数理解析研究所講究録
1150巻 2000年 11-13

Recent Development in Geometry of Parahermitian Symmetric Spaces

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§1. Parahermitian symmetric spaces

Parahermitian symmetric spaces (PHSS's for short) are a class of (affine) symmetric spaces which are interesting from the view point of geometry and harmonic analysis. A one-sheeted hyperboloid in R^3 is the simplest example of <u>parahermitian symmetric spaces</u>. By a PHSS $(M,\omega,F^{\pm})$ we mean a symplectic symmetric space (M,ω) with a double Lagrangian foliation $F^{\pm}$ ([2]). For a PHSS $(M,\omega,F^{\pm})$ one has two kinds of automorphism groups:

$$\mathrm{Aut}(M,F^{\pm}) = \{\varphi \in \mathrm{Diffeo}(M) : \varphi_* F^{\pm} = F^{\pm}\},$$

$$\mathrm{Aut}(M,\omega,F^{\pm}) = \{\varphi \in \mathrm{Aut}(M,F^{\pm}) : \varphi^*\omega = \omega\}.$$

The latter one is always a finite-dimensional Lie group, but the former one is not in general.

Let us start with a real simple (-1,1)-GLA $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, and let $Z \in \mathfrak{g}_0$ be the element such that $\mathrm{ad}\, Z = k1$ on $\mathfrak{g}_k$. Let $\sigma = \mathrm{Ad}\exp \pi i Z$. Then $(\mathfrak{g},\mathfrak{g}_0,\sigma)$ is a symmetric triple. Let G_0 be the centralizer of Z in the automorphism group $\mathrm{Aut}\,\mathfrak{g}$. Let G be the open subgroup of $\mathrm{Aut}\,\mathfrak{g}$ generated by G_0 and $\mathrm{Ad}\,\mathfrak{g}$. Then the coset space $M = G/G_0$ is a symmetric space corresponding to $(\mathfrak{g},\mathfrak{g}_0,\sigma)$. M has a natural parahermitian structure $(\omega,F^{\pm})$. $(M,\omega,F^{\pm})$ is called the PHSS associated to the GLA $\mathfrak{g}$. There exists a one-to-one correspondence between the set of local isomorphism classes of PHSS's of simple Lie groups and the isomorphism classes of simple (-1,1)-GLA's ([3]).

§2. Automorphism groups

Consider the parabolic subgroups $U^{\pm} = G_0 \exp \mathfrak{g}_{\pm 1}$ of G. The flag manifolds $M^{\pm} = G/U^{\pm}$ are called symmetric R-spaces. The product manifold $\tilde{M} = M^- \times M^+$ has the natural double foliation $\mathcal{M}^{\pm}$ whose leaves are G-translates of $M^{\pm}$. The group G acts on $\tilde{M}$ diagonally. Let r be the split rank of the symmetric pair $(\mathfrak{g},\mathfrak{g}_0)$. Then there are exactly $r+1$ G-orbits $M_r, M_{r-1}, \cdots,$ M_0 with $\dim M_k > \dim M_{k-1}$. M_r is open dense and M_0 is closed in $\tilde{M}$. The PHSS $(M,F^{\pm})$ is imbedded in $\tilde{M}$ as M_r in such a way that $F^{\pm}$ are the restrictions of $\mathcal{M}^{\mp}$. G-orbits $M_r,\cdots,M_0$ give a stratification on $\tilde{M}$. It is proved that the action of $Aut(M,F^{\pm})$ extends to $\tilde{M}$ as automorphisms of the stratification. The restriction of the extended action to M_0 is the symmetry group of a certain geometric structure on M_0. When the split root system of M is of BC_r-type, the restriction of $\mathfrak{M}^{\mp}$ to M_0 gives a double fibration $F_0^{\pm}$. Let $Aut(M_0,F_0^{\pm})$ be the automorphism group of $F_0^{\pm}$. When the split root system is of C_r-type, M_0 coincides with M^-. Let $\mathcal{K}$ be the generalized conformal structure on M^- obtained from the cone of singular G_0-orbits in g_1 (= the tangent space at the origin of M^-). The automorphism group $Aut(M^-,\mathcal{K})$ was determined by Gindikin-Kaneyuki [1].

<u>Theorem 1</u> ([6]). Let $(M=G/G_0,\omega,F^{\pm})$ be the PHSS associated to a simple (-1,1)-GLA g. Let $\tilde{\Delta}$ be the split root system of (g,g_0,σ). Suppose $\tilde{\Delta}$ is of BC_r-type. Then $Aut(M,F^{\pm}) = Aut(M_0,F_0^{\pm}) = G$. Suppose $\tilde{\Delta}$ is of C_r-type, $r \geqslant 2$. Then $Aut(M,F^{\pm}) = Aut(M^-,\mathcal{K}) = G$. In the case where $\tilde{\Delta}$ is of C_1-type, $Aut(M,F^{\pm}) = Diffeo(M^-)$.

Under the assumption that G is classical, the above theorem has been obtained by Tanaka [7].

§3. Parahermitian symmetric spaces with causal structures

In this paragraph we assume that a simple (-1,1)-GLA g is of Hermitian type. The corresponding PHSS M is called a <u>symmetric space of Cayley type</u>. $\tilde{\Delta}$ is of C_r-type in this case. Harmonic analysis on this type of symmetric spaces have been extensively studied. There exists an irreducible bounded symmetric domain D of tube type such that g is the Lie algebra of of the holomorphic automorphism group G(D) of D. M^- can be identified with the Shilov boundary of D. G(D) acts on M^- effectively and transitively, and hence it is a subgroup of G. Let V be the homogeneous open convex cone of positive definite (in Jordan terminology) elements in g_{-1}. The automorphism group G(V) of V is considered to be an open subgroup of G_0. Let τ be a grade-reversing Cartan involution on g, and let $V^+ = (-\tau)V \subset g_{+1}$. G(V) acting on g_1 is the automorphism group of V^+. The closures C^-, C^+ of V, V^+ respectively are causal cones in $g_{\mp 1}$. By using the action of G(D) on $M^{\mp}$, we extend $C^{\pm}$ to the cone fields $\mathcal{C}^{\pm}$ on $M^{\mp}$. Consider the product cone field $\tilde{\mathcal{C}} = \mathcal{C}^{+} \times \mathcal{C}^{-}$ on $\tilde{M} = M^{-} \times M^{+}$. By restricting $\tilde{\mathcal{C}}$ to M ($= M_r$), we have a $G(D)$-invariant causal structure $\mathcal{C}$ on M. Note that if the split rank r of M is equal to 1, then M is a one-sheeted hyperboloid in R^3. By using Theorem 1, we have

Theorem 2([6]). Let $(M,\mathcal{C})$ be a symmetric space of Cayley type with split rank r, and let $\mathrm{Aut}(M,\mathcal{C})$ be the causal automorphism group. Then $\mathrm{Aut}(M,\mathcal{C}) = G(D)\cdot Z_2$ for $r \geqslant 2$ and $\mathrm{Aut}(M,\mathcal{C}) = \mathrm{Diffeo}^{+}(S^1)\cdot Z_2$ for $r = 1$. Here Z_2 is generated by the restriction $\theta|_M$, and θ is an involutive transformation of $\tilde{M}$ which interchanges $\mathcal{C}^{+}$ to $\mathcal{C}^{-}$. $\mathrm{Diffeo}^{+}(S^1)$ denotes the group of orientation-preserving diffeomorphisms of the circle S^1.

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