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Stationary waves for the discrete Boltzmann equations in the half space with reflection boundary conditions

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1 Introduction

1.1 Problem

The discrete Boltzmann equation appears in the discrete kinetic theory of rarefied gases. This system of equations describes the motion of gas particles with a finite number of velocities. It is interesting and important to analyze the asymptotic behavior of the solution under boundary effects not only as a purely mathematical problem, but also from the physical point of view.

The aim of this research is to show the unique existence and the stability of a stationary solution to the system in the half space $\mathbb{R}_+ := \{x > 0\}$ with the reflective boundary condition.

$$\nu_i(\partial_t F_i + v_i \partial_x F_i) = Q_i(F) \quad \text{for} \quad i \in \Lambda, \tag{1.1}$$

$$\nu_i F_i(0,t) = \sum_{j\in\Lambda_-} \mathcal{B}_{ij} F_j(0,t) \quad \text{for} \quad i \in \Lambda_+, \tag{1.2}$$

$$F_i(x,0) = F_{i0}(x) \quad \text{for} \quad i \in \Lambda, \tag{1.3}$$

where each $F = (F_i)_{i\in\Lambda}$ is an unknown function representing the mass density of gas particles; Λ is a finite set $\{1, 2, \ldots m\}$, $\Lambda_\pm := \{i \in \Lambda : v_i \gtrless 0\}$ and $\Lambda_0 := \{i \in \Lambda : v_i = 0\}$; each ν_i is a positive integer; each v_i is a constant representing the x-component of the i-th velocity and v_i's are not necessarily distinct and not necessarily non-zero; each $Q_i(F)$ is a given function called the collision term; each $\mathcal{B}_{ij}$ is a nonnegative constant. We assume that the compatibility condition holds, that is, the initial data $F_0 = (F_{i0})_{i\in\Lambda}$ satisfies (1.2) at $x = 0$. Moreover, it is assumed that the initial data F_0 satisfies the spatial asymptotic condition,

$$F_{i0}(x) \to M_i \quad \text{as} \quad x \to \infty \quad \text{for} \quad i \in \Lambda, \tag{1.4}$$

where $M = (M_i)_{i\in\Lambda}$ is a Maxwellian, i.e. $Q(M) = 0$ and $M_i > 0$ for $i \in \Lambda$.

The Boltzmann equation (1.1), the reflective boundary condition (1.2) and the initial data (1.3) are expressed in a vector form as

$$I^\nu (F_t + VF_x) = Q(F), \tag{1.5}$$
$$R^+ I^\nu F(0,t) = \mathfrak{B} R^- F(0,t), \tag{1.6}$$
$$F(x,0) = F_0(x) \to M \quad \text{as} \quad x \to \infty, \tag{1.7}$$

where $I^\nu = \mathrm{diag}(\nu_i)_{i\in\Lambda}$, $V = \mathrm{diag}(v_i)_{i\in\Lambda}$ and $\mathfrak{B} = (\mathfrak{B}_{ij})_{(i,j)\in\Lambda_+\times\Lambda_-}$; $\mathfrak{B}$ is called a boundary matrix in this paper. $R^\pm$ means the restriction to the subspace corresponding to $\Lambda_\pm$, respectively:

$$R^\pm\phi = R^\pm(\phi_i)_{i\in\Lambda} = (\phi_i)_{i\in\Lambda_\pm}.$$

A stationary solution is a function $\tilde{F}(x) = (\tilde{F}_i(x))_{i\in\Lambda}$ in $\mathfrak{B}^0[0,\infty)$ satisfying (1.5), (1.6) and (1.7). Precisely,

$$V^\nu \tilde{F}_x = Q(\tilde{F}), \tag{1.8}$$
$$R^+ I^\nu \tilde{F}(0) = \mathfrak{B} R^- \tilde{F}(0), \tag{1.9}$$
$$\tilde{F}(x) \to M \quad \text{as} \quad x \to \infty, \tag{1.10}$$

where $V^\nu := I^\nu V = V I^\nu$.

The existence of a stationary solution in the half space is first considered in [6] to (1.8) and (1.10) with a pure diffusive boundary condition,

$$F_i(0,t) = \mathfrak{B}_i \quad \text{for} \quad i \in \Lambda_+ \tag{1.11}$$

where each $\mathfrak{B}_i$ is a constant, under the additional assumption that $v_i \neq 0$. This result is developed in [2] to the general system including the possibility that $v_i = 0$. Also, it is proved in [2] that the stationary solution approaches the asymptotic Maxwellian exponentially fast. The stability of this stationary solution is discussed in [4].

Obviously, the pure diffusive boundary condition (1.11) is more easily handled by mathematical analysis than the reflective boundary condition (1.2). However, the latter (1.2) seems more realistic than the former (1.11) from the physical point of view. The reason is that while the pure diffusive boundary condition (1.11) requires that the behaviors of particles on the boundary $\{x = 0\}$ are known a priori in gas dynamic context, the reflective boundary condition (1.2) only assumes the rules of reflection on the boundary $\{x = 0\}$.

Applying results in [2], we prove the existence of the stationary solution with the reflective boundary condition (1.2). First, we obtain the existence of a stationary solution to the linearized system and show that it is expressed by a certain explicit formula. We then define a functional by this formula on a certain Banach space with a suitably weighted supremum norm. The existence and the uniqueness of the stationary solution to (1.8), (1.9) and (1.10) are established by showing that this functional is a contraction mapping. These discussions also show that the stationary solution approaches the asymptotic Maxwellian state M exponentially fast as the spatial variable x tends to infinity.

The stability of the stationary solution is proved by the energy method. Here, we adopt the idea in [4]. This idea makes it possible to handle some error terms, arising from the energy method, by utilizing the exponential convergence of the stationary solutions to the Maxwellian M at the spatial asymptotic point.

In conclusion, it is worth noting that our theory is general enough to cover concrete models of the Boltzmann equation such as Cabannes' 14-velocity model and the 6-velocity model with multiple collisions. The readers are referred to [3] for these applications.

1.2 Basic results and reformulation

A vector ϕ which is orthogonal to the collision term $Q(F)$ for each $F \in \mathbb{R}^m$ is called a collision invariant. The set of the collision invariants is denoted by $\mathfrak{M}$:

$$\mathfrak{M} = \{\phi \in \mathbb{R}^m; \langle \phi, Q(F) \rangle = 0 \ \text{ for } \ {}^{\forall}F \in \mathbb{R}^m\}. \tag{1.12}$$

$\mathfrak{M}$ is not an empty set nor the total space $\mathbb{R}^m$ owing to the formulation of the collision term $Q(F)$. Thus, let d $(1 \leq d \leq m-1)$ denote the dimension of $\mathfrak{M}$, $\{\phi_i\}_{i=1,\dots,d}$ the basis of the subspace $\mathfrak{M}$ and $\{\phi_i\}_{i=d+1,\dots,m}$ the basis of the orthogonal complement $\mathfrak{M}^\perp$ of $\mathfrak{M}$;

$$\mathfrak{M} = \operatorname{span}\{\phi_1,\ \phi_2, \dots, \phi_d\}, \quad \mathfrak{M}^\perp = \operatorname{span}\{\phi_{d+1},\ \phi_{d+2}, \dots, \phi_m\}. \tag{1.13}$$

Taking the inner product of (1.1) and a collision invariant $\phi = (\phi_i)_{i\in\Lambda} \in \mathfrak{M}$, we have a conservation law:

$$\partial_t \sum_{i=1}^m \nu_i \phi_i F_i + \partial_x \sum_{i=1}^m \nu_i \phi_i F_i = 0.$$

Also, the direct computation yields Boltzmann H-theorem:

$$\partial_t \sum_{i\in\Lambda} \nu_i F_i \log F_i + \partial_x \sum_{i\in\Lambda} \nu_i v_i F_i \log F_i = \langle \log F, Q(F) \rangle \leq 0 \tag{1.14}$$

where $\log F := (\log F_i)_{i\in\Lambda}$. The last equality in (1.14) holds if and only if F is a Maxwellian, i.e., $Q(F) = 0$.

It is convenient to introduce an unknown function $\tilde{f}$ and express solutions to (1.8) by

$$\tilde{F} = M + I_M \tilde{f}, \tag{1.15}$$

where $I_M = \operatorname{diag}(M_i)$. Substituting (1.15) in (1.8), (1.9) and (1.10), we have

$$V_M \tilde{f}_x + L_M \tilde{f} = \Gamma_M(\tilde{f}), \tag{1.16}$$
$$(R^+ I^\nu - \mathcal{B} R^-) I_M \tilde{f}(0) = -\mu, \tag{1.17}$$
$$\tilde{f}(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{1.18}$$

where

$$V_M = I^\nu V I_M = \mathrm{diag}(\nu_i v_i M_i) \tag{1.19}$$
$$L_M = -D_F Q(M) I_M, \tag{1.20}$$
$$\Gamma_M(\tilde{f}) = Q(M + I_M\tilde{f}) - Q(M) - D_F Q(M) I_M \tilde{f}. \tag{1.21}$$

It is known that the linearized collision operator L_M is real symmetric and non-negative definite. Moreover, it holds that

$$\mathfrak{N}(L_M) = \mathfrak{M}, \quad \mathfrak{R}(L_M) = \mathfrak{M}^\perp,$$
$$\Gamma_M(\varphi) \in \mathfrak{M}^\perp \quad \text{for} \quad {}^\forall\varphi \in \mathbb{R}^m.$$

The quantity μ in the right hand side of (1.17) is given by

$$\mu = (\mu_i)_{i\in\Lambda_+} := (R^+ I^\nu - \mathcal{B}R^-)M, \quad \mu_i = \nu_i M_i - \sum_{j\in\Lambda_-} \mathcal{B}_{ij} M_j \quad \text{for} \quad i \in \Lambda_+. \tag{1.22}$$

μ measures the distance between the prescribed asymptotic Maxwellian state M and a boundary state satisfying the reflective boundary condition (1.6). It is shown in Theorem 2.1 that if the stationary solution exists then the consistency condition (1.23) holds:

$$\mu \in (R^+ I^\nu - \mathcal{B}R^-)(V^\nu \mathfrak{M})^\perp. \tag{1.23}$$

2 Assumptions and main results

First, we state assumptions necessary in showing the existence of a stationary solution.

[S.1] If $L_M\phi = 0$ and $V_M\phi = 0$ for $\phi \in \mathbb{R}^m$, then $\phi = 0$.

$$\dim R^+ \mathfrak{N}_M^\perp = {}^\#\{\gamma < 0;\ \det(\gamma V_M + L_M) = 0\}, \tag{2.1}$$

where we count the multiplicity of generalized eigenvalues γ.

$$\mathcal{B}R^-\,(V^\nu\mathfrak{M})^\perp \subset R^+ I^\nu\,(V^\nu\mathfrak{M})^\perp. \tag{2.2}$$
$$\nu_j v_j + \sum_{i\in\Lambda_+} v_i \mathcal{B}_{ij} \lneqq 0 \quad \text{for} \quad j \in \Lambda_-. \tag{2.3}$$
$$m_- \mathcal{B}_{ij} M_j + \mu_i \geq 0 \quad \text{for} \quad (i,j) \in \Lambda_+ \times \Lambda_-. \tag{2.4}$$

where $m_- := {}^\#\Lambda_-$.

We use the notations:

$$|\mu| = \sum_{i\in\Lambda_+} |\mu_i|, \qquad |g|_\sigma = \sup_{x\geq 0} e^{\sigma x}|g(x)|, \tag{2.5}$$

where σ is an arbitrary positive constant satisfying

$$\sigma \lneqq \bar{\sigma} = \min\{|\gamma|;\ \gamma < 0,\ \det(\gamma V_M + L_M) = 0\}. \tag{2.6}$$

Theorem 2.1. (i) *Suppose that the stationary problem,* (1.8), (1.9) *and* (1.10), *admits a solution. Then the asymptotic Maxwellian state* M *satisfies the consistency condition* (1.23).
(ii) *Suppose that conditions* **[S.1]**, (2.1), (2.2), (2.3) *and* (2.4) *hold. Also, let the consistency condition* (1.23) *hold. Then, there exists a positive constant* $\bar{\mu}$ *such that if* $|\mu| \leq \bar{\mu}$, *the stationary problem,* (1.8), (1.9) *and* (1.10), *has a unique solution* $\tilde{F} = (\tilde{F}_i)_{i\in\Lambda}$ *in a small neighborhood of the Maxwellian state* M *with respect to the norm* $|\cdot|_\sigma$ *defined by* (2.5). *Furthermore, this solution* $\tilde{F}(x)$ *belongs to* $C^\infty[0,\infty)$ *and verifies the estimate*

$$|\partial_x^k(\tilde{F}(x) - M)| \leq C_k|\mu|e^{-\sigma x} \tag{2.7}$$

for each integer $k \geq 0$, *where* C_k *is a positive constant depending on* k *and* σ.

The stronger condition than **[S.1]** is necessary to prove the stability of the stationary solution:

[S.2] If $L_M\phi = 0$ and $V\phi = \gamma\phi$ for $\phi \in \mathbb{R}^m$, $^\exists\gamma \in \mathbb{R}$, then $\phi = 0$.

Theorem 2.2. *Suppose that conditions* **[S.2]**, (2.2), (2.3) *and* (2.4) *hold as well as the stationary solution* $\tilde{F}(x)$ *exists. Then there exists a positive constant* δ_0 *such that if* $\|F_0 - M\|_1 \leq \delta_0$, *the initial boundary value problem* (1.5), (1.6) *and* (1.7) *has a unique global solution* $F(x,t)$ *in the class of functions,* $F - M \in C^0([0,\infty); H^1(\mathbb{R}_+)) \cap C^1([0,\infty); L^2(\mathbb{R}_+))$. *Furthermore, the solution* $F(x,t)$ *is asymptotically stable. Namely, it holds that*

$$\sup_{x\in\mathbb{R}_+} |F(x,t) - \tilde{F}(x)| \to 0 \quad \textit{as} \quad t \to \infty. \tag{2.8}$$

3 Outline of proofs

3.1 Existence of stationary solutions

Proof of (i) in Theorem 2.1. Taking the inner product of $\phi \in \mathfrak{M}$ and the equation (1.8) yields that

$$\langle I^\nu V\phi, F\rangle_x = 0. \tag{3.1}$$

Integrating (3.1) over $[0,\infty)$, we obtain that

$$\langle I^\nu V\phi, F(0) - M\rangle = 0. \tag{3.2}$$

This equality (3.2) implies $F(0) - M \in (I^\nu V\mathfrak{M})^\perp$. Then, by using the boundary condition (1.9) we have

$$(R^+I^\nu - \mathcal{B}R^-)M \in (R^+I^\nu - \mathcal{B}R^-)(I^\nu V\mathfrak{M})^\perp.$$

This is the consistency condition (1.23). □

Outline of proof of (ii) in Theorem 2.1. As this proof needs algebraic preparation, we state the outline only. For details, see [2] and [3]. The proof is divided into three steps.
1st step. We consider the linearized system with diffusive boundary.

$$V_M \tilde{f}_x + L_M \tilde{f} = h, \tag{3.3}$$
$$R^+ \tilde{f}(0) = b, \tag{3.4}$$
$$\tilde{f}(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{3.5}$$

where $h(x) \in \mathfrak{M}^\perp$. It is shown in [2] that the solution to this problem is given by the formula:

$$\tilde{f} = \Theta(b,h)(x).$$

The explicit formula of Θ is given in [2].
2nd step. We consider the linearized system with reflective boundary, (3.3), (1.17) and (3.5). It is shown that $R^+\tilde{f}(0)$ is uniquely determined by the problem (3.3), (1.17) and (3.5) for a fixed M. Thus, we may regard $b := R^+\tilde{f}(0)$ as the function of μ and obtain the solution formula to the reflection boundary problem as

$$\tilde{f} = \Theta(b(\mu),h)(x). \tag{3.6}$$

3rd step. Replacing h by $\Gamma_M(\tilde{f})$ in (3.6), we have

$$\tilde{f} = \Theta(b(\mu),\Gamma_M(\tilde{f}))(x). \tag{3.7}$$

Thus, the stationary wave $\tilde{f}$ to (1.16), (1.17) and (1.18) is a solution to (3.7). The existence of a solution to (3.7) is confirmed by the contraction mapping principle. To this end, we introduce a Banach space and its closed subset,

$$\mathfrak{X} = \{\tilde{f} \in \mathfrak{B}^0[0,\infty); |\tilde{f}|_\sigma < \infty\},$$
$$\mathfrak{S}_R = \{\tilde{f} \in \mathfrak{X}; |\tilde{f}|_\sigma \le R|\mu|\}.$$

Then, it is shown that Θ is a contraction map in $\mathfrak{S}_R$ with suitably chosen R, provided that $|\mu| \ll 1$. □

3.2 Stability of stationary solutions

We introduce new known function $f = (f_i)_{i\in\Lambda}$ by

$$F = \tilde{F} + I_M f = M + I_M(\tilde{f} + f),$$

and obtain from (1.5), (1.6) and (1.7) that

$$I^\nu I_M(f_t + Vf_x) + L_M f + L(x)f = N(x,f), \tag{3.8}$$
$$(R^+I^\nu - \mathfrak{B}R^-)I_M f(0,t) = 0, \tag{3.9}$$
$$f(x,0) = f_0(x) := I_M^{-1}(F_0(x) - M) - \tilde{f}(x), \tag{3.10}$$

where

$$L(x) = \left(D_F Q(M) - D_F Q(M + I_M \tilde{f})\right) I_M, \tag{3.11}$$

$$N(x, f) = Q(M + I_M \tilde{f} + I_M f) - Q(M + I_M \tilde{f}) - D_F Q(M + I_M \tilde{f}) I_M f. \tag{3.12}$$

Sometimes it is convenient to rewrite (3.8) as

$$I^\nu I_M (f_t + V f_x) = Q(F) - Q(\tilde{F}). \tag{3.13}$$

The following norms are used.

$$N(t) = \sup_{0 \le \tau \le t} \|f(\tau)\|_1,$$

$$M(t)^2 = \int_0^t \|f_x(\tau)\|^2 + \|f_t(\tau)\|^2 + |f^-(0,\tau)|^2 + |f_t^-(0,\tau)|^2 \, d\tau,$$

where $f^- = R^- f$.

Theorem 2.2 follows from the next proposition.

Proposition 3.1. *Suppose that the stability condition* **[S.2]** *holds. Furthermore, assume the conditions* (2.4) *and* (2.3). *Let* $f = (f_i)_{i \in \Lambda}$ *be a solution to the problem* (3.8), (3.9) *and* (3.10), *satisfying*

$$f \in C^0([0,T]; H^1(\mathbb{R}_+)) \cap C^1([0,T]; L^2(\mathbb{R}_+))$$

for a certain $T > 0$. *Then there is a positive constant* $\bar{\delta}$ *independent of* T *and* $|\mu|$ *such that if* $N(T) + |\mu| \le \bar{\delta}$, *then it verifies the estimate:*

$$\|f(t)\|_1^2 + \int_0^t \|f_x(\tau)\|^2 + \|f_t(\tau)\|^2 \, d\tau \le \bar{C} \|f_0\|_1^2, \tag{3.14}$$

where $0 \le t \le T$ *and* $\bar{C} > 1$ *is a constant independent of* T *and* $|\mu|$.

The difficulty of proving the above proposition arises from the fact that we have no information of the monotonicity of the stationary solution $\tilde{f}$. Usually, the monotonicity of the traveling wave plays the essential role to estimate the error terms in the energy method. This difficulty is overcome by taking advantage of the exponential convergence at the spatial asymptotic point proved in Theorem 2.1.

Actually, the following estimates hold since the stationary solution decays sufficiently fast.

Lemma 3.2.

$$\int_0^\infty |\partial_x^k \tilde{f}| |f^1|^2 \, dx \le C|\mu| \left(|f^-(0,\tau)|^2 + \|f_x^1\|^2\right) \tag{3.15}$$

$$\int_0^\infty |\partial_x^k \tilde{f}| |f^0|^2 dx \le C|\mu| \left(|f^-(0,\tau)|^2 + \|f_x^1\|^2 + \|f_t^0\|^2\right) \tag{3.16}$$

$$\int_0^\infty |\partial_x^k \tilde{f}| |f|^2 dx \le C|\mu| \left(|f^-(0,\tau)|^2 + \|f_x^1\|^2 + \|f_t^0\|^2\right) \tag{3.17}$$

for $k = 0, 1, 2, \ldots$, *where* $f^1 = P_1 f$ *and* $f^0 = P_0 f$. P_1 *and* P_0 *are the projections on* $\mathfrak{R}(V)$ *and* $\mathfrak{N}(V)$, *respectively.*

Proof. First, observe the elemental equality:

$$f^1(x,t) = f^1(0,t) + \int_0^x 1 \cdot \frac{d}{dy} f^1(x,t)\,dy.$$

Thus, we obtain that

$$|f^1(x,t)| \le |f^1(0,t)| + \sqrt{x}\|f_x^1\|. \tag{3.18}$$

Square (3.18), multiply by $|\partial_x^k \tilde{f}| \le C|\mu|e^{-\sigma x}$ and then integrate the resulting inequality over $x > 0$. Consequently,

$$\begin{aligned}\int_0^\infty |\partial_x^k f^\infty||f^1|^2\,dx &\le \int_0^\infty C|\mu|e^{-\sigma x}\left(|f^1(0,\tau)|^2 + x\|f_x^1(\tau)\|^2\right)\,dx \\ &\le C|\mu|\left(|f^1(0,\tau)|^2 + \|f_x^1\|^2\right).\end{aligned}$$

Then applying the equality $|f^1(0,\tau)|^2 \le C|f^-(0,\tau)|^2$, which is due to (3.9), we have the estimate (3.15).

Solve (3.8) with respect to f^0 by the implicit function theorem and estimate the resultant equality to obtain that

$$|f^0| \le C(|f^1| + |f_t^0|).$$

Then, apply the estimate (3.15). This gives the estimate (3.16). Adding estimates (3.15) and (3.16) yields (3.17). □

Outline of proof of Proposition 3.1. Proposition (3.1) is proved by the energy method, which is divided into the following 4 steps.

1st step: Estimate of f, (3.19).

2nd step: Estimate of f_t, (3.25).

3rd step: Estimate of f_x, (3.26).

4th step: Estimate of the remained terms, (3.37).

Summing up these four estimates yields the estimate (3.14).

Lemma 3.3 (1st step).

$$\|f(t)\|^2 + \int_0^t |f^-(0,\tau)|^2\,d\tau + \int_0^t \|Q(F) - Q(\tilde{F})\|^2\,d\tau \le C\|f_0\|^2 + C|\mu|M(t)^2. \tag{3.19}$$

Proof. Substitute $\tilde{F} = (\tilde{F}_i)_{i\in\Lambda}$ in (1.14) to obtain that

$$\partial_t \sum_{i\in\Lambda} \nu_i \tilde{F}_i \log \tilde{F}_i + \partial_x \sum_{i\in\Lambda} \nu_i v_i \tilde{F}_i \log \tilde{F}_i = \langle \log \tilde{F}, Q(\tilde{F})\rangle. \tag{3.20}$$

Multiply (1.1) by $1 + \log \tilde{F}_i(x)$ and sum up with respect to $i \in \Lambda$. The result is that

$$\partial_t \sum_{i\in\Lambda} \nu_i (1 + \log \tilde{F}_i)(F_i - \tilde{F}_i) + \partial_x \sum_{i\in\Lambda} \nu_i v_i \left(\tilde{F}_i \log \tilde{F}_i + (1 + \log \tilde{F}_i)(F_i - \tilde{F}_i)\right) - \\ - \sum_{i\in\Lambda} \nu_i v_i \frac{(F_i - \tilde{F}_i)}{\tilde{F}_i} \partial_x \tilde{F}_i = \langle \log \tilde{F}, Q(F) \rangle \quad (3.21)$$

Subtracting (3.20) and (3.21) from (1.14),

$$\begin{aligned} \partial_t \sum_{i\in\Lambda} \nu_i \Phi(F_i, \tilde{F}_i) + \partial_x \sum_{i\in\Lambda} \nu_i v_i \Phi(F_i, \tilde{F}_i) - \sum_{i\in\Lambda} \nu_i v_i \Psi(F_i, \tilde{F}_i) \partial_x \tilde{F}_i \\ = \langle \log F - \log \tilde{F}, Q(F) - Q(\tilde{F}) \rangle \\ \le -c|Q(F) - Q(\tilde{F})|^2 + C|\tilde{F} - M|^2 |f|^2, \end{aligned} \quad (3.22)$$

where

$$\Phi(F_i, \tilde{F}_i) = F_i \log F_i - \tilde{F}_i \log \tilde{F}_i - (1 + \log \tilde{F}_i)(F_i - \tilde{F}_i) \sim |F_i - \tilde{F}_i|^2 \sim |f_i|^2 \quad (3.23)$$

$$\Psi(F_i, \tilde{F}_i) = \log F_i - \log \tilde{F}_i - \frac{F_i - \tilde{F}_i}{\tilde{F}_i} = O(|F_i - \tilde{F}_i|^2) = O(|f_i|^2). \quad (3.24)$$

The inequality in (3.22) is obtained from estimating the collision term $Q(F)$.

Integrate (3.22) over $[0, t] \times (0, \infty)$ and estimate the integration with respect to t on the boundary $x = 0$ by using (2.3) to obtain (3.19). □

Lemma 3.4 (2nd step).

$$\|f_t(t)\|^2 + \int_0^t |f_t^-(0, \tau)|^2 + \|P_L f_t(\tau)\|^2 \, d\tau \le C_2 \|f_t(0)\|^2 + C_2(|\mu| + N(t)) M(t)^2 \quad (3.25)$$

where P_L is the projection on $\mathfrak{R}(L_M) = \mathfrak{M}^\perp$.

Proof. Apply ∂_t to (3.8), take the inner product with f_t, and integrate over $[0, t] \times (0, \infty)$. Then, use (2.3) to estimate integration in t on $x = 0$ and obtain the desired inequality (3.25). □

Lemma 3.5 (3rd step).

$$\|f_x(t)\|^2 + \int_0^t \|f_x(t)\|^2 \, d\tau \le C\left(\|f_0\|_1^2 + \|f_t(0)\|^2\right) + C\left(|\mu| + N(t)\right) M(t)^2. \quad (3.26)$$

Proof. The estimate (3.26) is given by summing up the following 3 estimates.

$$\|f_x^1(t)\|^2 \le C(\|f_0\|^2 + \|f_t(0)\|^2) + C\left(|\mu| + N(t)\right) M(t)^2, \quad (3.27)$$

$$\int_0^t \|f_x^1(t)\|^2 \, d\tau \le C(\|f_0\|^2 + \|f_t(0)\|^2) + C\left(|\mu| + N(t)\right) M(t)^2, \quad (3.28)$$

$$\|f_x^0(t)\|^2 + \int_0^t \|f_x^0(t)\|^2 \, d\tau \le C(\|f_0\|^2 + \|f_t(0)\|^2) + C\left(|\mu| + N(t)\right) M(t)^2. \quad (3.29)$$

Derivation of (3.27). From (3.13), it holds that

$$Vf_x = -f_t + (I^\nu I_M)^{-1}(Q(F) - Q(\tilde{F})). \tag{3.30}$$

Square (3.30) and integrate the resultant equality over $[0,t] \times (0,\infty)$. Using Lemma 3.3, we have the estimate (3.27).
Derivation of (3.28). It is proved in citeSK that the stability condition **[S.2]** implies that there exists a skew-symmetric matrix K_0 such that

$$K_0 = O \text{ on } \mathfrak{N}, \tag{3.31}$$

$$\langle (K_0V - VK_0)\phi, \phi\rangle + \langle VL_M V\phi, \phi\rangle \geq c|P^1\phi|^2. \tag{3.32}$$

Thus, it holds from (3.31) that

$$\langle (K_0V - VK_0)\phi, \phi\rangle \geq c|P^1\phi|^2 - C|P_L V\phi|. \tag{3.33}$$

Multiply (3.13) by $(I^\nu I_M)^{-1}$

$$f_t + Vf_x = (I^\nu I_M)^{-1}\left(Q(F) - Q(\tilde{F})\right). \tag{3.34}$$

Multiply the equality (3.34) by $2K_0$ and take the inner product with f_x to obtain

$$\begin{aligned}\langle K_0 f, f_x\rangle_t + \langle K_0 f, f_t\rangle_x + \langle (K_0V - VK_0)f_x, f_x\rangle& \\ = -\langle 2K_0(I^\nu I_M)^{-1}(Q(F) - Q(\tilde{F}), f_x\rangle&. \end{aligned}\tag{3.35}$$

Integrate (3.35) over $[0,t] \times (0,\infty)$ and apply (3.33). Then, estimate integration in t on the boundary $x = 0$ with using (2.3) to obtain (3.28).
Derivation of (3.29). Apply P_0 on the equation (3.8) to obtain

$$I^\nu I_M f_t^0 + P_0 L_M P_0 f = -P_0 L_M(I - P_0)f + P_0(-L(x)f + N(x,f)). \tag{3.36}$$

$P_0 L_M P_0$ is real symmetric and positive definite on $\mathfrak{N}(V)$ owing to the stability condition **[S.1]**. Apply ∂_x on (3.36), take the inner product the resultant equality with f_x^0 and integrate over $[0,t] \times (0,\infty)$. Then applying the estimates (3.17) and (3.28), we obtain the desired estimate (3.29). □

Lemma 3.6 (4th step).

$$\int_0^t \|f_t(\tau)\|^2 \, d\tau \leq C(\|f_0\|_1^2 + \|f_t(0)\|^2) + C(|\mu| + N(t))M(t)^2. \tag{3.37}$$

Proof. From (3.13), we have

$$f_t = -Vf_x + (I^\nu I_M)^{-1}(Q(F) - Q(\tilde{F})).$$

Square this equality, integrate the resultant equality over $[0,\infty) \times [0,t]$, and apply the estimates (3.19) and (3.28). Consequently, we have the inequality (3.37). □

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References

[1] H. CABANNES, The discrete Boltzmann equation (Theory and applications), *Lecture Notes, Univ. of California, Berkeley,* (1980).

[2] S. KAWASHIMA AND S. NISHIBATA, Existence of a stationary wave for the discrete Boltzmann equation in the half space, *Comm. Math. Phys.* to appear.

[3] S. KAWASHIMA AND S. NISHIBATA, Stationary waves for the discrete Boltzmann in the half space with reflective boundaries, to appear.

[4] Y. NIKKUNI AND S. KAWASHIMA , Stability of stationary solutions to the half-space problem for the discrete Boltzmann equation with multiple collisions, *Kyushu J. Math.* to appear.

[5] Y. SHIZUTA AND S. KAWASHIMA, Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation, *Hokkaido Math. J.*, **14** (1985), 249-275.

[6] S. UKAI, On the Half-Space Problem for Discrete Velocity Model of the Boltzmann Equation, *Advances in Nonlinear Partial Differential Equations and Stochastics, S. Kawashima and T. Yanagisawa (ed.) Series on Advances in Mathematics for Applied Sciences*, **48** (1998), 160-174.