

## 準周期的 Duffing 方程式の解の存在と一意性および近似解の 精度保証について

By

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The present paper is concerned with the existence and uniqueness of the quasiperiodic solutions to the Duffing's General Type equation

$$(1) \quad \frac{d^2x}{dt^2} + 2\mu \frac{dx}{dt} + \nu^2 x = \varepsilon x^3 + \sum_{k=1}^m (a_k \cos \nu_k t + b_k \sin \nu_k t),$$

where  $\mu, \nu, \nu_k = 2\pi/w_k$  and,  $w_k (k = 1, 2, \dots, m)$  are all positive, and  $1/w_k$  are rationally linearly independent. Putting  $y = dx/dt$ ,

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -\nu^2 & -2\mu \end{pmatrix}, \quad \phi(t) = \begin{pmatrix} 0 \\ \sum_{k=1}^m (a_k \cos \nu_k t + b_k \sin \nu_k t) \end{pmatrix},$$

$$\eta(z) = \begin{pmatrix} 0 \\ x^3 \end{pmatrix},$$

the equation (1) can be written in the vector form as follows:

$$(2) \quad \frac{dz}{dt} = Az + \phi(t) + \varepsilon \eta(z).$$

The linear differential equation

$$(3) \quad Lz = \phi(t), \quad L = \frac{d}{dt} - A$$

satisfies the generalized exponential dichotomy for  $\mu \neq 0$ . Let  $\Phi(t)$  be the fundamental matrix of the equation  $Lz = 0$  such that  $\Phi(0) = E$  (unit matrix), we have

$$(4) \quad \|\Phi(t)\| \leq K_0 e^{-\sigma_0 t}.$$

In the paper we use the  $\ell_\infty$ -norm  $\|\cdot\|$  in Euclidean space and denote that  $\|f\| = \sup_{t \in \mathbb{R}} \|f(t)\|$  for any bounded function  $f = f(t)$ .

Then there exist a projection matrix  $P$  (Duffing:  $P = E$ ), positive numbers  $\sigma_1, \sigma_2$  and non-negative functions  $C_1(t, s), C_2(t, s)$  such that

$$(5) \begin{cases} (i) P^2 = P \\ (ii) \|\Phi(t)P\Phi^{-1}(s)\| \leq C_1(t, s)e^{-\sigma_1(t-s)} & \text{for } t \geq s, \\ (iii) \|\Phi(t)(E - P)\Phi^{-1}(s)\| \leq C_2(t, s)e^{-\sigma_2(s-t)} & \text{for } t < s, \\ (iv) \int_{-\infty}^t C_1(t, s)e^{-\sigma_1(t-s)} ds + \int_t^{+\infty} C_2(t, s)e^{-\sigma_2(s-t)} ds \leq M, \end{cases}$$

where

$$M = \begin{cases} \frac{K_0}{\mu - \sqrt{\mu^2 - \nu^2}} & \text{if } \mu > \nu > 0, \\ \int_{-\infty}^t K_0(t-s)e^{-\mu(t-s)} ds & \text{if } \mu = \nu, \\ \frac{K_0}{\mu} & \text{if } 0 < \mu < \nu. \end{cases}$$

**Theorem 1** Consider a nonlinear differential equation

$$(6) \quad \frac{dz}{dt} = X(t, z),$$

where  $z$  and  $X(t, z)$  are vectors and  $X(t, z)$  is quasiperiodic in  $t$  with periods  $w_1, w_2, \dots, w_m$  and is continuously differentiable with respect to  $z$  belonging to a region  $D$  of  $z$ -space.

Suppose that there is a continuously differentiable quasiperiodic function  $z_0(t)$  with periods  $w_1, w_2, \dots, w_m$  such that

$$\begin{cases} z_0(t) \in D, \\ \left\| \frac{dz_0(t)}{dt} - X[t, z_0(t)] \right\| \leq r, \end{cases}$$

for all  $t \in \mathbb{R}$ .

Further suppose that there are a positive number  $\delta$ , a non-negative number  $\kappa < 1$  and a quasiperiodic matrix  $A(t)$  with periods  $w_1, w_2, \dots, w_m$  such that

(i) the equation (3) satisfies a generalized exponential dichotomy,

(ii)  $D_\delta = \{z; \|z - z_0(t)\| \leq \delta \text{ for some } t \in \mathbb{R}\} \subset D$ ,

(iii)  $\|\Psi(t, z) - A(t)\| \leq \frac{\kappa}{M}$  whenever  $\|z - z_0(t)\| \leq \delta$ ,

and

(iv)  $\frac{Mr}{1 - \kappa} \leq \delta$ ,

where  $\Psi(t, z)$  is the Jacobian matrix of  $X(t, z)$  with respect to  $z$ .

Then the equation (6) possesses a solution  $z = \hat{z}$  quasiperiodic in  $t$  with periods  $w_1, w_2, \dots, w_m$  such that

$$(7) \quad \|z_0(t) - \hat{z}(t)\| \leq \frac{Mr}{1 - \kappa}.$$

for all  $t \in R$ . Furthermore, to the equation (6) there is no other quasiperiodic solution belonging to  $D_\delta$  besides  $z = \hat{z}(t)$ .

Let  $z_0(t)$  be the quasiperiodic solution to the linear equation (3) and bounded by  $K$  as  $\|z_0(t)\| \leq K$ .

**Theorem 2** *If the parameter  $\varepsilon$  satisfies the inequality*

$$(8) \quad |\varepsilon| \leq \frac{1}{13K^2M},$$

*the equation (1) possesses a quasiperiodic solution  $z = \hat{z}(t)$  with periods  $w_1, w_2, \dots, w_m$  such that*

$$(9) \quad \|z_0(t) - \hat{z}(t)\| \leq K$$

*for all  $t \in R$ .*

If the inequality (8) does not hold, or the error estimation (9) is too crude, we should compute a more accurate approximation than  $z_0(t)$ . For this purpose the Galerkin method is usefull.

Starting from the solution  $z = z_0(t)$  of the equation (3), we can compute the following Galerkin approximation of  $n$ -th order to the solution  $x(t)$  of the equation (1) :

$$(10) \quad x_n(t) = \alpha(0, 0) + \sum_{r=1}^n \sum_{|p|=r} \{\alpha_p \cos(p, \nu)t + \beta_p \sin(p, \nu)t\},$$

$$(p, \nu) = \sum_{k=1}^m p_k \nu_k, |p| = \sum_{k=1}^m |p_k|.$$

Theorem 1 in the paper makes us to verify the existence and uniqueness of an exact quasiperiodic solution and know the error bound of the approximation (10) to the quasiperiodic solution of Duffing's general type equation (1) which is a fundamental equation in nonlinear oscillations.

## 参考文献

- [1] Y.Shinohara, Investigation of the quasiperiodic solutions to Duffing's general type equations, Bulletin of the Faculty of Engineering, The University of Tokushima, Vol.44(1999), 1-13.