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Recent Results on Developable Submanifolds: Examples and Classifications

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We give non-trivial examples of compact developable submanifolds in real projecitve spaces and in spheres. Also their partial classifications are provided.

This survey article is based on the joint work with Prof. T. Morimoto [20], and on the joint work (now in progress) with Prof. M. Kimura and Prof. R. Miyaoka.

The detailed proofs will be given in a forthcoming paper.

A connected C^∞ submanifold M^m in $\mathbf{R}^n$, $\mathbf{R}P^n$ or S^n $(m < n)$ is called *developable* (or, *tangentially degenerate*, or, in the Riemannian geometry, *strongly parabolic*) if its Gauss map

$$\gamma : M \to \mathrm{Gr}(m+1, \mathbf{R}^{n+1})$$

has rank $< m$. Here the rank of γ is, by definition, the maximal value over M of the rank of the differential maps, or the linearlizations, $\gamma_* : T_xM \to T_{\gamma(x)}\mathrm{Gr}(m+1, \mathbf{R}^{n+1})$ at $x \in M$ of γ.

Remark that the developability is a notion of projective geometry; the image of a developable submanifold under a projective transformation is again developable. Also remark that a compact connected manifold with the projective structure is a real projective space or a sphere. (Consider the "developping mapping", which is a covering mapping, to the real projective space. See, for instance, [27].)

Then our viewpoint of understanding developable submanifolds is as follows: We do not need the metric structures on them for the formulation of the results, but only their projective structures, while, for the proofs of the results, we use freely the metric structures.

The developable hypersurfaces are regarded as global solutions of Monge-Ampère equations of special type [25][26]: A developable hypersurface lifts to a Legendre submanifold in the incident manifold of the projective duality, endowed with the canonical contact structure, and it is projected to the degenerate projective dual. Also a developable submanifold has a Legendre lifting with a degenerate projection.

As classical examples of developable surfaces in the three dimensional space, we have cylinders, cones and tangent developables of space curves [11][18]. Among them, only the planes have no singularities in the projective space.

Observing the singularities of developable submanifolds, we expect, also in the general case, that non-singular and compact developable submanifols in $\mathbf{R}P^n$ or S^n are heavily restrictive. In fact, it is known that a non-singular compact complex developable submanifold in $\mathbf{C}P^n$ is necessarily a projective subspace [2][15]. Also in the real case, we see that, for a C^∞ compact developable submanifold M^m in $\mathbf{R}P^n$, the maximal rank $r = \mathrm{rank}(\gamma)$ of the Gauss mapping $\gamma : M \to \mathrm{Gr}(m+1, \mathbf{R}^{n+1})$, is an even integer and it satisfies the inequality $m + 1 < \dfrac{r(r+3)}{2}$, provided $r \neq 0$ (cf. [20]).

Moreover, using [14], we see the following result: Let M^m be compact and connected, and $f : M \to \mathbf{R}P^n$ a developable immersion. Then there eixsts a number $F(m)$ (Ferus number), depending only on the dimension m of M, such that, if $r < F(m)$ then $r = 0$ and so $M = \mathbf{R}P^m$ and f is an inclusion of an m-dimensional projective subspace in $\mathbf{R}P^n$, or $M = S^m$ and f is a double covering on a projective subspace of $\mathbf{R}P^n$. The Ferus number $F(m)$ is defined by

$$F(m) := \min\{\ell \mid A(\ell) + \ell > m\},$$

where $A(\ell)$ is the Adams number: $A(\ell)$ is the maximal number of linearly independent vector fields over the sphere $S^{\ell-1}$.

In particular, if $r \leq 1$, then $r = 0$. If m is a power of 2, namely if $m = 2, 4, 8, 16, 32, \ldots$, then $r = 0$. If $m = 3, 5, 6, 7$, then $r < 4$ implies $r = 0$. If $m = 9, 10, 11, 12, 13, 14, 15$, then $r < 8$ implies $r = 0$. If $m = 17, 18, 19, 20, 21, 22, 23, 24$, then $r < 16$ implies $r = 0$. If $m = 25, 26, 27, 28, 29, 30, 31$, then $r < 24$ implies $r = 0$. See [6].

The proof of the above result is achieved by considering the Levi-Civita connection of the ordinary metric on $\mathbf{R}P^n$ or on S^n, the co-nullity operator, and a matrix Riccati-type equation [13][14].

Problem: Is the inequality $r < F(m)$ best possible for the implication $r = 0$. Do there exist developable immersions

$M^m \to \mathbf{R}P^n$ with $r = F(m)$, M being compact? Moreover can we classify developable immersions $M^m \to \mathbf{R}P^n$ with $r = F(m)$ and M compact?

There are several papers on the general theory of developable submanifolds ([6][7][28][29]). However, few results are known about examples of compact developable submanifolds, so far.

Then we have non-trivial examples of developable submanifolds: For $n = 4, 7, 13, 25$, there exists a real algebraic cubic non-singular developable hypersurface in $\mathbf{R}P^n$. These developable hypersurfaces have the structure of homogeneous spaces of groups $SO(3), SU(3), Sp(3), F_4$, respectively. Their projective duals are linear projections of Veronese embeddings of projective planes $\mathbf{K}P^2$, for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$ (the Cayley's octonians).

A C^∞ hypersurface $M \subset \mathbf{R}P^n$ is called a *Cartan hypersurface* if, $n = 4, 7, 13$ or 25, and M is projectively equivalent to one of above examples. Cartan hypersurfaces appears in the theory of isoparametric hypersurfaces [10]. Also they are obtained from real forms of Severi varieties classified by Zak [30] in the context of algebraic geometry.

Each Cartan hypersurface admits C^∞ deformations among developable hypersurfaces with $2, 3, 5, 9$ functional parameters, via C^∞ deformations of its projective dual. Remark that the family of deformations contain finite-dimensional subfamily consisting of hypersurfaces projectively equivalent to the original hypersurface. Also compare with the following fact: Any projective subspace is *rigid* among developable submanifolds. Namely any developable immersion sufficiently near to the inclusion of a projective subspace in $\mathbf{R}P^n$, relatively to the C^∞ topology, is projectively equivalent to the inclusion.

Remark that a Cartan hypersurface in $\mathbf{R}P^n$ is a G-orbit for a compact Lie subgroup $G \subset \mathrm{GL}(n+1,\mathbf{R})$. In general, we call a (projectively) homogeneous submanifold $M \subset \mathbf{R}P^n$ *of compact type* if M is a G-orbit for a compact Lie subgroup G of $\mathrm{GL}(n+1,\mathbf{R})$, under the action on $\mathbf{R}P^n$ induced by the natural linear action on $\mathbf{R}^{n+1}$. If M is of compact type, then M is compact.

Theorem([19]): Let M be a homogeneous C^∞ developable hypersurface of compact type. Then M is a projective hyperplane or a Cartan hypersurface.

Recently, R. Miyaoka has observed that focal submanifolds of isoparametric hypersurfaces with even number of principal curvatures (4 and 6) provide other examples of developable immersions. (cf. [12], page 248, Cor. 2.2.) For example, we have a developable immersion $f : M^{10} \to \mathbf{R}P^{13}, r = 8$, M being compact, from the adjoint action of G_2: $(m,r) = (10,8), F(10) = 8$.

Also we have examples of compact developable submanifolds as the pull-backs by the Hopf fibration $\pi : S^{2n+1}(\subset \mathbf{C}^{n+1}) \to \mathbf{C}P^n$ of compact complex manifolds in $\mathbf{C}P^n$. In some cases also this construction provides examples of compact developable submanifolds satisfying the equality $r = F(m)$:
$(m,r) = (3,2),(5,4),(9,8),(17,16),(25,24)$.

We have also a result, in the simplest case $n = 4, m = 3, r = 2$, on the classification of developable immersions $M^3 \to \mathbf{R}P^4$: Any developable immersion $M^3 \to \mathbf{R}P^4$ from a compact connected manifold M^3 of dimension 3, with *constant rank* 2 of Gauss mapping, are re-parametrized by a developable immersion from the "doubled Cartan hypersurface". So, for this subclass, the classification problem is reduced, in some sense, to the study on the space of developable immersions from the doubled Cartan hypersurface to $\mathbf{R}P^4$.

Moreover we construct an example of developable immersions from a compact submanifold M of dimension 3 to $\mathbf{R}P^4$, the rank of whose Gauss mapping is *not constant*: $r = 2$, but there exists a point where the rank is less than 2. For this we use the result of Kimura [22], and the results on first-order isotropic holomorphic mappings from S^2 to the complex quadric Q^3 in $\mathbf{C}P^4$ due to Bryant and Peng [9]. This type of construction can be generalised into the higher dimensional cases [24].

The classification problem of developable submanifolds is far from being solved: It seems fruitful to study the relation between developable submanifolds, isoparametric submanifolds, minimal submanifolds, hyperbolic immersions (in the sense of Gromov) [16], and so on, in metric geometry, and, besides, to establish the projective differential geometry of developable submanifolds (cf. [4]).

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