Superconvergence and Nonsuperconvergence of the Shortley-Weller Approximation for Dirichlet Problems

YAMAMOTO Tetsuro (山本 哲朗)†
FANG Qing (方 青)†
CHEN Xiaojun (陳 小君)‡
†Department of Mathematical Sciences, Ehime University
‡Department of Mathematics and Computer Science, Shimane University

1. Introduction

Consider the Dirichlet problem

$-\Delta u = f(x,y) \quad \text{in } \Omega,$

$u = g(x,y) \quad \text{on } \Gamma = \partial \Omega,$

(1.1)
(1.2)

where $\Omega$ is a bounded domain of $\mathbb{R}^2$ and $f, g$ are given functions. We assume the (unique) existence of a solution $u$ for (1.1)-(1.2).

It was recently shown by Yamamoto [8] and Matsunaga-Yamamoto [7] that the Shortley-Weller approximation applied to (1.1)-(1.2) had a superconvergence property and numerical examples illustrating this fact were also given there.

To state the result, we construct a net over $\overline{\Omega} = \Omega \cup \Gamma$ by the grid points $P_{ij} = (x_i, y_j)$ in $\overline{\Omega}$ with the mesh size $h$. The set of the grid points is denoted by $\Omega_h$. We denote by $\mathcal{P}_\Gamma$ the set of points $P_{ij}$ such that at least one of $(x_i \pm h, y_j), (x_i, y_j \pm h)$ does not belong to $\Omega$ and put $\mathcal{P}_0 = \Omega_h \setminus \mathcal{P}_\Gamma$. Furthermore, we denote by $\Gamma_h$ the set of points of intersection of grid lines with $\Gamma$ and $\mathcal{S}_h(\kappa)$ by the set of points $P_{ij} \in \Omega_h$ which satisfy $\text{dist}(P_{ij}, \Gamma) \leq \kappa h$, where $\kappa$ is a constant with $\kappa > 1$, which is arbitrary chosen independently of $h$. We define the neighbors of $P \in \Omega_h$
to be four points in $\Omega_h \cup \Gamma_h$ which are adjacent to $P$ and on horizontal and vertical grid lines through $P$. As is shown in Figures 1.1 and 1.2, these points are denoted by $P_E, P_W, P_S, P_N$ and their distance to $P$ by $h_E, h_W, h_S, h_N$. Note that at least one of $P_E, P_W, P_S, P_N$ is on $\Gamma$ if and only if $P \in \mathcal{P}_\Gamma$ and that all of them are in $\Omega$ if and only if $P \in \mathcal{P}_0$, in which case we have $h_E = h_W = h_S = h_N = h$.

![Figure 1.1](image1.png) ![Figure 1.2](image2.png)

We denote by $U(P)$ an approximate solution at $P \in \Omega_h$. Then the Shortley-Weller (S-W) approximation $-\Delta_h^{(SW)}$ for $-\Delta$ at $P$ is defined by

$$-\Delta_h^{(SW)} U(P) = \left( \frac{2}{h_E h_W} + \frac{2}{h_S h_N} \right) U(P) - \frac{2}{h_E (h_E + h_W)} U(P_E) \right)$$

$$- \frac{2}{h_W (h_E + h_W)} U(P_W) - \frac{2}{h_S (h_S + h_N)} U(P_S)$$

$$- \frac{2}{h_N (h_S + h_N)} U(P_N),$$

which includes the usual centered five point formula

$$-\Delta_h U(P) = \frac{1}{h^2} \left[ 4U(P) - U(P_E) - U(P_W) - U(P_S) - U(P_N) \right]$$

as a special case $h_E = h_W = h_S = h_N = h$. Hence, if $P \in \mathcal{P}_0$, then the S-W approximation means the centered five point approximation.

As is easily seen, if $u \in C^{3,1}(\bar{\Omega})$, then the local truncation error $\tau^{(SW)}(P) \equiv -[\Delta_h^{(SW)} u(P) - \Delta u(P)]$ of the S-W formula at $P$ is estimated by

$$|\tau^{(SW)}(P)| \leq \begin{cases} \frac{2h^2}{3} O(h^2) & \text{if } h_E = h_W = h_S = h_N = h \\ \frac{2M_3}{3} h = O(h) & \text{otherwise,} \end{cases}$$
where $L$ is a Lipschitz constant common to all third order derivatives $\partial^3/\partial x^i\partial y^{3-i}$, $0 \leq i \leq 3$ and

$$M_3 = \sup_{P \in \Omega} \left\{ \left| \frac{\partial^3 u(P)}{\partial x^i\partial y^{3-i}} \right| : i = 0, 1, 2, 3 \right\}.$$  

Then the following result holds for the S-W approximation.

**Theorem 1.1** (Superconvergence of the S-W approximation [8], [7])

Let $\Omega$ be a bounded convex domain with a piecewise $C^{2,\alpha}$ boundary. If $u \in C^{l+2,\alpha}(\overline{\Omega})$, $l = 0$ or 1, $\alpha \in (0,1]$, then

$$|u(P) - U(P)| \leq \begin{cases} O(h^{l+1+\alpha}) & P \in S_h(\kappa) \\ O(h^{l+\alpha}) & \text{otherwise} \end{cases}$$

This implies that if $u \in C^{3,1}(\overline{\Omega})$, then we have

$$u(P) - U(P) = O(h^3) \quad \text{at } P \in S_h(\kappa)$$

even if $\tau^{(SW)}(P) = O(h)$ and $u(P) - U(P) = O(h^2)$ at other grid points.

Theorem 1.1 is a refinement of the following result due to Bramble-Hubbard [1]:

**Theorem 1.2.** If $u \in C^{4}(\overline{\Omega})$, then

$$|u(P) - U(P)| \leq \frac{M_4}{96}d^2 h^2 + \frac{2M_3}{3} h^3 = O(h^2) \quad \forall P \in \Omega_h,$$

where

$$M_4 = \sup_{P \in \Omega} \left\{ \left| \frac{\partial^4 u(P)}{\partial x^i\partial y^{4-i}} \right| : i = 0, 1, 2, 3, 4 \right\}$$

and $d$ denotes the diameter of the smallest circle containing $\Omega$.

It is also known by Matsunaga's numerical experiments [4] that even if $u \in C^{4}(\overline{\Omega})$, the Bramble and the Collatz approximations do not have the superconvergence property like Theorem 1.1, although both have $O(h^2)$ accuracy at every $P \in \Omega_h$.

Now, we are interested in the behavior of the S-W approximate solution for the case $u \notin C^{l+2,\alpha}(\overline{\Omega})$. Has the S-W approximation any superconvergence property for such a case? The purpose of this paper is to answer
this question: Three examples with $\Omega = (0, 1) \times (0, 1)$ are given in § 2, which show three kinds of different behavior: (i) nonsuperconvergence at any point of $\Omega_h$, (ii) superconvergence near a part of $\Gamma$ and (iii) superconvergence in a neighborhood of a point of $\Gamma$. Furthermore, in § 3, we shall give two theorems by which the above phenomena can be illustrated.

2. Numerical Examples

In this section, we give three examples in which the S-W approximations applied to (1.1)-(1.2) show different behaviors.

Example 2.1. Let $f$ and $g$ be chosen so that the function

$$u = \sqrt{x(1-x)} + \sqrt{y(1-y)}$$

is the solution of (1.1)-(1.2). Observe that $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$, but $u \notin H^1(\Omega)$. Then, as is shown in Table 2.1, we see

$$u(P) - U(P) = O(h^{1/2}) \quad \forall P \in \Omega_h$$

(2.1)

and nonsuperconvergence occurs at any point in $\Omega_h$.

| $\alpha$ | $\max_{P \in \Omega_h} |u(P) - U(P)| / h^\alpha$ |
|---------|---------------------------------|
| $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.6$ |
| $h = 1.0e-01$ | 2.942673010e-01 | 3.704605831e-01 | 4.663822421e-01 |
| $h = 5.0e-02$ | 2.730515389e-01 | 3.684237579e-01 | 4.971078573e-01 |
| $h = 2.0e-02$ | 2.478926487e-01 | 3.665731474e-01 | 5.420728410e-01 |
| $h = 1.0e-02$ | 2.308381843e-01 | 3.658538669e-01 | 5.798393031e-01 |
| $h = 5.0e-03$ | 2.151573948e-01 | 3.654763480e-01 | 6.208151064e-01 |

$\alpha = 0.4$ | $\max_{P \in S_h(2)} |u(P) - U(P)| / h^\alpha$ |
| $\alpha = 0.6$ |
| $h = 1.0e-01$ | 2.804064720e-01 | 3.530108332e-01 | 4.444143086e-01 |
| $h = 5.0e-02$ | 2.596048461e-01 | 3.502803661e-01 | 4.726272898e-01 |
| $h = 2.0e-02$ | 2.358898589e-01 | 3.488239302e-01 | 5.158260507e-01 |
| $h = 1.0e-02$ | 2.197937251e-01 | 3.483495786e-01 | 5.520968758e-01 |
| $h = 5.0e-03$ | 2.049350740e-01 | 3.481122390e-01 | 5.913196240e-01 |

Table 2.1

It should also be remarked that $u^{(4)}(Q) = O(h^{1/2-4})$ if $Q$ is close to the boundary $\Gamma$ and the local truncation error $\tau(Q)$ approaches to infinity as
$Q$ approaches to $\Gamma$. The distribution of errors $|u(P) - U(P)|$ in the case $h = 1.0e-002$ is shown in Figure 2.1.

Example 2.2. Let $f$ and $g$ be chosen so that the function

$$u = \sqrt{x} + y$$

is the solution of (1.1)-(1.2). Then

$$|u(P) - U(P)| = \begin{cases} O(h^{3/2}) & \text{near } \{(1, y) | 0 \leq y \leq 1\} \\ O(h^{1/2}) & \text{otherwise.} \end{cases} \quad (2.2)$$

The results are shown in Table 2.2 and Figure 2.2 for $h = 1.0e-002$.

<table>
<thead>
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<td>2.743224392e-001</td>
</tr>
<tr>
<td>$h = 5.0e-002$</td>
<td>2.299787622e-001</td>
<td>3.103063992e-001</td>
</tr>
<tr>
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<td>2.271939190e-001</td>
<td>3.285509309e-001</td>
</tr>
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<td>2.246577933e-001</td>
<td>3.322144274e-001</td>
</tr>
<tr>
<td>$h = 1.0e-002$</td>
<td>2.166254535e-001</td>
<td>3.433282065e-001</td>
</tr>
</tbody>
</table>

max $|u(P) - U(P)|/h^\alpha$ $(P \in \Omega_h)$

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max $|u(P) - U(P)|/h^\alpha$ $(P \in S_h(2)$ and away from $x = 1$)

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<th>$\alpha$ = 1.6</th>
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<td>7.969656240e-002</td>
<td>1.263105392e-001</td>
</tr>
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</table>

Table 2.2

In this case, a superconvergence occurs near the side $x = 1$ of $\Gamma$. 
Example 2.3. Let $f$ and $g$ be chosen so that the function $u = \sqrt{x} + \sqrt{y}$ is the solution of (1.1)-(1.2). Then

$$|u(P) - U(P)| = \begin{cases} O(h^{3/2}) & \text{near the corner (1,1)}, \\ O(h^{1/2}) & \text{otherwise.} \end{cases} \quad (2.3)$$

The superconvergence occurs only near the corner (1,1). (See Figure 2.3 for the case $h = 1.0e-002$).

In the above examples, observe that the S-W approximation works well, although

$$\max_{P \in \Omega_h} |\tau^{SW}(P)| \rightarrow +\infty \quad \text{as } h \rightarrow 0.$$

This is a nice feature of the finite difference method.

3. Convergence Theorems

It is possible to give mathematical proofs for the error estimates (2.1)-(2.3). We can first prove the following results for the two-point boundary value problem

$$-u''(x) = \varphi(x), \quad 0 < x < 1 \quad (3.1)$$
$$u(0) = \alpha, \quad u(1) = \beta, \quad (3.2)$$

where $\varphi$ is a given function and $\alpha, \beta$ are given constants.

**Theorem 3.1.** Let $d(x) = \min(x, 1-x), \quad 0 < x < 1$. If $0 < p < 1$, and the solution $u(x)$ of (3.1)-(3.2) belongs to $C^4(0,1)$ and satisfies

$$\sup_{x \in (0,1)} \frac{d(x)^k |u^{(k)}(x)|}{d(x)^p} < \infty, \quad k = 0, 1, 2, 3, 4,$$

then

$$|u_i - U_i| = O(h^p) \quad \forall i,$$

where $\{U_i\}$ is the finite difference solution for (3.1)-(3.2) and $u_i = u(x_i)$, $x_i = ih, \quad i = 0, 1, 2, \cdots, n + 1, \quad h = 1/(n + 1)$. That is, superconvergence does not occur at any $x_i \in \Omega_h$.\"
Theorem 3.2. If the solution $u(x)$ of (3.1)-(3.2) satisfies
\[ \sup_{x \in (0, 1)} \frac{x^k |u^{(k)}(x)|}{x^p} < \infty, \quad k = 0, 1, 2, 3, 4 \] (3.3)
with some constant $p \in (0, 1)$, then
\[ |u_i - U_i| \leq \begin{cases} O(h^{p+1}) & \text{near } x = 1 \\ O(h^p) & \text{otherwise.} \end{cases} \]

That is, superconvergence occurs near $x = 1$.

Theorems 3.1 and 3.2 can be derived with the use of the fact (e.g. Yamamoto-Ikebe [9]) that the inverse of the $n \times n$ tridiagonal matrix
\[ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 & -1 \\ & \ddots & \ddots & \ddots \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix} \]
is given by
\[ A^{-1} = (\alpha_{ij}), \quad \alpha_{ij} = \begin{cases} i(1 - \frac{j}{n+1}) & (i \leq j) \\ j(1 - \frac{i}{n+1}) & (i > j) \end{cases} \]
so that
\[ h \alpha_{ij} = \begin{cases} x_i(1 - x_j) & (i \leq j) \\ x_j(1 - x_i) & (i > j) \end{cases}. \]

Now, consider the Dirichlet problem
\[ -\Delta u = F_1(x) + F_2(y) \quad \text{in } \Omega = (0, 1) \times (0, 1), \] (3.4)
\[ u = G_1(x) + G_2(y) \quad \text{on } \Gamma \] (3.5)
and \{U_{ij}\} be the S-W approximation with the equal mesh size $h_E = h_W = h_S = h_N = h$ at every $P \in \Omega_h$. Let \{U_i^{(1)}\} and \{U_i^{(2)}\} be the usual finite difference solution for the two-point boundary value problems
\[ -u''(x) = F_1(x), \quad 0 < x < 1 \]
\[ u(0) = G_1(0), \quad u(1) = G_1(1) \]
and
\[-u''(y) = F_2(y),\quad 0 < y < 1\]
\[u(0) = G_2(0), \quad u(1) = G_2(1),\]
respectively. Then, by the uniqueness of the S-W approximate solution applied to (3.4)-(3.5), we have
\[U_{ij} = U_i^{(1)} + U_j^{(2)}, \quad \forall i, j.\]
Hence, all the phenomena stated in § 2 can now be illustrated with the use of Theorems 3.1 and 3.2 with \(p = 1/2\).

Note: Proofs of Theorems 3.1 and 3.2 will be given elsewhere.

References


