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Superconvergence and Nonsuperconvergence of the Shortley-Weller Approximation for Dirichlet Problems

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1. Introduction

Consider the Dirichlet problem

$$-\Delta u = f(x,y) \qquad \text{in } \Omega, \tag{1.1}$$

$$u = g(x,y) \qquad \text{on } \Gamma = \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^2$ and f, g are given functions. We assume the (unique) existence of a solution u for (1.1)-(1.2).

It was recently shown by Yamamoto [8] and Matsunaga-Yamamoto [7] that the Shortley-Weller approximation applied to (1.1)-(1.2) had a superconvergence property and numerical examples illustrating this fact were also given there.

To state the result, we construct a net over $\overline{\Omega} = \Omega \cup \Gamma$ by the grid points $P_{ij} = (x_i, y_j)$ in $\overline{\Omega}$ with the mesh size h. The set of the grid points is denoted by Ω_h. We denote by $\mathcal{P}_\Gamma$ the set of points P_{ij} such that at least one of $(x_i \pm h, y_j)$, $(x_i, y_j \pm h)$ does not belong to Ω and put $\mathcal{P}_0 = \Omega_h \backslash \mathcal{P}_\Gamma$. Furthermore, we denote by Γ_h the set of points of intersection of grid lines with Γ and $\mathcal{S}_h(\kappa)$ by the set of points $P_{ij} \in \Omega_h$ which satisfy $\mathrm{dist}(P_{ij}, \Gamma) \leq \kappa h$, where κ is a constant with $\kappa > 1$, which is arbitrary chosen independently of h. We define the neighbors of $P \in \Omega_h$

to be four points in $\Omega_h \cup \Gamma_h$ which are adjacent to P and on horizontal and vertical grid lines through P. As is shown in Figures 1.1 and 1.2, these points are denoted by P_E, P_W, P_S, P_N and their distance to P by h_E, h_W, h_S, h_N. Note that at least one of P_E, P_W, P_S, P_N is on Γ if and only if $P \in \mathcal{P}_\Gamma$ and that all of them are in Ω if and only if $P \in \mathcal{P}_0$, in which case we have $h_E = h_W = h_S = h_N = h$.

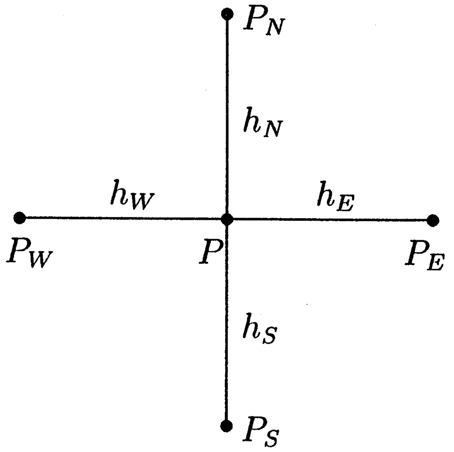

Figure 1.1

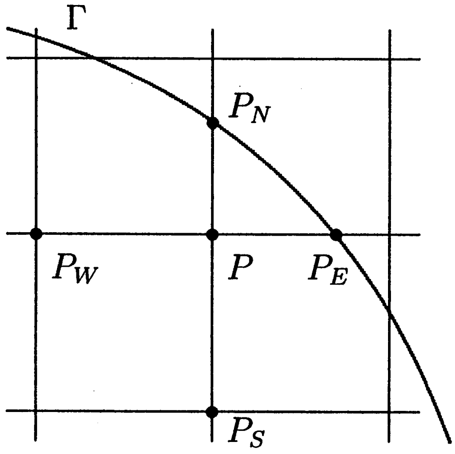

Figure 1.2

We denote by $U(P)$ an approximate solution at $P \in \Omega_h$. Then the Shortley-Weller (S-W) approximation $-\Delta_h^{(SW)}$ for $-\Delta$ at P is defined by

$$\begin{aligned} -\Delta_h^{(SW)}U(P) &= \left(\frac{2}{h_E h_W} + \frac{2}{h_S h_N}\right)U(P) - \frac{2}{h_E(h_E+h_W)}U(P_E) \\ &\quad - \frac{2}{h_W(h_E+h_W)}U(P_W) - \frac{2}{h_S(h_S+h_N)}U(P_S) \\ &\quad - \frac{2}{h_N(h_S+h_N)}U(P_N), \end{aligned}$$

which includes the usual centered five point formula

$$-\Delta_h U(P) = \frac{1}{h^2}[4U(P) - U(P_E) - U(P_W) - U(P_S) - U(P_N)]$$

as a special case $h_E = h_W = h_S = h_N = h$. Hence, if $P \in \mathcal{P}_0$, then the S-W approximation means the centered five point approximation.

As is easily seen, if $u \in C^{3,1}(\overline{\Omega})$, then the local truncation error $\tau^{(SW)}(P) \equiv -[\Delta_h^{(SW)}u(P) - \Delta u(P)]$ of the S-W formula at P is estimated by

$$|\tau^{(SW)}(P)| \le \begin{cases} \frac{2L}{3}h^2 = O(h^2) & \text{if } h_E = h_W = h_S = h_N = h \\ \frac{2M_3}{3}h = O(h) & \text{otherwise,} \end{cases}$$

where L is a Lipschitz constant common to all third order derivatives $\partial^3/\partial x^i \partial y^{3-i}$, $0 \leq i \leq 3$ and

$$M_3 = \sup_{P\in\Omega}\{|\frac{\partial^3 u(P)}{\partial x^i \partial y^{3-i}}| \mid i = 0,1,2,3\}.$$

Then the following result holds for the S-W approximation.

Theorem 1.1 (Superconvergence of the S-W approximation [8], [7])
Let Ω be a bounded convex domain with a piecewise $C^{2,\alpha}$ boundary. If $u \in C^{l+2,\alpha}(\overline{\Omega})$, $l = 0$ or 1, $\alpha \in (0,1]$, then

$$|u(P) - U(P)| \leq \begin{cases} O(h^{l+1+\alpha}) & P \in \mathcal{S}_h(\kappa) \\ O(h^{l+\alpha}) & \text{otherwise.} \end{cases}$$

This implies that if $u \in C^{3,1}(\overline{\Omega})$, then we have

$$u(P) - U(P) = O(h^3) \quad \text{at } P \in \mathcal{S}_h(\kappa)$$

even if $\tau^{(SW)}(P) = O(h)$ and $u(P) - U(P) = O(h^2)$ at other grid points.
Theorem 1.1 is a refinement of the following result due to Bramble-Hubbard [1]:

Theorem 1.2. *If $u \in C^4(\overline{\Omega})$, then*

$$|u(P) - U(P)| \leq \frac{M_4}{96}d^2h^2 + \frac{2M_3}{3}h^3 = O(h^2) \quad \forall P \in \Omega_h,$$

where

$$M_4 = \sup_{P\in\Omega}\{|\frac{\partial^4 u(P)}{\partial x^i \partial y^{4-i}}| \mid i = 0,1,2,3,4\}$$

and d denotes the diameter of the smallest circle containing Ω.

It is also known by Matsunaga's numerical experiments [4] that even if $u \in C^4(\overline{\Omega})$, the Bramble and the Collatz approximations do not have the superconvergence property like Theorem 1.1, although both have $O(h^2)$ accuracy at every $P \in \Omega_h$.

Now, we are interested in the behavior of the S-W approximate solution for the case $u \notin C^{l+2,\alpha}(\overline{\Omega})$. Has the S-W approximation any superconvergence property for such a case? The purpose of this paper is to answer this question: Three examples with $\Omega = (0,1) \times (0,1)$ are given in § 2, which show three kinds of different behavior: (i) nonsuperconvergence at any point of Ω_h, (ii) superconvergence near a part of Γ and (iii) superconvergence in a neighborhood of a point of Γ. Furthermore, in § 3, we shall give two theorems by which the above phenomena can be illustrated.

2. Numerical Examples

In this section, we give three examples in which the S-W approximations applied to (1.1)-(1.2) show different behaviors.

Example 2.1. Let f and g be chosen so that the function

$$u = \sqrt{x(1-x)} + \sqrt{y(1-y)}$$

is the solution of (1.1)-(1.2). Observe that $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$, but $u \notin H^1(\Omega)$. Then, as is shown in Table 2.1, we see

$$u(P) - U(P) = O(h^{1/2}) \quad \forall P \in \Omega_h \tag{2.1}$$

and nonsuperconvergence occurs at any point in Ω_h.

$\max_{P\in\Omega_h} \lvert u(P) - U(P)\rvert / h^\alpha$			
	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$
h = 1.0e-001	2.942673010e-001	3.704605831e-001	4.663822421e-001
h = 5.0e-002	2.730515389e-001	3.684237579e-001	4.971078573e-001
h = 2.0e-002	2.478926487e-001	3.665731474e-001	5.420728410e-001
h = 1.0e-002	2.308381843e-001	3.658538669e-001	5.798393031e-001
h = 5.0e-003	2.151573948e-001	3.654763480e-001	6.208151064e-001
$\max_{P\in S_h(2)} \lvert u(P) - U(P)\rvert / h^\alpha$			
	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$
h = 1.0e-001	2.804064720e-001	3.530108332e-001	4.444143086e-001
h = 5.0e-002	2.596048461e-001	3.502803661e-001	4.726272898e-001
h = 2.0e-002	2.358898589e-001	3.488239302e-001	5.158260507e-001
h = 1.0e-002	2.197937251e-001	3.483495786e-001	5.520968758e-001
h = 5.0e-003	2.049350740e-001	3.481122390e-001	5.913196240e-001

Table 2.1

It should also be remarked that $u^{(4)}(Q) = O(h^{1/2-4})$ if Q is close to the boundary Γ and the local truncation error $\tau(Q)$ approaches to infinity as

Q approaches to Γ. The distribution of errors $|u(P) - U(P)|$ in the case $h = 1.0\text{e-}002$ is shown in Figure 2.1.

Example 2.2. Let f and g be chosen so that the function

$$u = \sqrt{x} + y$$

is the solution of (1.1)-(1.2). Then

$$|u(P) - U(P)| = \begin{cases} O(h^{3/2}) & \text{near } \{(1,y)|0 \leq y \leq 1\} \\ O(h^{1/2}) & \text{otherwise.} \end{cases} \tag{2.2}$$

The results are shown in Table 2.2 and Figure 2.2 for $h = 1.0\text{e-}002$.

$\max|u(P) - U(P)|/h^\alpha \quad (P \in \Omega_h)$			
	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$
h = 1.0e-001	2.179020589e-001	2.743224392e-001	3.453514897e-001
h = 5.0e-002	2.299787622e-001	3.103063992e-001	4.186911020e-001
h = 2.5e-002	2.271939190e-001	3.285509309e-001	4.751258955e-001
h = 2.0e-002	2.246577933e-001	3.322144274e-001	4.912646216e-001
h = 1.0e-002	2.166254535e-001	3.433282065e-001	5.441385373e-001
$\max|u(P) - U(P)|/h^\alpha \quad (P \in S_h(2)$ and away from $x = 1)$			
	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$
h = 1.0e-001	2.179020589e-001	2.743224392e-001	3.453514897e-001
h = 5.0e-002	2.299787622e-001	3.103063992e-001	4.186911020e-001
h = 2.5e-002	2.271939190e-001	3.285509309e-001	4.751258955e-001
h = 2.0e-002	2.246577933e-001	3.322144274e-001	4.912646216e-001
h = 1.0e-002	2.142439333e-001	3.395537514e-001	5.381564290e-001
$\max|u(P) - U(P)|/h^\alpha \quad (P \in S_h(2)$ and near $x = 1)$			
	$\alpha = 1.4$	$\alpha = 1.5$	$\alpha = 1.6$
h = 1.0e-001	1.035198677e-001	1.303237921e-001	1.640679336e-001
h = 5.0e-002	9.450738537e-002	1.275171941e-001	1.720567627e-001
h = 2.5e-002	8.756602147e-002	1.266314609e-001	1.831249910e-001
h = 2.0e-002	8.554801688e-002	1.265047833e-001	1.870699143e-001
h = 1.0e-002	7.969656240e-002	1.263105392e-001	2.001887137e-001

Table 2.2

In this case, a superconvergence occurs near the side $x = 1$ of Γ.

Example 2.3. Let f and g be chosen so that the function $u = \sqrt{x} + \sqrt{y}$ is the solution of (1.1)-(1.2). Then

$$|u(P) - U(P)| = \begin{cases} O(h^{3/2}) & \text{near the corner (1,1),} \\ O(h^{1/2}) & \text{otherwise.} \end{cases} \tag{2.3}$$

The superconvergence occurs only near the corner (1,1). (See Figure 2.3 for the case $h = 1.0\text{e-}002$).

In the above examples, observe that the S-W approximation works well, although

$$\max_{P\in\Omega_h} |\tau^{(SW)}(P)| \to +\infty \quad \text{as } h \to 0.$$

This is a nice feature of the finite difference method.

3. Convergence Theorems

It is possible to give mathematical proofs for the error estimates (2.1)-(2.3). We can first prove the following results for the two-point boundary value problem

$$-u''(x) = \varphi(x), \quad 0 < x < 1 \tag{3.1}$$

$$u(0) = \alpha, \; u(1) = \beta, \tag{3.2}$$

where φ is a given function and α, β are given constants.

Theorem 3.1. *Let* $d(x) = \min(x, 1-x)$, $0 < x < 1$. *If* $0 < p < 1$, *and the solution* $u(x)$ *of* (3.1)-(3.2) *belongs to* $C^4(0,1)$ *and satisfies*

$$\sup_{x\in(0,1)} \frac{d(x)^k |u^{(k)}(x)|}{d(x)^p} < \infty, \quad k = 0,1,2,3,4,$$

then

$$|u_i - U_i| = O(h^p) \quad \forall i,$$

where $\{U_i\}$ *is the finite difference solution for* (3.1)-(3.2) *and* $u_i = u(x_i)$, $x_i = ih$, $i = 0,1,2,\cdots,n+1$, $h = 1/(n+1)$. *That is, superconvergence does not occur at any* $x_i \in \Omega_h$.

Theorem 3.2. *If the solution $u(x)$ of* (3.1)-(3.2) *satisfies*

$$\sup_{x\in(0,1)} \frac{x^k|u^{(k)}(x)|}{x^p} < \infty, \quad k = 0,1,2,3,4 \tag{3.3}$$

with some constant $p \in (0,1)$, then

$$|u_i - U_i| \le \begin{cases} O(h^{p+1}) & \text{near } x = 1 \\ O(h^p) & \text{otherwise.} \end{cases}$$

That is, superconvergence occurs near $x = 1$.

Theorems 3.1 and 3.2 can be derived with the use of the fact (e.g. Yamamoto-Ikebe [9]) that the inverse of the $n \times n$ tridiagonal matrix

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

is given by

$$A^{-1} = (\alpha_{ij}), \quad \alpha_{ij} = \begin{cases} i(1 - \frac{j}{n+1}) & (i \le j) \\ j(1 - \frac{i}{n+1}) & (i > j) \end{cases}$$

so that

$$h\alpha_{ij} = \begin{cases} x_i(1 - x_j) & (i \le j) \\ x_j(1 - x_i) & (i > j). \end{cases}$$

Now, consider the Dirichlet problem

$$-\Delta u = F_1(x) + F_2(y) \qquad \text{in } \Omega = (0,1) \times (0,1), \tag{3.4}$$

$$u = G_1(x) + G_2(y) \qquad \text{on } \Gamma \tag{3.5}$$

and $\{U_{ij}\}$ be the S-W approximation with the equal mesh size $h_E = h_W = h_S = h_N = h$ at every $P \in \Omega_h$. Let $\{U_i^{(1)}\}$ and $\{U_i^{(2)}\}$ be the usual finite difference solution for the two-point boundary value problems

$$\begin{aligned} &-u''(x) = F_1(x), \qquad 0 < x < 1 \\ &u(0) = G_1(0), \ u(1) = G_1(1) \end{aligned}$$

and

$$\begin{aligned} -u''(y) &= F_2(y), \qquad\qquad 0 < y < 1 \\ u(0) &= G_2(0),\ u(1) = G_2(1), \end{aligned}$$

respectively. Then, by the uniqueness of the S-W approximate solution applied to (3.4)-(3.5), we have

$$U_{ij} = U_i^{(1)} + U_j^{(2)}, \quad \forall i, j.$$

Hence, all the phenomena stated in § 2 can now be illustrated with the use of Theorems 3.1 and 3.2 with $p = 1/2$.

Note: Proofs of Theorems 3.1 and 3.2 will be given elsewhere.

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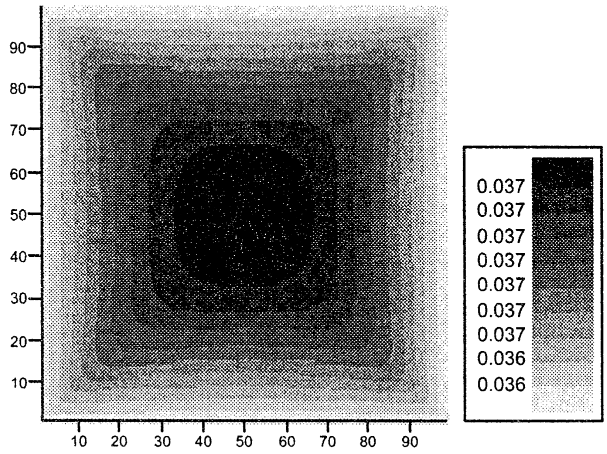

Figure 2.1

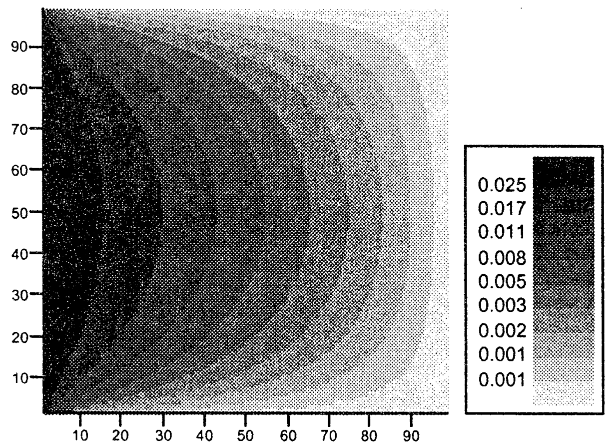

Figure 2.2

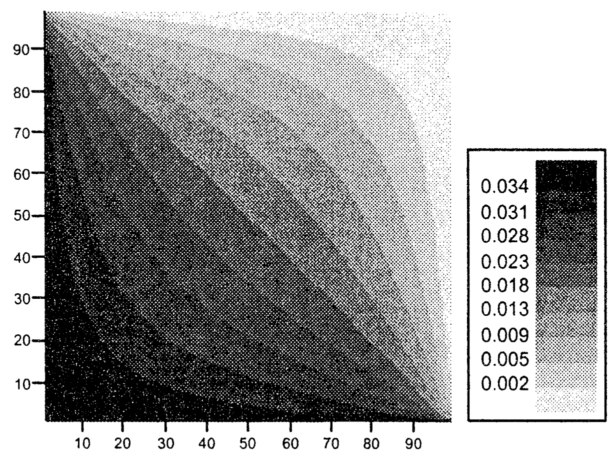

Figure 2.3