数理解析研究所講究録
1150巻 2000年 34-42

TRANSFORMATION GROUPS AND GEOMETRIC STRUCTURES (ON SOME SOPHUS LIE RESULTS TODAY)

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1. Classification of Lie algebras in their trinity: abstract, linear and Lie algebras of vector fields

1.1. General theory of transformation groups (continuous groups). Presented by Sophus Lie in books [53, 49]. Later developed into the abstract theory of Lie groups and Lie algebras (G. Campbell, E. Cartan, L.S. Pontryagin, N. Bourbaki) and into the theory of Lie transformation pseudogroups (E. Cartan, S. Sternberg [36, 86]).

1.2. Classification of Lie algebras of vector fields. A. Finite-dimensional Lie algebras of vector fields on the line and the plane over the fields $\mathbb{C}$ and $\mathbb{R}$ (S. Lie [55, 49]).

Complex classification of Lie algebras of vector fields on the plane was rewritten many times (G. Campbell, N.G. Tchebotarev [88], R. Hermann [37]) and is well-known. The real case was rewritten by A. Gonzalez-Lopez, N. Kamran, and P. Olver [34]. Global classification of all real two-dimensional homogeneous spaces was obtained by G. Mostow [64].

B. Infinite-dimensional Lie algebras of vector fields on the plane over $\mathbb{C}$ (S. Lie [52, pp. 396–493]). This result seems to be unknown today.

C. Primitive Lie algebras of vector fields in space over $\mathbb{C}$ (S. Lie [55]).

The complete local classification of primitive actions over $\mathbb{C}$ was derived in works of V.V. Morozov [62] and E.B. Dynkin [22, 23]. The global case was introduced by M. Golubitsky and considered over $\mathbb{C}$ for subalgebras of maximal rank [30, 31]. The complete local classification over $\mathbb{R}$ and the global classification over $\mathbb{C}$ and $\mathbb{R}$ was obtained by B.P. Komrakov [40, 42]. The complete English version of these results can be found in the book [ISLC1].

D. Finite- and infinite-dimensional Lie algebras of contact vector fields on $\mathbb{C}^3$ (S. Lie [54], [52, pp. 396–493]).

This result for real case was completed by B. Doubrov, B. Komrakov [18].

E. All Lie algebras of vector fields whose image of isotropic representation is either $\mathfrak{sl}(n)$ or $\mathfrak{gl}(n)$ and all Lie algebras of contact vector fields whose image of isotropic representation contains $\mathfrak{sp}(2n)$ (S. Lie [54, 52]).

In these works Sophus Lie uses essentially the standard filtrations on Lie algebras off all and contact vector fields. From modern point of view his results are simple exercises on the theory of filtered and graded Lie algebras, that appear as examples in works of S. Sternberg [36], T. Morimoto, N. Tanaka [61], and N. Tanaka [87]. Let us note, that even with lack modern language, Sophus Lie uses the technique of filtered and graded Lie algebras.

G. Partial results on the classification of vector fields in space with the statement that the complete classification was obtained, but it is impossible to publish because of lack of space (S. Lie [55]).

The complete classification of all Lie algebras of vector fields in space is unknown today. The classification of all non-solvable Lie algebras in $\mathbb{C}^3$ and $\mathbb{R}^3$ was obtained by V.V. Morozov and his student Kim Sen En [59] (with several mistakes). The archive of Sophus Lie papers [57] in the University library of Oslo contains Sophus Lie notes, where he considers various cases of this classification. Yet it seems to be unlikely that Sophus Lie completed this classification, since in general this problem contains, as one of the subcases, the "wild" problem of describing all ideals of finite codimension in the algebra of polynomials in two variables [13].

Let us point some results of the ISLC team in the classification of transitive actions in dimensions 3 and 4: three-dimensional isotropy faithful homogeneous spaces [43], four-dimensional pseudo-Riemannian homogeneous spaces [44, 45], transitive actions of quasi-reductive Lie algebras [90].

The modern versions of Sophus Lie results concerning two-dimensional homogeneous spaces (both local and global cases) as well as translations to English of most important papers of Sophus Lie in this topic can be found in the book [ISLC2].

1.3. **Classification of abstract Lie algebras.** Sophus Lie classified all abstract Lie algebras up to dimension 4 over $\mathbb{C}$ [49]. Today several powerful techniques for classification of Lie algebras in low dimensions are developed. The following results are known today in this direction:

- classification of nilpotent Lie algebras up to dimension 7 [93, 63, 1, 60, 83, 84, 46], as well as the classification of metabelian Lie algebras up to dimension 8 [29, 28] and filiform Lie algebras up to dimension 11 [7, 32, 33];
- classification of all solvable Lie algebras up to dimension 6 [67, 68, 69, 70, 91, 92, 89].

Most of classifications of abstract Lie algebras as well as computer packages and databases of the results are collected in [ISLC3].

1.4. **Classification of linear Lie algebras.** Sophus Lie obtained the complete classifications of subalgebras in the following linear Lie algebras: $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{gl}(2,\mathbb{C})$, $\mathfrak{gl}(2,\mathbb{C})\ltimes\mathbb{C}^2$, $\mathfrak{sl}(3,\mathbb{C})$, $\mathfrak{gl}(3,\mathbb{C})$, $\mathfrak{so}(4,\mathbb{C})$. These classifications are presented in his books [49, 55].

Today the classification of subalgebras plays the important role in theoretical physics and in constructing invariant solutions of partial differential equations. Many classifications of this type in Lie algebras appearing in physics where obtained in [79, 80, 81, 82, 3, 4, 5, 6, 66].

Most of classifications of linear Lie algebras as well as computer packages and databases of the results are collected in [ISLC3].

1.5. **Classification of homogeneous submanifolds.** Sophus Lie described all homogeneous curves in two-dimensional affine and projective geometries, as well as homogeneous surfaces in the three-dimensional projective geometry [52, pp. 494–538] (over $\mathbb{C}$).

Nowadays there exist many classifications of this type in various classical geometries. Classification of homogeneous surfaces in three-dimensional real projective space was obtained by K. Nomizu, T. Sasaki [73] (surfaces with non-vanishing Pick invariant) and by F. Dillen, T. Sasaki, and L. Vrancken [11] (surfaces with vanishing Pick invariant) and independently in [12]. Homogeneous surfaces in three-dimensional affine space where described by B. Doubrov, B. Komrakov and M. Rabinovich [14, 15] (see also [2, 24]). The corresponding classifications in unimodular and centroaffine geometries can be found in [25, 35, 39, 58, 71, 72]. Finally, all homogeneous submanifolds with non-trivial stabilizer in four-dimensional affine and projective geometries were recently described by N. Mozhei [65].

The general technique for classification of homogeneous submanifolds in arbitrary homogeneous spaces, based on algebraic model of homogeneous submanifolds, can be found in B. Doubrov, B. Komrakov [20]

2. Symmetries of differential equations

This topic is published by Lie in books [48, 50, 51, 52].

A. Sophus Lie introduced notions of global and infinitesimal symmetries of differential equations and obtained analytic formulas, that establish when a given vector field is an infinitesimal symmetry of a differential equation.

The notion of symmetry of differential equation plays a key role today in most areas of the theory of differential equations and mathematical physics. There are several monographs devoted to symmetries of differential equations, that translate Lie results to modern language [38, 75, 78, 8]. The book [ISLC4] contains lectures of Nordfjordeid Summer Schools and can be considered as a modern introduction to this topic.

B. Sophus Lie constructed the theory of differential and integral invariants of Lie algebras of vector fields (in both finite- and infinite-dimensional cases). In particular, he developed the detailed algorithms for describing all differential equations invariant with respect to a given Lie algebra of vector fields.

Geometric theory of differential and integral invariants (in the contrast to the analytic theory, developed by Sophus Lie) was developed by E. Cartan and is based on his moving frame method [9, 10]. Both analytic and geometric techniques are described on modern language by P. Olver [76, 77].

C. Sophus Lie described all ordinary differential equations invariant with respect to canonical representatives in his classification of Lie algebras of vector fields. (So-called Gruppenregister [51, pp. 240–310, 362–427, 432–448].) In particular, he proved that the symmetry algebra of any ODE of order ≥ 2 is finite-dimensional.

The equivalence problem for systems of ordinary differential equations is solved by B. Doubrov, B. Komrakov and T. Morimoto [19]. This, in particular, allows to find out whether a given ODE can be brought to one of the canonical forms from Lie's Gruppenregister and what set of operations is needed for this.

D. He developed various methods for solving differential equations by means of symmetries (for both ordinary and partial differential equations) [56]. In particular, he formulated the result (nowadays called Lie theorem) that any n-th order ODE with known n-dimensional solvable symmetry algebra can be solved in quadratures. It is thought that the notion of solvable Lie algebra is based on this result.

Sophus Lie results are translated by B. Doubrov, B. Komrakov [17, 16] to the modern language, based on the notion of Maurer-Cartan forms, and the general theory of integration of completely integrable distributions with transitive symmetry algebras is developed.

E. Sophus Lie described in details the geometry of first order jets. and first order partial differential equations. He proved that the complete solution of such equation can be constructed by quadratures from any n-dimensional commutative symmetry algebra, where n is a number of independent variables [50, Ch. 13].

A part of Sophus Lie results was rewritten by E. Noeter and is known today as Noeter theorem. Notion of integrable system of first order partial differential equations become classical today and is rewritten many times without any references to Sophus Lie.

F. Sophus Lie introduced the notion of ordinary differential equation with fundamental solutions, that is, essentially, a system of first order ODE's with non-linear superposition principle. He proved that a given ODE has this form if and only if it corresponds to a curve in the Lie algebra of vector fields that lies in a finite-dimensional subalgebra.

Modern version of the superposition principle is given in [16, 21]. Many examples of the superposition function for specific homogeneous spaces can be found in [85, 47, 27, 74, 41]. Let us note that the complete modern proof of this result seems to be absent in modern literature.

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ISLC BOOKS (TO APPEAR)

[ISLC1] Boris Komrakov, *Primitive actions and the Sophus Lie problem*, ISLC Press, 2000, 350p.

[ISLC2] Boris Doubrov, Boris Komrakov, *Two-dimensional homogeneous spaces*, ISLC Press, 2000, 270p.

[ISLC3] Boris Komrakov, Alexei Tchourioumov, *Low-dimensional abstract and linear Lie algebras*, ISLC Press, 2000, 342p. (with MapleV packages and databases).

[ISLC4] Boris Doubrov, Boris Komrakov, *Geometry of differential equations*, ISLC Press, 2000, 206p.

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