0} = a \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), \cdots, u^n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x,t) \in \boldsymbol{R}^n \times (0,\infty)$, respectively, while $a = (a^1(x), a^2(x), \cdots, a^n(x))$ is the given initial velocity vector.

It is proved by Fujita-Kato [10] that for every $a \in H^s_\sigma \equiv \{v \in H^s; \operatorname{div} v = 0\}$ with $s > n/2 - 1$, there exist $T > 0$ and a unique solution $u(t)$ of (N-S) on $[0,T)$ in the class

$$\text{(CN)}_s \qquad u \in C([0,T); H^s_\sigma) \cap C^1((0,T); H^s) \cap C((0,T); H^{s+2}).$$

Concerning the Euler equations, Kato-Lai [15] and Kato-Ponce [16] proved that for every $a \in W^{s,p}_\sigma$ for $s > n/p + 1$, $1 < p < \infty$, there are $T > 0$ and a unique solution u of (E) on the interval $[0,T)$ in the class

$$\text{(CE)}_{s,p} \qquad u \in C([0,T); W^{s,p}_\sigma) \cap C^1([0,T); W^{s-2,p}_\sigma),$$

where subindex σ means the divergence free. It is an interesting question whether the solution $u(t)$ really blows up as $t \uparrow T$. Giga [11] showed that if the strong solution u in $(CN)_s$ satisfies

(Se) $$\int_0^T \|u(t)\|_{L^r}^{\kappa} dt < \infty \quad \text{for } 2/\kappa + n/r = 1 \text{ with } n < r \leq \infty,$$

then u can be continued to the solution in the class $(CN)_s$ beyond $t = T$. Concerning the Euler equations, Beale-Kato-Majda [1] dealt with the vorticity $\omega = \text{rot } u$ and proved that under the condition

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < \infty$$

$u(t)$ can never break down its regularity at $t = T$. To prove this assertion, in [1] they made use of the logarithmic inequality such as

(0.1) $$\|\nabla u\|_{L^\infty} \leq C\left(1 + \|\text{rot } u\|_{L^\infty}(1 + \log^+ \|u\|_{W^{s+1,p}}) + \|\text{rot } u\|_{L^2}\right), \quad sp > n$$

for all vector functions u with div $u = 0$, where $\log^+ a = \log a$ if $a \geq 1$, $= 0$ if $0 < a < 1$.

The purpose of this paper is to extend these results to the marginal space BMO which is larger than L^∞.

1 Results.

Before stating our results, we introduce some function spaces. Let $C_{0,\sigma}^\infty$ denote the set of all C^∞ vector functions $\phi = (\phi^1, \phi^2, \cdots, \phi^n)$ with compact support in $\boldsymbol{R}^n$, such that div $\phi = 0$. L_σ^r is the closure of $C_{0,\sigma}^\infty$ with respect to the L^r-norm $\|\cdot\|_r$; $(\cdot,\cdot)$ denotes the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued) L^r-space over $\boldsymbol{R}^n$, $1 \leq r \leq \infty$. H_σ^s denotes the closure of $C_{0,\sigma}^\infty$ with respect to the H^s-norm $\|\phi\|_{H^s} = \|(1-\Delta)^{\frac{s}{2}}\phi\|_2, \quad s \geq 0$.

Our result on continuation of strong solutions of (N-S) now reads:

Theorem 1 *Let $s > n/2 - 1$ and let $a \in H_\sigma^s$. Suppose that u is the strong solution of (N-S) in the class $(CN)_s$ on $(0,T)$. If*

(1.1) $$\int_{\varepsilon_0}^T \|u(t)\|_{BMO}^2 dt < \infty \quad \textit{for some } 0 < \varepsilon_0 < T,$$

then u can be continued to the strong solution in the class $(CN)_s$ on $(0,T')$ for some $T' > T$.

Corollary 1 *Let u be the strong solution of (N-S) in the class $(CN)_s$ on $(0,T)$ for $s > n/2-1$. Suppose that T is maximal, i.e., u cannot be continued in the class $(CN)_s$ on $(0,T')$ for any $T' > T$. Then*

(1.2) $$\int_\varepsilon^T \|u(t)\|_{BMO}^2 dt = \infty \quad \textit{for all } 0 < \varepsilon < T.$$

For the space BMO, we refer to Stein [24]. Since $s > n/2 - 1$, there holds $H^{s+2} \subset BMO$, and hence for every u in the class $(\mathrm{CN})_s$ on $(0,T)$, we have $u \in C((0,T); BMO)$.

We next consider a criterion on uniqueness and regularity of weak solutions to (N-S). Our definition of a weak solution is as follows.

Definition 1. Let $a \in L^2_\sigma$. A measureble function u on $\boldsymbol{R}^n \times (0,T)$ is called a weak solution of (N-S) on $(0,T)$ if

(i) $u \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1_\sigma)$;

(ii) $u(t)$ is continuous on $[0,T]$ in the weak topology of L^2_σ;

(iii)

$$\int_s^t \{-(u, \partial_\tau \Phi) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)\} d\tau = -(u(t), \Phi(t)) + (u(s), \Phi(s)) \tag{1.3}$$

for every $0 \le s \le t < T$ and every $\Phi \in H^1((s,t); H^1_\sigma \cap L^n)$.

Our result on weak solutions of (N-S) now reads:

Theorem 2 *(1) (uniqueness) Let $a \in L^2_\sigma$ and let u, v be two weak solutions of (N-S) on $(0,T)$. Suppose that*

$$u \in L^2(0,T; BMO) \tag{1.4}$$

and that v satisfies the energy inequality

$$\|v(t)\|_2^2 + 2\int_0^t \|\nabla v\|_2^2 d\tau \le \|a\|_2^2, \quad 0 < t < T. \tag{1.5}$$

Then we have $u \equiv v$ on $[0,T]$.

(2) (regularity) Let $a \in L^2_\sigma$ and let u be a weak solution with the additional property (1.4). Then for every $0 < \varepsilon < T$, u is actually a strong solution of (N-S) on (ε, T) in the class $(CN)_s$ for $s > n/2 - 1$.

Remark. Theorem 2 may be regarded as an extension of Serrin's criterion [22], [23] on uniqueness and regularity of weak solutions u in the class

$$u \in L^\kappa(0,T; L^r) \quad \text{for } 2/\kappa + n/r = 1 \text{ with } n < r \le \infty\ . \tag{1.6}$$

Our class (1.4) is larger than the marginal case $L^2(0,T;L^\infty)$ in (1.6). Moreover, by virtue of the estimate $\|u\|_{BMO} \le C\|\nabla u\|_{M^n}$ of John-Nirenberg [13], we see that the weak solution u with $\nabla u \in L^2(0,T; M^n)$ becomes regular, where M^n denotes the Morrey space which is larger than L^n. See Beirão da Veiga [2].

We shall next investigate continuation of the strong solution in terms of the vorticity $\omega = \mathrm{rot}\ u \equiv (\partial_j u^k - \partial_k u^j)_{1 \le j,k \le n}$ and the deformation tensor $\mathrm{Def}\ u \equiv (\partial_j u^k + \partial_k u^j)_{1 \le j,k \le n}$.

Theorem 3 *Let $s > n/2 - 1$. Suppose that u is the strong solution of (N-S) in the class $(CN)_s$ on $(0,T)$. If either*

$$\int_{\varepsilon_0}^{T} \|\omega(t)\|_{BMO} dt < \infty \tag{1.7}$$

or

$$\int_{\varepsilon_0}^{T} \|\mathrm{Def}\, u(t)\|_{BMO} dt < \infty \tag{1.8}$$

holds for some $0 < \varepsilon_0 < T$, then u can be continued to the strong solution in the class $(CN)_s$ on $(0,T')$ for some $T' > T$.

Corollary 2 *Suppose that u is the strong solution of (N-S) in the class $(CN)_s$ on $(0,T)$ for $s > n/2 - 1$. Assume that T is maximal in the same sense as in Corollary 1. Then both*

$$\int_{\varepsilon}^{T} \|\omega(t)\|_{BMO} dt = \infty \quad \text{and} \quad \int_{\varepsilon}^{T} \|\mathrm{Def}\, u(t)\|_{BMO} dt = \infty \tag{1.9}$$

hold for all $0 < \varepsilon < T$.

Theorem 3 yields the following regularity criterion on weak solutions of (N-S) by mean of rot u and Def u.

Theorem 4 *Let $a \in L^2_\sigma$. Suppose that u is a weak solution of (N-S) on $(0,T)$. If either*

$$\omega \in L^1(0,T; BMO) \quad \text{or} \quad \mathrm{Def}\, u \in L^1(0,T; BMO) \tag{1.10}$$

holds, then for every $0 < \varepsilon < T$, u is actually a strong solution of (N-S) in the class $(CN)_s$ on (ε, T) for $s > n/2 - 1$.

Remark. Beirão da Veiga [2] proved the regularity criterion in the class $\nabla u \in L^\kappa(0,T; L^r)$ for $2/\kappa + n/r = 2$ with $1 < \kappa < \infty$, $n/2 < r < \infty$. Theorem 4 covers the borderline case $\kappa = 1$ and $r = \infty$.

Our result on (E) reads as follows.

Theorem 5 *Let $1 < p < \infty$, $s > n/p + 1$. Suppose that u is the solution of (E) in the class $(CE)_{s,p}$ on $(0,T)$. If either*

$$\int_0^T \|\omega(t)\|_{BMO} dt (\equiv M_0) < \infty \tag{1.11}$$

or

$$\int_0^T \|\mathrm{Def}\, u(t)\|_{BMO} dt (\equiv M_1) < \infty \tag{1.12}$$

holds, then u can be continued to the solution in the class $(CE)_{s,p}$ on $(0,T')$ for some $T' > T$.

Corollary 3 *Let u be the solution of (E) in the class $(CE)_{s,p}$ on $(0,T)$ for $1 < p < \infty$, $s > n/p + 1$. Assume that T is maximal, i.e., u cannot be continued to the solution in the class $(CE)_{s,p}$ on $(0,T')$ for any $T' > T$. Then both*

$$\int_0^T \|\mathrm{rot}\, u(t)\|_{BMO} dt = \infty \quad \text{and} \quad \int_0^T \|\mathrm{Def}\, u(t)\|_{BMO} dt = \infty$$

hold.

2 Bilinear estimates and critical Sobolev inequality in BMO.

In this section we shall prepare some lemmas. In what follows we shall denote by C various constants. In particular, $C = C(*, \cdots, *)$ denotes constants depending only on the quantities appearing in the parenthesis.

We first prove the following key estimate.

Lemma 2.1 (Bilinear estimates) *Let $1 < r < \infty$. Then we have*

(i)

$$\|f \cdot \nabla g\|_r \leq C(\|f\|_r \|(-\Delta)^{\frac{1}{2}} g\|_{BMO} + \|(-\Delta)^{\frac{1}{2}} f\|_{BMO} \|g\|_r) \tag{2.1}$$

for all $f, g \in W^{1,r}$ with $\nabla f, \nabla g \in BMO$ with $C = C(n, r)$.

(ii) Let $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$ be multi-indices with $|\alpha| = \alpha_1 + \cdots + \alpha_n \geq 1$ and $|\beta| = \beta_1 + \cdots + \beta_n \geq 1$. Then

$$\|\partial^\alpha f \cdot \partial^\beta g\|_2 \leq C(\|f\|_{BMO} \|(-\Delta)^{\frac{|\alpha|+|\beta|}{2}} g\|_2 + \|(-\Delta)^{\frac{|\alpha|+|\beta|}{2}} f\|_2 \|g\|_{BMO}) \tag{2.2}$$

for all $f, g \in BMO \cap H^{|\alpha|+|\beta|}$ with $C = C(n, \alpha, \beta)$, where $\partial^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.

The proof of this lemma is based on the following proposition due to Coifman-Meyer [6, Chapter V. Proposition 2].

Proposition 2.1 (Coifman-Meyer) *Let $\sigma = \sigma(\xi, \eta) \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n \setminus \{(0,0)\})$ satisfy*

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq C(|\xi| + |\eta|)^{-|\alpha|-|\beta|}, \quad (\xi, \eta) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \setminus \{(0,0)\}$$

for all mluti-indices α, β with $C = C(\alpha, \beta)$. Suppose that

$$\sigma(\xi, 0) = 0.$$

Then the bilinear operator $\sigma(D)(\cdot, \cdot)$ defined by

$$\sigma(D)(f, g)(x) \equiv \iint_{\boldsymbol{R}^n \times \boldsymbol{R}^n} e^{ix\cdot(\xi+\eta)} \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta, \quad x \in \boldsymbol{R}^n \tag{2.3}$$

satisfies

$$\|\sigma(D)(f, g)\|_p \leq C\|f\|_p \|g\|_{BMO} \quad (1 < p < \infty) \tag{2.4}$$

with $C = C(n, p)$.

Proof of Lemma 2.1. Here we prove only (2.2). The proof of (2.1) is similar to that of (2.2). Let Φ_1 be a C^∞-function on $[0, \infty)$ such that supp $\Phi_1 \subset [0, 1)$, $0 \leq \Phi_1 \leq 1$, $\Phi_1(t) \equiv 1$ for $0 \leq t \leq 1/2$, and let $\Phi_2 = 1 - \Phi_1$. Then we have

$$\begin{aligned} & \partial^\alpha f(x) \partial^\beta g(x) \\ = \; & C \iint_{\boldsymbol{R}^n \times \boldsymbol{R}^n} e^{ix\cdot(\xi+\eta)} \xi^\alpha \eta^\beta \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta \end{aligned}$$

$$
\begin{aligned}
&= C \iint_{\boldsymbol{R}^n \times \boldsymbol{R}^n} e^{ix\cdot(\xi+\eta)} \times \\
&\quad \left(\frac{\xi^\alpha \eta^\beta}{|\eta|^{|\alpha|+|\beta|}} \Phi_1(|\xi|/|\eta|) \widehat{f}(\xi) |\eta|^{|\alpha|+|\beta|} \widehat{g}(\eta) + \frac{\xi^\alpha \eta^\beta}{|\xi|^{|\alpha|+|\beta|}} \Phi_2(|\xi|/|\eta|) |\xi|^{|\alpha|+|\beta|} \widehat{f}(\xi) \widehat{g}(\eta) \right) d\xi d\eta \\
&= C \left(\sigma_1(D)(f, (-\Delta)^{\frac{|\alpha|+|\beta|}{2}} g)(x) + \sigma_2(D)((-\Delta)^{\frac{|\alpha|+|\beta|}{2}} f, g)(x) \right),
\end{aligned}
$$

where

$$
\sigma_1(\xi,\eta) = \frac{\xi^\alpha \eta^\beta}{|\eta|^{|\alpha|+|\beta|}} \Phi_1(|\xi|/|\eta|), \quad \sigma_2(\xi,\eta) = \frac{\xi^\alpha \eta^\beta}{|\xi|^{|\alpha|+|\beta|}} \Phi_2(|\xi|/|\eta|).
$$

Since $|\alpha| \geq 1$ ans $|\beta| \geq 1$, we see that

$$
\sigma_1(0,\eta) = 0, \quad \sigma_2(\xi, 0) = 0
$$

and that σ_1 and σ_2 satisfy the hypotheses of Proposition 2.1. Hence there holds

$$
\begin{aligned}
\|\sigma_1(D)(f, (-\Delta)^{\frac{|\alpha|+|\beta|}{2}} g)\|_2 &\leq C\|f\|_{BMO} \|(-\Delta)^{\frac{|\alpha|+|\beta|}{2}} g\|_2, \\
\|\sigma_2(D)((-\Delta)^{\frac{|\alpha|+|\beta|}{2}} f, g)\|_2 &\leq C\|(-\Delta)^{\frac{|\alpha|+|\beta|}{2}} f\|_2 \|g\|_{BMO},
\end{aligned}
$$

which yields (2.2). This proves Lemma 2.1.

The next lemma plays an important role to show the energy identity of weak solutions in the class (1.4) and (1.10).

Lemma 2.2 *(i) Let $w \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1_\sigma)$ and $u \in L^2(0,T; H^1_\sigma \cap BMO)$. Then we have*

(2.5)
$$
\int_0^T (w \cdot \nabla u, u) d\tau = 0.
$$

(ii) Let $w, u \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1_\sigma)$. Suppose that either

$$
\text{rot } w, \text{rot } u \in L^1(0,T; BMO)
$$

or

$$
\text{Def } w, \text{Def } u \in L^1(0,T; BMO)
$$

holds. Then we have

(2.6)
$$
\int_0^T (w \cdot \nabla u, u) d\tau = 0.
$$

To prove (2.5), we use the estimate of Coifman-Lions-Meyer-Semmes [5]:

(2.7)
$$
w \cdot \nabla u \in \mathcal{H}^1 \quad \text{with } \|w \cdot \nabla u\|_{\mathcal{H}^1} \leq C\|w\|_2 \|\nabla u\|_2,
$$

where $\mathcal{H}^1$ denotes the Hardy space on $\boldsymbol{R}^n$. For detail, see [5].

To prove (2.6), we use (2.1) and the Biot-Savart law. Indeed, by the Biot-Savart law, we have the representation

$$
\begin{aligned}
&(2.8) \quad \frac{\partial u}{\partial x_j} = R_j(R \times \omega), \quad j = 1, \cdots, n, \quad \text{where } \omega = \text{rot } u; \\
&(2.9) \quad \frac{\partial u^l}{\partial x_j} = R_j\left(\sum_{k=1}^n R_k \text{Def } u_{kl}\right), \quad j, l = 1, \cdots, n, \quad \text{where Def } u_{kl} = \frac{\partial u^k}{\partial x_l} + \frac{\partial u^l}{\partial x_k}.
\end{aligned}
$$

Here $R = (R_1, \cdots, R_n)$, and $R_j = \dfrac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ denote the Riesz transforms. Since R is a bounded operator in BMO, we have by (2.8), (2.9) and assumption that

$$\nabla u, \nabla w \in L^1(0,T; BMO). \tag{2.10}$$

It follows from Lemma 2.1 (2.1) and (2.10) that $\int_0^T (w \cdot \nabla u, u) d\tau$ is well-defined. For details of the proof of Lemma 2.2 we refer to [17].

Using the usual mollifier argument, by Lemma 2.2, we have the following energy identity for weak solutions with (1.4) or (1.10).

Lemma 2.3 *Let $n \geq 3$ and let $a \in L^2_\sigma$. Suppose that u is a weak solution of (N-S) on $(0,T)$ satisfying one of the additional conditions (1.4) and (1.10). Then u fulfills the energy identity*

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u\|_2^2 d\tau = \|u(s)\|_2^2 \quad \text{for all } 0 \leq s \leq t < T. \tag{2.11}$$

Now we prove the following lemma which is an extension of (0.1).

Lemma 2.4 (Critical Sobolev Inequality) *Let $1 < p < \infty$ and let $s > n/p$. There is a constant $C = C(n,p,s)$ such that the estimate*

$$\|f\|_\infty \leq C\left(1 + \|f\|_{BMO}(1 + \log^+ \|f\|_{W^{s,p}})\right) \tag{2.12}$$

holds for all $f \in W^{s,p}$.

Remark. Compared with (0.1), we do not need to add $\|f\|_{L^2}$ to the right hand side of (2.12). This makes it easier to derive an apriori estimate of solutions to the Euler equations than Beale-Kato-Majda [1].

Proof of Lemma 2.4.

We shall make use of the Littlewood-Paley decomposition; there exists a non-negative function $\varphi \in \mathcal{S}$ ($\mathcal{S}$; the Schwartz class) such that $\operatorname{supp}\varphi \subset \{2^{-1} \leq |\xi| \leq 2\}$ and such that $\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1$ for $\xi \neq 0$. See Bergh-Löfström [3, Lemma 6.1.7]. Let us define ϕ_0 and ϕ_1 as

$$\phi_0(\xi) = \sum_{k=1}^{\infty} \varphi(2^k\xi) \quad \text{and} \quad \phi_1(\xi) = \sum_{k=-\infty}^{-1} \varphi(2^k\xi),$$

respectively. Then we have that $\phi_0(\xi) = 1$ for $|\xi| \leq 1/2$, $\phi_0(\xi) = 0$ for $|\xi| \geq 1$ and that $\phi_1(\xi) = 0$ for $|\xi| \leq 1$, $\phi_1(\xi) = 1$ for $|\xi| \geq 2$. It is easy to see that for every positive integer N there holds the identity

$$\phi_0(2^N\xi) + \sum_{k=-N}^{N} \varphi(2^{-k}\xi) + \phi_1(2^{-N}\xi) = 1, \quad \xi \neq 0. \tag{2.13}$$

Since C_0^∞ is dense in $W^{s,p}$ and since $W^{s,p}$ is continuously embedded in BMO, implied by $s > n/p$, it suffices to prove (2.12) for $f \in C_0^\infty$. For such f we have the representation

$$f(x) = \int_{y \in \mathbf{R}^n} K(x-y) \cdot \nabla f(y) dy \quad \text{with} \quad K(y) = \frac{1}{n\omega_n}\frac{y}{|y|^n},$$

for all $x \in \boldsymbol{R}^n$, where ω_n denotes the volume of the unit ball in $\boldsymbol{R}^n$. By (2.13) we decompose f into three parts:

$$\begin{aligned} f(x) &= \int_{y\in\boldsymbol{R}^n} K(x-y) \times \\ &\quad \times \left(\phi_0(2^N(x-y)) + \sum_{k=-N}^{N} \varphi(2^{-k}(x-y)) + \phi_1(2^{-N}(x-y))\right) \cdot \nabla f(y)dy \\ (2.14) \qquad &\equiv f_0(x) + g(x) + f_1(x) \end{aligned}$$

for all $x \in \boldsymbol{R}^n$.

We can show that

$$(2.15) \qquad |f_0(x)| \le C2^{-\beta N} \|f\|_{W^{s,p}}$$

for all $x \in \boldsymbol{R}^n$, where $\beta = \beta(n,p,s)$ is a positive constant. For detail, see [18].

By integration by parts we have

$$g(x) = \sum_{k=-N}^{N} (\text{div } \Psi)_{2^k} * f(x), \quad x \in \boldsymbol{R}^n,$$

where $\Psi(x) = K(x)\varphi(x)$ and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Since $\Psi \in \mathcal{S}$ with the property that

$$\int_{\boldsymbol{R}^n} \text{div } \Psi(x)dx = 0,$$

it follows from Stein [24, Chap. IV, 4.3.3] that

$$\begin{aligned} \|g\|_\infty &\le \sum_{k=-N}^{N} \|(\text{div } \Psi)_{2^k} * f\|_\infty \\ &\le \sum_{k=-N}^{N} \sup_{t>0} \|(\text{div } \Psi)_t * f\|_\infty \\ (2.16) \qquad &\le CN\|f\|_{BMO}, \end{aligned}$$

where $C = C(n)$ is independent of N.

Integrating by parts, we have by a direct calculation

$$\begin{aligned} |f_1(x)| &= \left|\int_{y\in\boldsymbol{R}^n} \text{div}_y \left(K(x-y)\phi_1(2^{-N}(x-y))\right) f(y)dy\right| \\ (2.17) \qquad &\le C2^{-N\cdot\frac{n}{p}}\|f\|_p \end{aligned}$$

for all $x \in \boldsymbol{R}^n$, where $C = C(n,p)$ is independent of N.

Now it follows from (2.14) and (2.15)-(2.17) that

$$(2.18) \qquad \|f\|_\infty \le C(2^{-\gamma N}\|f\|_{W^{s,p}} + N\|f\|_{BMO})$$

with $\gamma = \text{Min.}\{\beta, n/p\}$, where $C = C(n,s,p)$ is independent of N and f. If $\|f\|_{W^{s,p}} \le 1$, then we may take $N = 1$; otherwise, we take N so large that the first term of the right hand side of (2.18) is dominated by 1, i.e., $N \equiv \left[\frac{\log \|f\|_{W^{s,p}}}{\gamma \log 2}\right] + 1$ ([·]; Gauss symbol) and (2.18) becomes

$$\|f\|_\infty \le C\left\{1 + \|f\|_{BMO}\left(\frac{\log \|f\|_{W^{s,p}}}{\gamma \log 2} + 1\right)\right\}.$$

In both cases, (2.12) holds. This proves Lemma 2.4.

3 Proof of Theorems 1-4

3.1 Proof of Theorem 1.

It is proved by Kato [14] and Giga [11] that, for the initial data $a \in H^s$ with $s > n/2 - 1$, the local existence time interval T of the strong solution u of (N-S) in the class $(\mathrm{CN})_s$ can be estimated from below as

$$T \ge \frac{C}{\|a\|_{H^s}^{\frac{2}{s-(n/2-1)}}}, \tag{3.1}$$

where $C = C(n, s)$. Actually, for $a \in L^r$ with $r > n$, Giga [11, Theorem 1 (ii)] gave T in such a way that

$$T = \frac{C}{\|a\|_r^{2r/(r-n)}}, \tag{3.2}$$

so from the continuous embedding $H^s \subset L^r$ for $1/r = 1/2 - s/n$, we obtain (3.1). Hence by the standard argument of continuation of local solutions, it suffices to prove the following apriori estimate

$$\sup_{\varepsilon_0 < t < T} \|u(t)\|_{H^{[s]+1}} \le \|u(\varepsilon_0)\|_{H^{[s]+1}} \exp\left(C\int_{\varepsilon_0}^T \|u\|_{BMO}^2 dt\right), \tag{3.3}$$

where $C = C(n, s)$ is independent of T.

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index with $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n \le [s] + 1$, and let $v = \partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. Applying ∂^α to (N-S), we have for v the equation

$$\frac{\partial v}{\partial t} - \Delta v + u \cdot \nabla v + \nabla q = F, \quad \varepsilon_0 < t < T, \tag{3.4}$$

where $q = \partial^\alpha p$ and

$$F = -\sum_{|\beta| \le |\alpha| - 1} {}_\alpha C_\beta \partial^{\alpha-\beta} u \cdot \nabla(\partial^\beta u). \tag{3.5}$$

Taking the inner product in L^2 between (3.4) and v, and then integrating the result identity on the time interval (ε_0, t), we obtain

$$\|v(t)\|_2^2 + 2\int_{\varepsilon_0}^t \|\nabla v\|_2^2 d\tau \le \|v(\varepsilon_0)\|_2^2 + 2\int_{\varepsilon_0}^t \|F\|_2 \|v\|_2 d\tau. \tag{3.6}$$

On the other hand, by (2.2), we have

$$\|F\|_2 \le C\|u\|_{BMO}\|(-\Delta)^{\frac{|\alpha|+1}{2}} u\|_2,$$

from which and (3.6) it follows that

$$\begin{aligned}
&\|\partial^\alpha u(t)\|_2^2 + 2\int_{\varepsilon_0}^t \|\nabla(\partial^\alpha u)\|_2^2 d\tau \\
\leq\ & \|\partial^\alpha u(\varepsilon_0)\|_2^2 + \int_{\varepsilon_0}^t \|(-\Delta)^{\frac{|\alpha|+1}{2}} u\|_2^2 d\tau + C\int_{\varepsilon_0}^t \|u\|_{BMO}^2 \|\partial^\alpha u\|_2^2 d\tau
\end{aligned}$$

with C independent of t. Summing over α with $0 \leq |\alpha| \leq [s]+1$, we have

$$\|u(t)\|_{H^{[s]+1}}^2 \leq \|u(\varepsilon_0)\|_{H^{[s]+1}}^2 + C\int_{\varepsilon_0}^t \|u\|_{BMO}^2 \|u\|_{H^{[s]+1}}^2 d\tau$$

for all $\varepsilon_0 \leq t < T$. Now the Gronwall inequality yields (3.3). This proves Theorem 1.

3.2 Proof of Theorem 2.

(1) Let us first prove uniqueness. We follow the argument of Masuda [20, Theorems 2, 3]. We can show that

$$\int_0^t \{2(\nabla u, \nabla v) + (v\cdot\nabla v, u) - (u\cdot\nabla v, u)\} d\tau = -(u(t), v(t)) + \|a\|_2^2. \tag{3.7}$$

See Masuda [20, p.640 (4.4)]. By Lemma 2.3, u satisfies the energy identity

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 d\tau = \|a\|_2^2. \tag{3.8}$$

Addition of (3.7) (multiplied by -2), (3.8) and (1.5) yields

$$\|w(t)\|_2^2 + 2\int_0^t \|\nabla w\|_2^2 d\tau \leq 2\int_0^t (w\cdot\nabla v, u) d\tau = 2\int_0^t (w\cdot\nabla w, u) d\tau, \tag{3.9}$$

where $w = v - u$. In the last identity, we have used (2.5). By (2.7) we have

$$\begin{aligned}
\text{RHS of (3.9)} &\leq C\int_0^t \|w\cdot\nabla w\|_{\mathcal{H}^1} \|u\|_{BMO} d\tau \\
&\leq C\int_0^t \|w\|_2 \|\nabla w\|_2 \|u\|_{BMO} d\tau \\
&\leq \int_0^t \|\nabla w\|_2^2 d\tau + C\int_0^t \|w\|_2^2 \|u\|_{BMO}^2 d\tau.
\end{aligned}$$

Hence by (3.9)

$$\|w(t)\|_2^2 \leq C\int_0^t \|w\|_2^2 \|u\|_{BMO}^2 d\tau, \quad 0 \leq t < T.$$

Since $u \in L^2(0,T;BMO)$, the Gronwall inequality yields

$$\|w(t)\|_2^2 = 0, \quad 0 \leq t < T$$

from which we get the desired uniqueness.

(2) We next prove regularity. Since $u \in L^2(0,T;H^1_\sigma \cap BMO)$, for every $0<\varepsilon<T$, there is $0<\delta<\varepsilon$ such that $u(\delta) \in H^1_\sigma \cap BMO \subset L^2_\sigma \cap L^r_\sigma$ for $n<r<\infty$. Hence it follows from the local existence theorem of Kato [14] and Giga [11] that there are $T_* > \delta$ and a unique solution $\tilde{u}$ on $[\delta, T_*)$ with $\tilde{u}|_{t=\delta} = u(\delta)$, such that

$$\tilde{u} \in C([\delta,T_*);H^1_\sigma \cap L^r_\sigma) \cap C^1((\delta,T_*);H^{s+2}) \quad \text{for } s > n/2-1. \tag{3.10}$$

Since u satisfies the energy identity

$$\|u(t)\|_2^2 + 2\int_\delta^t \|\nabla u\|_2^2 d\tau = \|u(\delta)\|_2^2, \quad \delta \le t < T, \tag{3.11}$$

implied by Lemma 2.3, we have by the uniqueness criterion of Serrin-Masuda [23], [20]

$$u \equiv \tilde{u} \quad \text{on } [\delta, T_*). \tag{3.12}$$

By (3.10) and (3.12), we may regard u as a strong solution in the class $(\mathrm{CN})_s$ on (δ', T_*) for $\delta < \delta' < \varepsilon$.

In fact, there holds $T_* = T$. Suppose that $T_* < T$. Then there exists $T_0 < T$ such that u is a strong solution in the class $(\mathrm{CN})_s$ on (δ', T_0), but cannot be continued in the class $(\mathrm{CN})_s$ on $(\delta', \tilde{T})$ for any $\tilde{T} > T_0$. By assumption, we have

$$\int_{\delta'}^{T_0} \|u\|_{BMO}^2 d\tau \le \int_0^T \|u\|_{BMO}^2 d\tau < \infty. \tag{3.13}$$

This contradicts Corollary 1, so we get $T_* = T$. This proves Theorem 2.

3.3 Proof of Theorems 3-4.

Proof of Theorem 3:

On account of (3.2), it suffices to prove

$$\sup_{\varepsilon_0 < t < T} \|u(t)\|_r \le \|u(\varepsilon_0)\|_r \exp\left(C\int_{\varepsilon_0}^T \|\nabla u\|_{BMO} d\tau \right), \quad r > n. \tag{3.14}$$

In the same way as in (2.10), we see that the hypothesis (1.7) or (1.8) yields

$$\int_{\varepsilon_0}^T \|\nabla u\|_{BMO} dt < \infty. \tag{3.15}$$

Since $u \in C([\varepsilon_0,T);H^{s+2}) \subset C([\varepsilon_0,T);W^{1,\infty})$, u is actually the solution in $C([\varepsilon_0,T);W^{1,r}) \cap C^1((\varepsilon_0,T);W^{1,r}) \cap C((\varepsilon_0,T);W^{3,r})$ for all $2 \le r < \infty$ and has the integral representation:

$$u(t) = e^{(t-\varepsilon_0)\Delta}u(\varepsilon_0) - \int_{\varepsilon_0}^t e^{(t-s)\Delta}P(u\cdot\nabla u)(s)ds, \quad \varepsilon_0 < t < T. \tag{3.16}$$

See Kato [14]. Here $e^{t\Delta}$ is the well-known heat oparator and $P = \{P_{kl}\}_{k,l=1,\dots,n}$ is the Helmholtz projection defined by $P_{kl} = \delta_{kl} + R_k R_l$.

Since $\|e^{t\Delta}\|_{B(L^r,L^r)} \le 1$ for all $t > 0$, it follows from Lemma 2.1(i) and (3.16) that

$$\begin{aligned} \|u(t)\|_r &\le \|u(\varepsilon_0)\|_r + C\int_{\varepsilon_0}^t \|u\cdot\nabla u\|_r d\tau \\ &\le \|u(\varepsilon_0)\|_r + C\int_{\varepsilon_0}^t \|\nabla u\|_{BMO}\|u\|_r d\tau, \quad \varepsilon_0 < t < T. \end{aligned}$$

From this and the Gronwall inequality, we obtain the desired apriori estimate (3.14), which proves Theorem 3.

Proof of Theorem 4:

The proof of Theorem 4 is parallel to that of Theorem 2.

4 Proof of Theorem 5.

We follow the argument of Beale-Kato-Majda [1]. It is proved by Kato-Lai [15] and Kato-Ponce [16] that for the given initial data $a \in W^{s,p}$ for $s > 1 + n/p$, the time interval T of the existence of the solution u to (E) in the class $(\mathrm{CE})_{s,p}$ depends only on $\|a\|_{W^{s,p}}$. Hence by the standard argument of continuation of local solutions, it suffices to establish an apriori estimate for u in $W^{s,p}$ in terms of a, T, M_0 or a, T, M_1 according to (1.11) or (1.12). Indeed, we shall show that the solution $u(t)$ in the class $(\mathrm{CE})_{s,p}$ on $(0,T)$ is subject to the following estimate:

$$\sup_{0<t<T} \|u(t)\|_{W^{s,p}} \le (\|a\|_{W^{s,p}} + e)^{\alpha_j} \exp(CT\alpha_j) \quad \text{with } \alpha_j = e^{CM_j}, \quad j = 0, 1, \tag{4.17}$$

where $C = C(n,p,s)$ is a constant independent of a and T.

We shall first prove (4.17) under (1.11). It follows from the commutator estimate in L^p given by Kato-Ponce [16, Proposition 4.2] that

$$\|u(t)\|_{W^{s,p}} \le \|a\|_{W^{s,p}} \exp\left(C\int_0^t \|\nabla u(\tau)\|_\infty d\tau\right), \quad 0 < t < T, \tag{4.18}$$

where $C = C(n,p,s)$.

By the Biot-Savard law (2.8), we have

$$\|\nabla u\|_{BMO} \le C\|\omega\|_{BMO} \tag{4.19}$$

with $C = C(n)$. Hence it follows from (4.19) and Lemma 2.4 that

$$\|\nabla u(t)\|_\infty \le C\left(1 + \|\omega(t)\|_{BMO}(1 + \log^+ \|u(t)\|_{W^{s,p}})\right) \tag{4.20}$$

for all $0 < t < T$ with $C = C(n,p,s)$. Substituting (4.20) to (4.18), we have

$$\begin{aligned} &\|u(t)\|_{W^{s,p}} + e \\ \le\ & (\|a\|_{W^{s,p}} + e) \exp\left(C\int_0^t \{1 + \|\omega(\tau)\|_{BMO} \log(\|u(\tau)\|_{W^{s,p}} + e)\} d\tau\right) \end{aligned}$$

for all $0 < t < T$. Defining $z(t) \equiv \log(\|u(t)\|_{W^{s,p}} + e)$, we obtain from the above estimate

$$z(t) \leq z(0) + CT + C \int_0^t \|\omega(\tau)\|_{BMO} z(\tau) d\tau, \quad 0 < t < T.$$

Now (1.11) and the Gronwall inequality yield

$$\begin{aligned} z(t) &\leq (z(0) + CT) \exp \left(C \int_0^t \|\omega(\tau)\|_{BMO} d\tau \right) \\ &\leq (z(0) + CT)\, \alpha_0 \end{aligned}$$

for all $0 < t < T$ with $C = C(n, p, s)$, which implies (4.17) for $j = 0$.

Similarly we prove (4.17) for $j = 1$ under (1.12). This proves Theorem 5.

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