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LOCAL SUBGROUPS AND GROUP ALGEBRAS OF FINITE p-SOLVABLE GROUPS

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1. Introduction

Let k be an algebraically closed field of prime characteristic p, and let G and H be finite groups with Sylow p-subgroups P and Q, respectively. In representation theory of finite groups it seems important to consider a problem that if the two group algebras kG and kH are isomorphic as k-algebras, then which kind of properties of G can be heritable to H?

In this talk we consider this problem for a property that $\mathrm{N}_G(P)/P$ is abelian, where $\mathrm{N}_G(P)$ is the normalizer of P in G. Namely, we want to know whether the property $\mathrm{N}_G(P)/P$ is abelian implies that $\mathrm{N}_H(Q)/Q$ is abelian under the case that $kG \cong kH$ as k-algebras. Here, actually, we consider the above problem for p-nilpotent groups and groups of p-length 1. It seems that this problem is difficult even if groups are p-nilpotent. For a p-nilpotent group G, we give some necessary conditions for $\mathrm{N}_G(P)/P$ to be abelian, but they cannot be sufficient conditions since there exist trivial counter examples. For a group G of p-length 1, we give some necessary and sufficient conditions for $\mathrm{N}_G(P)/P$ to be abelian, but they contain some group theoretic condition. It seems that the problem for groups of p-length 1 can be reduced to one for p-nilpotent groups.

This is a joint work with Professor Shigeo Koshitani.

2. Preliminary

Let H be a finite group, and let K be a finite group acting on H. Then $\mathrm{Irr}(H)$ denotes the set of all irreducible ordinary characters of H, $\mathrm{LIrr}(H)$ denotes the set of all linear ordinary characters of H, and $\mathrm{Irr}_K(H)$ and $\mathrm{LIrr}_K(H)$ denote the set of all K-invariant irreducible characters and the set of K-invariant linear characters of H, respectively. We fix a prime p and an algebraically closed field k of characteristic p, and $\mathrm{IBr}(H)$ denotes the set of all irreducible p-Brauer characters of H. For a k-algebra A, $\mathrm{IRR}(A)$ denotes the set of all non-isomorphic irreducible (simple) A-modules, and $\mathrm{IRR}^0(A)$ denotes the set of all non-isomorphic irreducible A-modules whose k-dimensions are not divisible by p. For the group algebra kG of a p-solvable group G and $S \in \mathrm{IRR}(kG)$, it follows from [2, Theorem 2.1] that S is in $\mathrm{IRR}^0(kG)$ if and only if the vertex of S is a Sylow p-subgroup of G. We write $[G, G]$ for the commutator subgroup of G and $|\mathrm{IRR}(A)|$ for the number of elements of $\mathrm{IRR}(A)$ for a k-algebra A. For other notation and terminology see the books of Isaacs [3] and Nagao and Tsushima [6]. Throughout this paper groups mean always finite groups.

First we introduce some results related to our problem.

Proposition 2.1. *Let G and H be finite groups, and let P and Q be Sylow p-subgroups of G and H, respectively. Assume that $kG \cong kH$ as k-algebras. Then*

(1) *if G is p-nilpotent, then so is H,*
(2) [Okuyama-Michler] *if G is p-closed, then so is H,*
(3) [Morita] *if $G/\mathrm{O}_{p',p}(G)$ is abelian, then so is $H/\mathrm{O}_{p',p}(H)$,*
(4) [Navarro] *if G is q-nilpotent, then so is H, for $p \neq q$,*
(5) *if G is of p-length 1, then so is H.*

Proof. (1) Well known.

(2) Okuyama [9, Theorem 2] for $p = 2$, and Michler [4, Theorem 5.5] for $p \neq 2$. It should be noted that in his proof the classification of finite simple groups is used in the proof of Michler [4].

(3) Morita [5, Theorem 6].

(4) Navarro [7, Theorem].

(5) is proved essentially by almost the same argument in [9] and (2). It seems that the proof is unpublished, but we omit it here since we do not need this result for our argument. □

Let A be a k-algebra. We say A is *primary* if $A/\mathrm{J}(\mathrm{A})$ is a simple ring, and A is *quasi-primary* if $A/\mathrm{J}(\mathrm{A})$ is a direct sum of isomorphic simple rings.

Theorem 2.2. [5, Theorem 6, 7] *A finite group G is p-nilpotent iff every block of the groups algebra kG is primary, and $G/\mathrm{O}_{\mathrm{p}',\mathrm{p}}(\mathrm{G})$ is abelian iff every block of the groups algebra kG is quasi-primary.*

A block B of kG is quasi-primary if and only if all irreducible B-modules have the same dimensions.

We prepare one more easy group theoretic lemma.

Lemma 2.3. *Assume that G is a finite group of p-length 1 with Sylow p-subgroup P. Then*

(1) $G = \mathrm{N}_G(P)\mathrm{O}_{p'}(G)$,
(2) *if $\mathrm{N}_G(P)/P$ is abelian, then so is $G/\mathrm{O}_{p',p}(G)$.*

Proof. (1) By Frattini argument, we have $G = \mathrm{N}_G(P)\mathrm{O}_{p',p}(G)$. Now the result holds clearly.

(2) By (1), $G/\mathrm{O}_{p',p}(G) \cong \mathrm{N}_G(P)/(\mathrm{N}_G(P) \cap \mathrm{O}_{p',p}(G))$. Since $\mathrm{N}_G(P) \cap \mathrm{O}_{p',p}(G)$ contains P, there is an epimorphism from $\mathrm{N}_G(P)/P$ to $G/\mathrm{O}_{p',p}(G)$. □

3. p-NILPOTENT CASE

Now we consider the condition that $\mathrm{N}_G(P)/P$ is abelian for a finite group G with a Sylow p-subgroup P. Note that $N_G(P)/P \cong \mathrm{C}_{\mathrm{O}_{p'}(G)}(P)$ for a p-nilpotent group G with Sylow p-subgroup P. In this section, we use character theoretic descriptions.

Theorem 3.1. *Let H be a finite p'-group, and P a finite p-group acting on H. Assume that $\mathrm{C}_H(P)$ is abelian, $\chi \in \mathrm{Irr}_P(H)$, and $\phi \in \mathrm{LIrr}_P(H)$ which is non-trivial. Then $\chi \neq \chi\phi$.*

Proof. Put $M = \mathrm{C}_H(P)$. Then there exists the Glauberman correspondence $\pi : \mathrm{Irr}_P(H) \to \mathrm{Irr}(M)$ (See [3, §13]). By [3, Theorem 13.1(c)], $\pi(\chi\phi) = \pi(\chi)\phi_M$. Since M is abelian, $\pi(\chi)$ is linear. So if ϕ_M is non-trivial, then $\pi(\chi) \neq \pi(\chi\phi)$ and thus $\chi \neq \chi\phi$.

By [1, Exercise 8.8], $H = M[H, P]$. Since ϕ is P-invariant and linear, $[H, P]$ is contained in the kernel of ϕ. So if ϕ_M is trivial, then ϕ must be trivial. Now the result is proved. □

Corollary 3.2. *Let G be a p-nilpotent group with a Sylow p-subgroup P. If $\mathrm{N}_G(P)/P$ is abelian, then the number of linear characters of G divides the number of irreducible characters of G of degree d for any positive integer d with $p \nmid d$.*

Proof. Put $H = \mathrm{O}_{p'}(G)$. Then $\mathrm{N}_G(P)/P \cong \mathrm{C}_H(P)$. Every P-invariant character of H is extendible to G and the number of its extensions is $|P : [P, P]|$. So $|\mathrm{LIrr}(G)| = |\mathrm{LIrr}_P(H)||P : [P, P]|$. Let $\chi \in \mathrm{Irr}_P(H)$. Then, by Theorem 3.1, there are $|\mathrm{LIrr}_P(H)|$ distinct characters of the form $\chi\phi$, $\phi \in \mathrm{LIrr}_P(H)$, and each of them has $|P : [P, P]|$ extensions. Thus the assertion holds. □

The converse of Corollary 3.2 is true for groups of small order, for example, for 3-nilpotent groups of order $2^n \cdot 3$, $n \leq 7$. But there exists a trivial counter example of it, consider a simple group of p'-order with the trivial action of an arbitrary p-group.

4. p-LENGTH 1 CASE

In this section, we use module theoretic descriptions.

Theorem 4.1. *Let G be a finite group of p-length 1 with a Sylow p-subgroup P. The following are equivalent.*

(1) $\mathrm{N}_G(P)/P$ *is abelian.*

(2) $\mathrm{N}_G(P) \cap \mathrm{O}_{p'}(G)$ *is abelian, every block of kG is quasi-primary, and the restriction S to $\mathrm{O}_{p'}(G)$ is irreducible for every irreducible kG-module S with $p \nmid \dim_k S$.*

Proof. Put $N = \mathrm{N}_G(P)$, $E = \mathrm{O}_{p'}(G)$, and $M = N \cap E$.

Assume (2). We can define the restriction map $R : \mathrm{IRR}^0(kG) \to \mathrm{IRR}_P(kE)$. First we shall show that R is surjective. Let $X \in \mathrm{IRR}_P(E)$. Then X can be extended to PE. Let $S \in \mathrm{IRR}(kG)$ such that S_E has X as a direct summand. Since G is p-solvable, by [3, Corollary 11.29] and Fong-Swan's theorem, we have $p \nmid \dim_k S$. Thus $S_E = X$, and R is surjective. Also R is a $|G : PE[G,G]|$ to 1 map.

Let $\pi : \mathrm{IRR}_P(kE) \to \mathrm{IRR}(kM)$ be the Glauberman correspondence. Let $X \in \mathrm{IRR}_P(kE)$. By Lemma 2.3(1) and [8, Theorem 4.9 (2)], X is extendible to G if and only if $\pi(X)$ is extendible to N. Since every $X \in \mathrm{IRR}_P(kE)$ is extendible to G, so is every $Y \in \mathrm{IRR}(kM)$ to N, and the number of extensions of Y to N is $|N : PM[N,N]|$. But $|G : PE[G,G]| = |N : PM[N,N]|$ since $G/E \cong N/M$. By [8, Theorem 4.1], $|\mathrm{IRR}^0(kG)| = |\mathrm{IRR}(kN)|$. This yields that every irreducible kN-module restricts irreducibly to M. Since M is abelian, every irreducible kN-module is of dimension one, and thus N/P is abelian.

Assume (1). By Lemma 2.3(2), G/PE and M are both abelian. Let $X \in \mathrm{IRR}_P(E)$. Since N/P is abelian, $\pi(X)$ is extendible to N, and so is X to G. Similar argument as the above yields (2). □

Corollary 4.2. *Let G be a finite group of p-length 1 with a Sylow p-subgroup P. Assume $\mathrm{N}_G(P)/P$ is abelian. Then the number of irreducible kG-modules of k-dimension one divides the number of irreducible kG-modules of k-dimension d for any positive integer d with $p \nmid d$.*

Proof. Put $E = \mathrm{O}_{p'}(G)$. Let S be an irreducible kG-module with $p \nmid \dim_k S$. Then S_E is irreducible by Theorem 4.1. So we can define the restriction map $R : \mathrm{IRR}^0(kG) \to \mathrm{Irr}_P(E)$. As in the proof of Theorem 4.1, R is surjective and for any element $\chi \in \mathrm{Irr}_P(E)$ there are exactly $|G : PE|$ distinct elements in $\mathrm{IRR}^0(kG)$ which are sent to χ through R, and clearly R preserves the degrees. Now Corollary 3.2 yields the result. □

Theorem 4.3. *Let G be a finite group of p-length 1 with a Sylow p-subgroup P. Then the following are equivalent.*

(1) $\mathrm{N}_G(P)/P$ *is abelian.*

(2) $\mathrm{N}_G(P) \cap \mathrm{O}_{p'}(G)$ *is abelian, every block of* kG *is quasi-primary, and all full defect blocks of* kG *have the same numbers of irreducible modules.*

Proof. Put $N = \mathrm{N}_G(P)$ and $M = N \cap \mathrm{O}_{p'}(G)$.

Let B be a block of kG of full defect, and let b be the block of kN which is the Brauer correspondent of B. By [5], all irreducible kG-modules in B have the same degrees, and by [8, Theorem 4.9], we have $|\mathrm{IRR}(B)| = |\mathrm{IRR}(b)|$.

Assume (1). Let β be a block of kM. Since M is central in N, only one block b of kN covers β. By the assumption that $\mathrm{N}_G(P)/P$ is abelian, we have $|\mathrm{IRR}(b)| = |N : PM|$. Thus (2) holds.

Assume (2). Let b_0 is the principal block of kN. Since N/PM is abelian, $|\mathrm{IRR}(b_0)| = |N : PM|$. Thus $|\mathrm{IRR}(b)| = |N : PM|$ for any kN-block b. We know that N-conjugacy classes of $\mathrm{Irr}(M)$ correspond to blocks kN. Let $\xi \in \mathrm{Irr}(M)$, let b be a block of kN which covers blocks $\{\xi\}$ of kM, and let T be the inertial group of ξ in N. If $T \lneq N$ then $|\mathrm{IRR}(b)| \leq |T : PM| \lneq |\mathrm{IRR}(b_0)|$. So ξ is N-invariant. Since $|\mathrm{IRR}(b)| = |N : PM|$, ξ must be extendible to N and any irreducible Brauer character in b is a extension of ξ. Since M is abelian, ξ is of degree 1, and so is any irreducible Brauer character in b. Now the proof is complete. □

In Theorem 4.3(2), the conditions except $\mathrm{N}_G(P) \cap \mathrm{O}_{p'}(G)$ being abelian are characterized by the structure of kG as a k-algebra. So it seems for us that the problem for groups of p-length 1 can be reduced to one for p-nilpotent groups.

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