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On cusps in the boundary of the Maskit slice for once punctured torus groups

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Introduction.

The aim of this paper is to explain analytic and geometric properties of the Maskit slice for once punctured torus groups which are obtained in [8]. We will investigate the Maskit slice via the horocyclic coordiate of the Teichmüller space of once punctured tori. The computer graphic of this image of the embedding is drawn by Professor David J.Wright ([11]). In drowing his picture, he conjectured some properties of the figure. This paper will treat one of his conjectures.

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1 Notation and definition

1.1 Simple closed curves on a once punctured torus

Let Σ be a once punctured torus. Let α and β be oriented simple closed curves on Σ such that the algebraic intersection number of α and β is $+1$. Then the fundamental group $\pi_1(\Sigma)$ of Σ is generated freely on α and β.

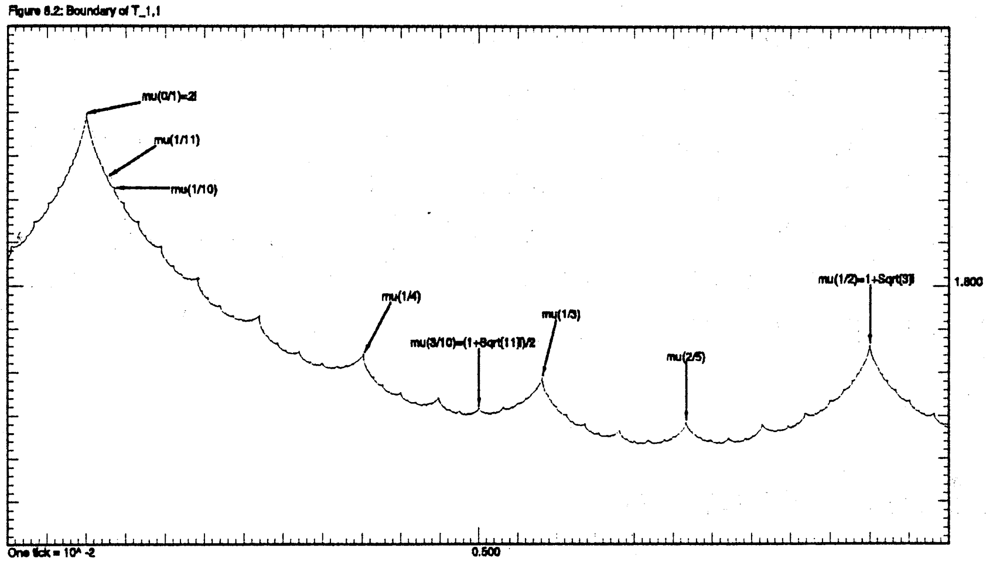

Figure 1: The Maskit embedding *Courtesy of David J.Wright*

Let $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{1/0\}$. In this talk, all rational numbers are used by the form $p/q \in \mathbb{Q}$ satisfying that p and q are relatively prime integers with $q > 0$. For $p/q \in \hat{\mathbb{Q}}$, we define $\gamma(p/q) \in \pi_1(\Sigma)$ as the following recursive operation: Let us first $\gamma(1/0) = \alpha^{-1}$ and $\gamma(0/1) = \beta$. Then we put $\gamma((p+r)/(q+s)) = \gamma(r/s)\gamma(p/q)$ where $p/q, r/s \in \hat{\mathbb{Q}}$ with $ps - rq = -1$. It can be shown that the homology class of $\gamma(p/q)$ is equal to $-p[\alpha] + q[\beta]$, and hence $\gamma(p/q)$ represents a simple closed curve on Σ. Furthermore, every simple closed curve on Σ is represented by $\gamma(p/q)$ for some $p/q \in \hat{\mathbb{Q}}$.

1.2 The model domain for the Maskit slice

Next, we define the model domain $\mathcal{M} \subset \mathbb{C}$ of the Maskit slice after Keen and Series [2] and Wright [11]. For $\mu \in \mathbb{C}$, we put

$$S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, T_\mu = \begin{bmatrix} -i\mu & -i \\ -i & 0 \end{bmatrix}.$$

Let $G_\mu = \langle S, T_\mu \rangle$. We define the homomorphism χ_μ from $\pi_1(\Sigma)$ to G_μ by $\chi_\mu(\alpha) = S$ and $\chi_\mu(\beta) = T_\mu$. Then we say that $\mu \in \mathbb{C}$ is contained in $\mathcal{M}$ if $\mathrm{Im}\mu > 0$, χ_μ is an isomorphism, and G_μ is a terminal regular b-group. This $\mathcal{M}$ is known as the figure drawn by D.J.Wright. Recently, Y.N.Minsky proved that $\mathcal{M}$ is a Jordan domain in the Riemann sphere (cf. Minsky [6]). By a Theorem of Jørgensen, for every point μ in the closure of $\mathcal{M}$ in $\mathbb{C}$,

G_μ is a Kleinian group and χ_μ is an isomorphism. For the Maskit slice or embedding, consult Kra [4] and Maskit [5].

For $p/q \in \mathbb{Q}$, let $W_{p/q,\mu} = \chi_\mu(\gamma(p/q))$. Then there exists $\mu(p/q) \in \partial\mathcal{M} \setminus \{\infty\}$ such that $W_{p/q,\mu(p/q)}$ is parabolic and that $W_{r/s,\mu(p/q)}$ is loxodromic unless $r/s = p/q$. It is known that for $\mu \in \partial\mathcal{M} \setminus \{\infty\}$, G_μ is geometrically finite if and only if $\mu = \mu(p/q)$ for some $p/q \in \mathbb{Q}$, and $G_{\mu(p/q)}$ is a maximally parabolic group with A.P.T.s $W_{p/q,\mu(p/q)}$ and S, see Keen, Maskit and Series [3].

2 Main Theorem

2.1 Main theorem

The main theorem of this talk is the following assertion.

Main Theorem. *For $\partial\mathcal{M} \setminus \{\infty\}$. if G_μ is geometrically finite, then μ is an inward-pointing cusp of $\mathcal{M}$.*

For a boundary point x_0 of a domain D in $\mathbb{C}$, the point x_0 is called an inward-pointing cusp if there exists a disk B such that $0 \in \partial B$ and $x_0+t^2 \in D$ for all $t \in B$ (Figure 2).

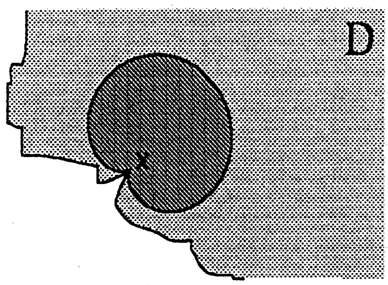

Figure 2: An inward-pointing cusp.

To show the main theorem, we will prove the following two theorems:

Theorem A. *For $p/q \in \mathbb{Q}$, if the derivative of $\mathrm{tr}^2 W_{p/q,\mu}$ does not vanish at $\mu = \mu(p/q)$, then $\mu(p/q)$ is an inward-pointing cusp of $\mathcal{M}$.*

Theorem B. *For any $p/q \in \mathbb{Q}$, the derivative of $\mathrm{tr}^2 W_{p/q,\mu}$ does not vanish at $\mu = \mu(p/q)$.*

We note that Theorem B gives an affirmative answer of one of conjectures of D.Wright appearing in his unpublished paper [11]:

Theorem. *For any $p/q \in \mathbb{Q}$, the point $\mu = \mu(p/q)$ is a simple root of of the polynomial* $\mathrm{tr}^2 W_{p/q,\mu} - 4$.

2.2 Proof of theorems

Theorem A is proved by applying a Theorem of Minsky, called Pivot theorem (cf. [6] and [7]).

Next, we explain the proof of Theorem B. To prove this, we deeply use the notion of the pleating ray in $\mathcal{M}$ introduced by L.Keen and C.Series in their paper [2].

Let $\mathcal{P}_{p/q}$ be the p/q-pleating ray in $\mathcal{M}$, that is, for each $\mu \in \mathcal{P}_{p/q}$, the connected component of the boundary of convex hull of the limit set of G_μ facing to the invariant component of G_μ is bent along the axis of $W_{p/q,\mu}$. Notice that $\mathcal{P}_{p/q}$ is a simple curve in $\mathcal{M}$ whose end points are ∞ and $\mu(p/q)$. Further, we know that $W_{p/q,\mu}$ is hyperbolic on $\mathcal{P}_{p/q}$. Denote by $\ell(\mu) > 0$ the translation length of $W_{p/q,\mu}$. Then this ℓ is a diffeomorphism from $\mathcal{P}_{p/q}$ to $\mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$. For $r/s \in \mathbb{Q}$, $\lambda_{r/s}$ the complex translation length of $W_{r/s,\mu}$. We assume that $\lambda_{r/s}$ is holomorphic on $\mathcal{M}$. It is easy to see that if $r/s \neq p/q$, $\lambda_{r/s}$ can be extended holomorphically on a neighborhood of $\mu(p/q)$.

Then Theorem B is shown by the following lemma.

Main Lemma. *Let $r/s \in \mathbb{Q}$ with $r/s \neq p/q$. Then there exists $l_0, C_0 > 0$ such that*

$$\left|\frac{d}{dl}(\lambda_{r/s} \circ \ell^{-1})(l)\right| \leq C_0 l$$

whenever $l < l_0$.

In fact, Theorem B is proved by Main Lemma as follows: Since $\mathrm{tr}^2 W_{n/1,\mu} = -(\mu - 2n)^2$ and $\mu(n/1) = 2n + 2i$, we may assume that $p/q \neq n/1$ for $n \in \mathbb{Z}$.

Let $r/s \in \mathbb{Q}$ with the properties that $r/s \neq p/q$ and the derivative of $\lambda_{r/s}$ does not vanish at $\mu = \mu(p/q)$. For example, the case where $r/s = n/1$, $n \in \mathbb{Z}$ satisfies this condition.

Take positive constants l_0 and C_0 for r/s as in Main Lemma. Then,

$$\left|\frac{d}{dl}(\lambda_{r/s}\circ \ell^{-1})(l)\right| \leq C_0 l \tag{1}$$

for $0 < l < l_0$. Let $\mu \in \mathcal{P}_{p/q}$ with $\ell(\mu) < l_0$. Integrating (1) from $l = 0$ to $l = \ell(\mu)$, we obtain

$$|\lambda_{r/s}(\mu) - \lambda_{r/s}(\mu(p/q))| \leq 2^{-1}C_1\ell(\mu)^2.$$

Since $W_{p/q,\mu(p/q)}$ is parabolic, the trace function of $W_{p/q,\mu}$ forms

$$\mathrm{tr}^2 W_{p/q,\mu} = 4 + \ell(\mu)^2 + o(\ell(\mu)^2)$$

for $\mu \in \mathcal{P}_{p/q}$ near $\mu(p/q)$. Hence there exists $C_0', l_0' > 0$ such that

$$|\lambda_{r/s}(\mu) - \lambda_{r/s}(\mu(p/q))| \leq C_0'|\mathrm{tr}^2 W_{p/q,\mu} - 4| \tag{2}$$

for $\mu \in \mathcal{P}_{p/q}$ with $\ell(\mu) < l_0'$. Dividing the inequality (2) by $|\mu - \mu(p/q)|$ and letting $\mu \to \mu(p/q)$, we conclude the assertion.

2.3 Quasiconformal deformation

We define a quasiconformal deformation of the group on a pleating ray, which is the central tool for proving Main Lemma. In this and the following section we fix a rational number p/q. Set ℓ as in previous subsection, and put $\pi - \theta(\mu)$ the bending angle along the axis of $W_{p/q,\mu}$.

Let $\mu \in \mathcal{P}_{p/q}$. Let H_1 and H_2 be the F-peripheral subgroups with respect to $W_{p/q,\mu}$ in G_μ. Namely, take $V \in G_\mu$ satisfying $G_\mu = \langle W_{p/q,\mu}, V\rangle$. Then we define $H_i = \langle W_{p/q,\mu}, V^{\epsilon_i} W_{p/q,\mu} V^{-\epsilon_i}\rangle$ where $\epsilon_i = (-1)^i$. We know that the pair $\{H_i\}_{i=1,2}$ is well-defined, that is, the definition of the pair $\{H_i\}_{i=1,2}$ is independent of the choice of V.

Since H_i acts on the peripheral disk $\Delta(H_i)$ of H_i, we can consider the axis ω_i of $W_{p/q,\mu}$ in $\Delta(H_i)$ as 2-dimensional hyperbolic geometry. By definition, each ω_i is a circular arc connecting between the fixed points of $W_{p/q,\mu}$. Further, ω_1 and ω_2 bound the sector F contained in $\Delta(H_1) \cup \Delta(H_2)$. Such F is uniquely determined, see Figure 3. Set $F_{[B]} = B^{-1}(F)$ for $[B] \in \langle W_{p/q,\mu}\rangle \backslash G_\mu$. (Notice that F is invariant under the action of $W_{p/q,\mu}$.) Then, for $[B_1], [B_2] \in \langle W_{p/q,\mu}\rangle \backslash G_\mu$, it holds $F_{[B_1]} \cap F_{[B_2]} = \emptyset$ if $[B_1] \neq [B_2]$.

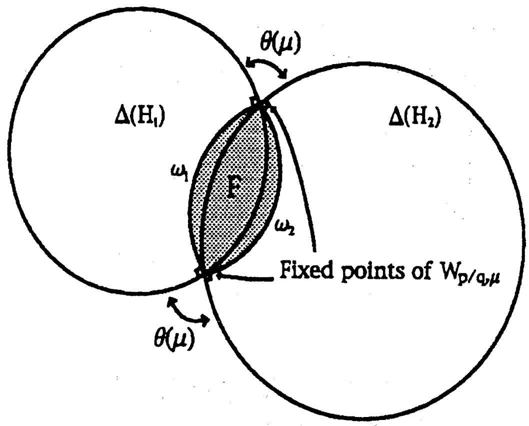

Figure 3: The set F.

Fix a Möbius transformation A sending the fixed points of $W_{p/q,\mu}$ to $\{0,\infty\}$. By definition, $A(F)$ is a sector with center at origin whose central angle is equal to $\pi-\theta(\mu)$. Set $\hat{\tau}(z)=A(z)\overline{A'(z)}/\overline{A(z)}A'(z)$ on F and $\hat{\tau}(z)=0$ otherwise. We can define the Bertrami differential τ_μ, $\mu\in\mathcal{P}_{p/q}$, compatible with G_μ by

$$\tau_\mu(z)=\frac{1}{\ell(\mu)}\sum_{[B]\in\langle W_{p/q,\mu}\rangle\backslash G_\mu}\hat{\tau}(B(z))\frac{\overline{B'(z)}}{B'(z)}.$$

The differential τ_μ satisfies that $\|\tau_\mu\|_\infty=1/\ell(\mu)$ and the support $\mathrm{Supp}(\tau_\mu)$ is $\cup_{[B]\in\langle W_{p/q,\mu}\rangle\backslash G_\mu}F_{[B]}$.

For $\epsilon\in\mathbb{C}$, $|\epsilon|<\ell(\mu)$, let w^ϵ be a solution on $\hat{\mathbb{C}}$ of the Bertrami equation $\overline{\partial}w^\epsilon=\epsilon\tau_\mu\partial w^\epsilon$. Then, there exists a holomorphic mapping Φ_μ from a disk $\{|\epsilon|<\ell(\mu)\}$ to $\mathcal{M}$ such that $\Phi_\mu(0)=\mu$ and $G_{\Phi_\mu(\epsilon)}$ is conjugate to $w^\epsilon G_\mu(w^\epsilon)^{-1}$ by an element in $\mathrm{PSL}_2(\mathbb{C})$.

2.4 Proof of Main Lemma

To prove Main Lemma, we shall show the following two propositions:

Proposition A. *For $\mu\in\mathcal{P}_{p/q}$, let Φ_μ as in previous subsection. Then, there exists $l_1>0$ such that if $\ell(\mu)<l_1$, then*

$$\left|\frac{d}{d\epsilon}(\lambda_{p/q}\circ\Phi_\mu)\Big|_{\epsilon=0}\right|\geq\frac{1}{2}.$$

Proposition B. *As in Proposition A, define the mapping Φ_μ for $\mu \in \mathcal{P}_{p/q}$. Take $r/s \in \mathbb{Q}$ with $r/s \neq p/q$. Then there exist l_2 and $C_2 > 0$ such that*

$$\left| \frac{d}{d\epsilon}(\lambda_{r/s} \circ \Phi_\mu)\Big|_{\epsilon=0} \right| \leq C_2 \ell(\mu)$$

for all $\mu \in \mathcal{P}_{p/q}$ with $\ell(\mu) < l_2$.

These propositions are showed by applying the Gardinar's differential formula for complex translation length (cf.§8 of Imayoshi and Taniguchi [1]):

Proposition. (F.Gardiner) *Let $g(w) = e^\lambda w$ with $\mathrm{Re}\lambda > 0$. Let ν be a Bertrami differential on $\mathbb{C}$ compatible with g. Denote by f^ϵ a solution on $\hat{\mathbb{C}}$ of the equation $\overline{\partial} f^\epsilon = \epsilon\nu\partial f^\epsilon$ for $|\epsilon| < 1/\|\nu\|_\infty$. Define a holomorphic function $\lambda(\epsilon)$ on $\{|\epsilon| < 1/\|\nu\|\}$ by $\mathrm{tr}^2 f^\epsilon g(f^\epsilon)^{-1} = 4\cosh^2(\lambda(\epsilon)/2)$ and $\lambda(0) = \lambda$. Then, it holds*

$$\frac{d\lambda}{d\epsilon}\Big|_{\epsilon=0} = \frac{1}{\pi}\int_{\{1<|\zeta|<e^{\mathrm{Re}\lambda}\}} \nu(\zeta)\frac{d\xi d\eta}{\zeta^2}, \quad \zeta = \xi + i\eta.$$

3 Further results

We also obtain an analytic property of the image. The next theorem concerns with the actions of the Teichmüller modular group on the boundary: Since it is known that boundary points corresponding to geometrically finite groups lie densely on the boundary, the main theorem tells us that this boundary is very complicate in the geometrical point of view. For instance, we can show from the main theorem that the image is not quasidisk. In addition to the complexities of the boundary, C.T.McMullen and D.J.Wright observed that the spiraling phenomena[1] occurs in the boundary of the Maskit slice[2].

On the other hand, Y.Minsky proved that the image is Jordan domain. Therefore, the actions of the Teichmüller modular group can be extended continuously not only on the boundary but also on the Riemann sphere. Hence, this result tells us that the complexity is studied via the actions of

[1] They observed more strongly result: There exist boundary points that require arbitrary large winding number to get to.

[2] The author knew these phenomena from Professor C.T.McMullen in oral communication, and from Professor D.J.Wright in e-mail communication, independently.

the Teichmüller modular group on the boundary. However we obtain no more information about regularities of the actions from topological properties on the boundary. The next observation is related to this subject.

Theorem C.[3] *Let $p/q, r/s \in \mathbb{Q}$. Let $h \in \mathrm{Aut}(\mathcal{M})$ with $h(\mu(p/q)) = \mu(r/s)$. Then h is conformal at $\mu(p/q)$ in the following sense: there exists $a \in \mathbb{C} \setminus \{0\}$ such that*

$$h(\mu) = \mu(r/s) + a(\mu - \mu(p/q)) + o(|\mu - \mu(p/q)|),$$

as $\mu \to \mu(p/q)$ in a cone with vertex at $\mu(p/q)$ (cf. Figure 4).

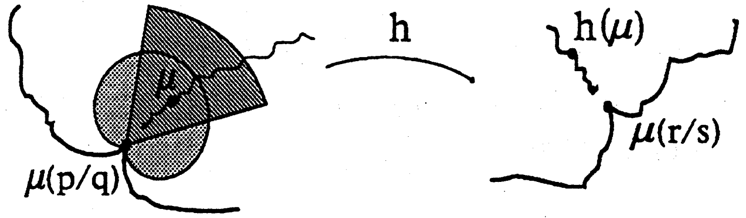

Figure 4: A cone with vertex at $\mu(p/q)$

Remark that elements in the Teichmüller modular group satisfy the condition in Theorem C. We also note that such cone domain alway exists since z_1 is an inward-pointing cusp. This theorem gives an expectation that the boundary may not be so complicated from the function theoretic point of view.

In [10] J.P.Otal proved the following remarkable fact: Let ρ be a once punctured torus group, that is, ρ is a faithful discrete representation from $\pi_1(\Sigma)$ to $\mathrm{PSL}(2, \mathbb{C})$. If $\rho(\gamma(p/q))$ is hyperbolic and its translation length is sufficiently small, then the boundary of convex core of $\mathbb{H}^3/\rho(\pi_1(\Sigma))$ is bent along the geodesic corresponding to $\rho(\gamma(p/q))$.

In our case, the following result is observed.

Theorem D. *Let $p/q \in \mathbb{Q}$. Then there exists a neighborfood U_0 of $\mu(p/q)$ in $\mathbb{C}$ such that for $\mu \in U_0$, if the element $\chi_\mu(\gamma(p/q))$ is hyperbolic, then G_μ is discrete (further $\mu \in \mathcal{M}$) and the boundary of convex core of $\mathbb{H}^3/G_\mu$ is bent along the geodesic corresponding to $\chi_\mu(\gamma(p/q))$.*

[3]The author hopes that this theorem becomes a step-stone for solving the problem on the self-similarity of the boundary of $\mathcal{M}$ (cf. McMullen [9],p.180).

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