数理解析研究所講究録
1155巻 2000年 1-15

Analytic extension formulas, integral transforms and reproducing kernels

Saburou Saitoh (群馬大工・斎藤三郎)

Department of Mathematics, Faculty of Engineering
Gunma University, Kiryu 376-8515, Japan
E-mail: ssaitoh@math.sci.gunma-u.ac.jp

Abstract. In this survey article, we shall present a general framework and applications of our recent results among reproducing kernels, linear transforms and analytic extension formulas.

1. Mystery of analytic extension

The most fundamental function e^x is extensible analytically onto the whole complex $z = x + iy$ plane and we have the mysteriously beautiful identity

$$e^{\pi i} = -1, \tag{1.1}$$

which states a relation among the basic numbers $-1, \pi, e,$ and i. Note that 0 and 1 may be arbitrarily fixed as two points on the real line and, π and e are irrational numbers. The author stated in [39] that the best result in mathematics is the Leonhard Euler formula (1.1) based on the idea that :

Mathematics is relations and the research in mathematics is to look for some relations. Good relations that we call theorems will mean that the relations are fundamental in mathematics, are beautiful and give good impacts to human beings.

In the Riemann ζ–function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$

we have, by its analytic extension

$$\begin{aligned} \zeta(-1) &= -\frac{1}{12} \\ (&= \ ?! \quad 1+2+3+\cdots \ ). \end{aligned} \tag{1.2}$$

In general, an analytic function is determined locally and we have the idea of the Riemann surface as its natural existence domain. An analytic function looks like having a life.

2. Reproducing kernel Hilbert spaces and decisive representation formulas

Since an analytic function is determined locally, we are intuitively interested in its analytic extensibility and representations. For these fundamental problems we firstly would like to refer that the theory of reproducing kernels will give a decisive method in some sense and in some general situation.

We consider any positive matrix $K(p,q)$ on E; that is, for an abstract set E and for a complex–valued function $K(p,q)$ on $E \times E$, it satisfies that for any finite points $\{p_j\}$ of E and for any complex numbers $\{C_j\}$,

$$\sum_j \sum_{j'} C_j \overline{C_{j'}} K(p_{j'}, p_j) \geqq 0.$$

Then, by the fundamental theorem by Moore–Aronszajn, we have:

Proposition 2.1. *For any positive matrix $K(p,q)$ on E, there exists a uniquely determined functional Hilbert space H_K comprising functions $\{f\}$ on E and admitting the reproducing kernel $K(p,q)$* (RKHS H_K) *satisfying and characterized by*

$$K(\cdot, q) \in H_K \ \text{ for any } \ q \in E \tag{2.1}$$

and, for any $q \in E$ and for any $f \in H_K$

$$f(q) = (f(\cdot), K(\cdot, q))_{H_K}. \tag{2.2}$$

For some general properties for reproducing kernel Hilbert spaces and for various constructions of the RKHS H_K from a positive matrix $K(p,q)$, see the recent book[38] and its Chapter 2, Section 5, respectively.

We shall assume that H_K is separable. Then, the functions $\{K(\cdot,q); q \in E\}$ generate H_K and there exists a countable set S of E such that $\{K(\cdot,q_j); q_j \in S\}$ is a family of linearly independent functions forming a basis for H_K. We set $S_n = \{q_1, q_2, \cdots, q_n\} \subset S$ and $\|\Gamma_{jj'n}\|_{1 \leqq j,j' \leqq n}$ is the inverse of $\|K(q_j, q_{j'})\|_{1 \leqq j,j' \leqq n}$. Then, we obtain

Proposition 2.2 ([20] and see Chapter 2, Section 5 in [38]). *For any $f \in H_K$, the sequence of functions f_n defined by*

$$f_n(p) = \sum_{j,j'=1}^{n} f(q_j) \Gamma_{jj'n} K(p, q_{j'}) \tag{2.3}$$

converges to f as $n \to \infty$ in both the senses in norm of H_K and everywhere on E.

Furthermore, for any function f defined on E satisfying

$$\lim_{n \to \infty} \sum_{j,j'=1}^{n} f(q_j) \Gamma_{jj'n} \overline{f(q_{j'})} < \infty, \quad q_j \in S, \tag{2.4}$$

the sequence of functions f_n defined by (2.3) is a Cauchy sequence in H_K whose limit coincides with f on E. Conversely, any member f of H_K is obtained in this way in terms of $\{f(q_j)\}$.

We see in Proposition 2.2 that extensibility and representation of f in terms of $f(q_j)$, $q_j \in S$ are established by means of the reproducing kernel $K(p,q)$.

On the millennium occasion, the author wonders Proposition 2.2 will become a powerful method connecting analytic functions and discrete sets in the next millennium.

3. Connection with linear transforms

We shall connect linear transforms in the framework of Hilbert spaces with reproducing kernels.

For an abstract set E and for any Hilbert (possibly finite–dimensional) space H, we shall consider an H–valued function $\boldsymbol{h}$ on E

$$\boldsymbol{h}: \quad E \longrightarrow H \tag{3.1}$$

and the linear transform for H

$$f(p) = (\boldsymbol{f}, \boldsymbol{h}(p))_H \quad \text{for} \quad \boldsymbol{f} \in H \tag{3.2}$$

into a linear space comprising functions $\{f(p)\}$ on E. For this linear transform (3.2), we form the positive matrix $K(p,q)$ on E defined by

$$K(p,q) = (\boldsymbol{h}(q), \boldsymbol{h}(p))_H \quad \text{on} \quad E \times E. \tag{3.3}$$

Then, we have the following fundamental results:

(I) For the RKHS H_K admitting the reproducing kernel $K(p,q)$ defined by (3.3), the images $\{f(p)\}$ by (3.2) for H are characterized as the members of the RKHS H_K.
(II) In general, we have the inequality in (3.2)

$$\|f\|_{H_K} \leqq \|\boldsymbol{f}\|_H, \tag{3.4}$$

however, for any $f \in H_K$ there exists a uniquely determined $\boldsymbol{f}^* \in H$ satisfying

$$f(p) = (\boldsymbol{f}^*, \boldsymbol{h}(p))_H \quad \text{on } E \tag{3.5}$$

and

$$\|f\|_{H_K} = \|\boldsymbol{f}^*\|_H. \tag{3.6}$$

In (3.4), the isometry holds if and only if $\{\boldsymbol{h}(p); p \in E\}$ is complete in H.
(III) We can obtain the inversion formula for (3.2) in the form

$$f \longrightarrow \boldsymbol{f}^*, \tag{3.7}$$

by using the RKHS H_K. However, this inversion formula will depend on, case by case, the realizations of the RKHS H_K.
(IV) Conversely, if we have an isometrical mapping $\tilde{L}$ from a RKHS H_K admitting a reproducing kernel $K(p,q)$ on E onto a Hilbert space H, then the Hilbert space H–valued function $\boldsymbol{h}$ satisfying (3.1) and (3.2) is given by

$$\boldsymbol{h}(p) = \tilde{L}K(\cdot, p) \;\; \text{on} \;\; E \tag{3.8}$$

and, then $\{\boldsymbol{h}(p); p \in E\}$ is complete in H. The isometrical inversion $\tilde{L}^{-1}$ is given by the transform (3.2).

When (3.2) is isometrical, sometimes we can use the isometrical mapping for a realization of the RKHS H_K, conversely—that is, if the inverse L^{-1} of the linear transform (3.2) is known, then we have $\|f\|_{H_K} = \|L^{-1}f\|_H$.

We shall state some general applications of the results (I)$\sim$(IV) to several wide subjects and their basic references:

(1) Linear transforms ([23],[35]).

(2) Integral tansforms among smooth functions ([42]).

(3) Nonharmonic integral transforms ([27]).

(4) Various norm inequalities ([27],[36]).

(5) Nonlinear transforms ([36],[39]).

(6) Linear integral equations ([43]).

(7) Linear differential equations with variable coefficients ([43]).

(8) Approximation theory ([10]).

(9) Representations of inverse functions ([37]).

(10) Various operators among Hilbert spaces ([40]).

(11) Sampling theorems ([38], Capter 4, Section 2).

(12) Interpolation problems of Pick-Nevanlinna type ([27],[28]).

In this survey article, we shall present

(13) Analytic extension formulas and their applications ([38]).

4. Typical examples for analytic extension formulas

We shall consider the Weierstrass transform

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\boldsymbol{R}} F(\xi) \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi \tag{4.1}$$

for functions $F \in L_2(\boldsymbol{R}, d\xi)$. Then, by using (I) and (II) we obtained in [24] simply and naturally the isometrical identity

$$\int_{\boldsymbol{R}} |F(\xi)|^2 d\xi = \frac{1}{\sqrt{2\pi t}} \int\int_{\boldsymbol{R}^2} |u(z,t)|^2 \exp\left[-\frac{y^2}{2t}\right] dxdy \tag{4.2}$$

for the analytic extension $u(z,t)$ of $u(x,t)$ to the entire complex $z = x + iy$ plane.

Of course, the image $u(x,t)$ of (4.1) is the solution of the heat equation

$$u_{xx}(x,t) = u_t(x,t) \quad \text{on} \quad \boldsymbol{R} \times \{t > 0\} \tag{4.3}$$

satisfying the initial condition

$$\lim_{t \to +0} \|u(x,t) - F(x)\|_{L_2(\boldsymbol{R},dx)} = 0.$$

On the other hand, by using the properties of the solution $u(x,t)$ of (4.3), N. Hayashi derived the identity

$$\int_{\boldsymbol{R}} |F(\xi)|^2 d\xi = \sum_{j=0}^{\infty} \frac{(2t)^j}{j!} \int_{\boldsymbol{R}} |\partial_x^j u(x,t)|^2 dx. \tag{4.4}$$

The two identities (4.2) and (4.4) were a starting point for obtaining our various analytic extension formulas and their applications.

As to the equality of (4.2) and (4.4), we obtained directly

Theorem 4.1 *([15]). For any analytic function $f(z)$ on the strip $S_r = \{|\mathrm{Im}z| < r\}$ with a finite integral*

$$\int\int_{S_r} |f(z)|^2 dxdy < \infty,$$

we have the identity

$$\int\int_{S_r} |f(z)|^2 dxdy = \sum_{j=0}^{\infty} \frac{(2r)^{2j+1}}{(2j+1)!} \int_{\boldsymbol{R}} |\partial_x^j f(x)|^2 dx. \tag{4.5}$$

Conversely, for a smooth function $f(x)$ with a convergence sum (4.5) on $\boldsymbol{R}$, there exists an analytic extension $f(z)$ onto S_r satisfying (4.5).

Theorem 4.2 *([15]). For any $\alpha > 0$ and for an entire function $f(z)$ with a finite integral*

$$\int\int_{\boldsymbol{R}^2} |f(z)|^2 \exp\left[-\frac{y^2}{\alpha}\right] dxdy < \infty,$$

we have the identity

$$\frac{1}{\sqrt{\alpha\pi}} \int\int_{\boldsymbol{R}^2} |f(z)|^2 \exp\left[-\frac{y^2}{\alpha}\right] dxdy = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} \int_{\boldsymbol{R}} |\partial_x^j f(x)|^2 dx. \tag{4.6}$$

Conversely, for a smooth function $f(x)$ with a convergence sum (4.6) on $\boldsymbol{R}$, there exists an analytic extension $f(z)$ on $\boldsymbol{C}$ satisfying the identity (4.6).

Our typical results of another type were obtained from the integral transform

$$v(x,t) = \frac{1}{t}\int_0^t F(\xi)\frac{x\exp\left\{\frac{-x^2}{4(t-\xi)}\right\}}{2\sqrt{\pi}\,(t-\xi)^{\frac{3}{2}}}\xi d\xi \tag{4.7}$$

in connection with the heat equation (4.3) for $x > 0$ satisfying the conditions, for $u(x,t) = tv(x,t)$

$$u(0,t) = tF(t) \quad \text{for } t \geqq 0$$

and

$$u(x,0) = 0 \quad \text{on} \quad x \geqq 0.$$

Then, we obtained

Theorem 4.3 *([1] and [30]). Let* $\Delta(\frac{\pi}{4})$ *denote the sector* $\left\{|\arg z| < \frac{\pi}{4}\right\}$. *Then, for any analytic function* $f(z)$ *on* $\Delta\left(\frac{\pi}{4}\right)$ *with a finite integral*

$$\int\int_{\Delta(\frac{\pi}{4})} |f(z)|^2 dxdy < \infty,$$

we have the identity

$$\int\int_{\Delta(\frac{\pi}{4})} |f(z)|^2 dxdy = \sum_{j=0}^{\infty} \frac{2^j}{(2j+1)!}\int_0^{\infty} x^{2j+1}\left|\partial_x^j f(x)\right|^2 dx. \tag{4.8}$$

Conversely, for any smooth function $f(x)$ *on* $\{x > 0\}$ *with a convergence sum in (4.8), there exists an analytic extension* $f(z)$ *onto* $\Delta\left(\frac{\pi}{4}\right)$ *satisfying (4.8).*

Let $\Delta(\alpha)$ be the sector $\{|\arg z| < \alpha\}$. Then, by using the conformal mapping e^z, H. Aikawa examined the relation between Theorem 4.1 and Theorem 4.3. Then, he used the Mellin transform and some expansion of Gauss' hypergeometric series $F(\alpha,\beta;\gamma;z)$ and we obtained a general version of Theorem 4.3 and a version for the Szegö space:

Theorem 4.4 *([2]). Let* $0 < \alpha < \frac{\pi}{2}$. *Then, for any analytic function* $f(z)$ *on* $\Delta(\alpha)$ *with a finite integral*

$$\int\int_{\Delta(\alpha)} |f(z)|^2 dxdy < \infty,$$

we have the identity

$$\begin{aligned}\int\int_{\Delta(\alpha)} |f(z)|^2 dxdy &= \sin(2\alpha)\sum_{j=0}^{\infty}\frac{(2\sin\alpha)^{2j}}{(2j+1)!} \\ &\quad \cdot \int_0^{\infty} x^{2j+1}|\partial_x^j f(x)|^2 dx. \end{aligned} \tag{4.9}$$

Conversely, for a smooth function $f(x)$ *with a convergence sum on* $x > 0$ *in (4.9), there exists an analytic extension* $f(z)$ *onto* $\Delta(\alpha)$ *satisfying the identity (4.9).*

Theorem 4.5 *([2]). Let* $0 < \alpha < \frac{\pi}{2}$. *Then, for any analytic function* $f(z)$ *on* $\Delta(\alpha)$ *satisfying*

$$\int_{|\theta|<\alpha} |f(re^{i\theta})|^2 dr < \infty,$$

we have the identity

$$\int_{\partial\Delta(\alpha)} |f(z)|^2 |dz| = 2\cos\alpha \sum_{j=0}^{\infty} \frac{(2\sin\alpha)^{2j}}{(2j)!} \int_0^{\infty} x^{2j} |\partial_x^j f(x)|^2 dx. \tag{4.10}$$

where $f(z)$ *mean Fatou's nontangentially boundary values of* f *on* $\partial\Delta(\alpha)$.

Conversely, for a smooth function $f(x)$ *on* $x > 0$ *with a convergence sum in (4.10), there exists an analytic extension* $f(z)$ *onto* $\Delta(\alpha)$ *satisfying the identity (4.10).*

As a general form of the right hand side of (4.9), we consider the infinite order Sobolev space $W(C_j; \boldsymbol{R}^+)$ on the positive real line $\boldsymbol{R}^+$ defined by

$$W(C_j; \boldsymbol{R}^+) = \left\{ f \ ; \ \sum_{j=0}^{\infty} C_j \int_{\boldsymbol{R}^+} x^{2j+1} |\partial_x^j f(x)|^2 dx < \infty \right\}$$

for a sequence $\{C_j\}$ of nonnegative numbers C_j. Then, Aikawa [4] proved that if $\alpha > \frac{\pi}{2}$, then for any $\{C_j\}$ with $W(C_j; \boldsymbol{R}^+) \neq \{0\}$, there is $f \in W(C_j; \boldsymbol{R}^+)$ that fails to have an analytic continuation to the "concave" sector $\Delta(\alpha)$. He also showed that $\frac{\pi}{2}$ is sharp.

5. Various analytic extension formulas and applications

We obtained various analytic extension formulas in the above line in [1, 2, 3, 4, 7, 8, 11, 12, 13, 14, 15, 16, 21, 22, 24, 25, 26, 29, 30, 31, 32, 33, 34] containing multi-dimensional spaces. As applications to nonlinear partial differential equations, the author expects Professor N. Hayashi to publish a survey article in this Koukyuroku, so the author would like to refer to applications to the Laplace transform and recent related results in the sequel.

6. Real inversion formulas of the Laplace transform

The inversion formula of the Laplace transform is, in general, given by complex forms. The observation in many cases however gives us real data only and so, it is important to establish the real inversion formula of the Laplace transform, because we have to extend the real data analytically onto a half complex plane. The analytic extension formula is, in general, very involved and makes the stability unclear. In particular, in the Reznitskaya transform combining the solutions of hyperbolic and parabolic partial differential equations, we need the real inversion formula, because the observation data of the solutions of hyperbolic partial differential equations are real-valued. See [41].

Since the image of the Laplace transform is, in general, analytic on a half-plane on the complex plane, in order to obtain the real inversion formula, we need a half plane version $\Delta\left(\frac{\pi}{2}\right)$ of Theorem 4.4 and Theorem 4.5, which is a crucial case $\alpha = \frac{\pi}{2}$ in those theorems. By using the famous Gauss summation formula and transformation properties in the Mellin transform we obtained, in a very general version containing the Bergman and the Szegö spaces:

Theorem 6.1 *([32]). For any $q > 0$, let $H_{K_q}(R^+)$ denote the Bergman–Selberg space admitting the reproducing kernel*

$$K_q(z, \bar{u}) = \frac{\Gamma(2q)}{(z+\bar{u})^{2q}}$$

on the right half plane $R^+ = \{z; \mathrm{Re} z > 0\}$. Then, we have the identity

$$\begin{aligned} \|f\|^2_{H_{K_q}(R^+)} &= \left(\frac{1}{\Gamma(2q-1)\pi} \int\int_{R^+} |f(z)|^2 (2x)^{2q-2} dx dy \ , \ q > \frac{1}{2}\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \\ &\quad \cdot \int_0^{\infty} |\partial_x^n (x f'(x))|^2 x^{2n+2q-1} dx. \end{aligned} \tag{6.1}$$

Conversely, any smooth function $f(x)$ on $\{x > 0\}$ with a convergence summation in (6.1) can be extended analytically onto R^+ and the analytic extension $f(z)$ satisfying $\lim_{x\to\infty} f(x) = 0$ belongs to $H_{K_q}(R^+)$ and the identity (6.1) is valid.

For the Laplace transform

$$f(z) = \int_0^{\infty} F(t) e^{-zt} dt, \tag{6.2}$$

we have, immediately, the isometrical identity, for any $q > 0$

$$\begin{aligned} \|f\|^2_{H_{K_q}(R^+)} &= \int_0^{\infty} |F(t)|^2 t^{1-2q} dt \\ (&:= \|F\|^2_{L^2_q}) \end{aligned} \tag{6.3}$$

from (I) and (II). By using (6.3) and (6.1), we obtain

Theorem 6.2 *([8]). For the Laplace transform (6.2), we have the inversion formula*

$$F(t) = \mathrm{s} - \lim_{\mathrm{N}\to\infty} \int_0^{\infty} f(x) e^{-xt} P_{N,q}(xt) dx \quad (t > 0) \tag{6.4}$$

where the limit is taken in the space L_q^2 *and the polynomials* $P_{N,q}$ *are given by*

$$\begin{aligned} P_{N,q}(\xi) &= \sum_{0 \leqq \nu \leqq n \leqq N} \frac{(-1)^{\nu+1}\Gamma(2n+2q)}{\nu!(n-\nu)!\Gamma(n+2q+1)\Gamma(n+\nu+2q)} \xi^{n+\nu+2q-1} \\ &\quad \cdot \left\{ \frac{2(n+q)}{n+\nu+2q}\xi^2 - \left(\frac{2(n+q)}{n+\nu+2q} + 3n + 2q \right) \xi \right. \\ &\quad \left. +(n+\nu+2q) \right\}. \end{aligned} \tag{6.5}$$

The truncation error is estimated by the inequality

$$\begin{aligned} &\left\| F(t) - \int_0^\infty f(x) e^{-xt} P_{N,q}(xt) dx \right\|_{L_q^2}^2 \\ \leqq &\sum_{n=N+1}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \left| \partial_x^n \left[x f'(x) \right] \right|^2 x^{2n+2q-1} dx. \end{aligned} \tag{6.6}$$

In order to obtain an inversion formula which converges pointwisely in (6.4), we considered an inversion formula of the Laplace transform for the Sobolev space satisfying

$$\int_0^\infty \left(|F(t)|^2 + |F'(t)|^2 \right) dt < \infty,$$

in [5]. In some subspaces of $H_{K_q}(R^+)$ and L_q^2, we established an error estimate for the inversion formula (6.4) in [6]. Some characteristics of the strong singularity of the polynomials $P_{N,q}(\xi)$ and some effective algorithms for the real inversion formula (6.4) are examined by J. Kajiwara and M. Tsuji [18,19]. Furthermore, they gave numerical experiments by using computers.

7. Representations and harmonic extension formulas on half spaces

Let $\boldsymbol{R}_+^{n+1} = \{(y,x); y > 0, x \in \boldsymbol{R}^n\}$ be the half space, where $x = (x_1, x'), x' = (x_2, \cdots, x_n)$. We consider the Poisson integral

$$U(y,x) = \int_{R^n} F(\xi) P(x-\xi, y) d\xi \tag{7.1}$$

for

$$\begin{aligned} P(x,y) &= \frac{1}{(2\pi)^n} \int_{R^n} e^{-y|t|} e^{-ix\cdot t} dt \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \cdot \frac{y}{(y^2+|x|^2)^{\frac{n+1}{2}}} \end{aligned}$$

and for functions $F \in L^2(\boldsymbol{R}^n, d\xi)$. For these harmonic functions $U(y,x)$ we obtained in [22] :

(A) *F and so, $U(y,x)$ are determined and simply represented by the functions*

$$\left.\frac{\partial U(y,x_1,x')}{\partial x_1}\right|_{x_1=0} \quad \text{and} \quad \left.\frac{\partial^2 U(y,x_1,x')}{\partial x_1^2}\right|_{x_1=0} \tag{7.2}$$

for $y>0$ and for $x' \in \boldsymbol{R}^{n-1}$, by using Fourier's integral and real inversion formulas for the Laplace transform,
and
(B) *characterization of the two functions in (7.2) on the hyperplane $x_1 = 0$ which are obtained from $U(y,x)$ in (7.1), by means of Fourier's transform and Laplace's transform; this will give a harmonic extension formula to $U(y,x)$ in (7.1) from the hyperplane $x_1 = 0$.*

8. Representations of initial heat distributions by means of their heat distributions as functions of time

In the Weierstrass transform (4.1), we obtained the isometrical identity, for any fixed $x \in \boldsymbol{R}$,

$$\begin{aligned} &\int_{-\infty}^{\infty} |F(\xi)|^2 d\xi \\ = \;& 2\pi \sum_{j=0}^{\infty} \frac{1}{j!\Gamma\left(j+\frac{3}{2}\right)} \int_0^{\infty} \left|\partial_t^j \left[t\partial_t u(x,t)\right]\right|^2 t^{2j-\frac{1}{2}} dt \\ &+2\pi \sum_{j=0}^{\infty} \frac{1}{j!\Gamma\left(j+\frac{5}{2}\right)} \int_0^{\infty} \left|\partial_t^j \left[t\partial_t \partial_x u(x,t)\right]\right|^2 t^{2j+\frac{1}{2}} dt. \end{aligned} \tag{8.1}$$

From this identity, we can obtain the inversion formula

$$u(x,t) \longrightarrow F(\xi) \quad \text{for any fixed } x. \tag{8.2}$$

We, in general, in multi–dimensional Weierstrass transform, established an exact and analytical representation formula of the initial heat distribution F by means of the observations

$$u(x_1,x',t) \text{ and } \frac{\partial(x_1,x',t)}{\partial x_1} \tag{8.3}$$

for $x' = (x_2, x_3, \cdots, x_n) \in \boldsymbol{R}^{n-1}$ and $t>0$, at any fixed point x_1, in [21].
We set

$$\sigma_F = \{\sup |x|,\; x \in \operatorname{supp} F\} \tag{8.4}$$

and supp$F$ denotes the smallest closed set outside which F vanishes almost everywhere. By using the isometrical identities (4.2), (4.4) and (8.1), we can solve the inverse source problem of determining the size σ_F of the initial heat distribution F from the heat flow $u(x,t)$ observed either at any fixed time t or at any fixed position x. See [44].

9. Representations of the solutions of partial differential equations of parabolic and hyperbolic types by means of time observations

In the problem (8.1), we can obtain a general result, in a very general situation.

Let D be a finitely-connected smoothy bounded domain in $\boldsymbol{R}^n$. We consider a partial differential equation of parabolic type

$$\frac{\partial u}{\partial t} = Au = \triangle u - q(x)u \quad (t > 0, x \in D) \tag{9.1}$$

subject to the boundary condition

$$\alpha(\xi)u + \{1 - \alpha(\xi)\}\frac{\partial u}{\partial \nu} = 0 \quad (t > 0, \text{ on } \partial D), \tag{9.2}$$

where $\partial/\partial\nu$ denotes the outer normal derivative on ∂D with respect to D. We assume that $q(x)$ is Hölder continuous on $\bar{D} = D \cup \partial D$, $\alpha \in C^2(\partial D)$ and $0 \leqq \alpha(\xi) \leqq 1$ on ∂D.

Let $U(t,x,y)$ be a fundamental solution for the equations (9.1) and (9.2). Then, in particular, recall that for any fixed $y \in \bar{D}$, $U(t,x,y) \in C^1((0,\infty) \times \bar{D})$, $U(t,x,y)$ satisfies (9.1) and (9.2).

Under the above situations, there exist eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ and eigenfunctions $\{\varphi_j\}_{j=0}^{\infty}$ satisfying

$$-\infty < \lambda_0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_j \leqq \cdots, \quad \lim_{j\to\infty} \lambda_j = \infty \tag{9.3}$$

$$\{\varphi_j\}_{j=0}^{\infty} \quad \text{forms a complete orthonormal system in } L_2(D), \tag{9.4}$$

$$\int_D U(t,x,y)\varphi_j(y)dy = e^{-\lambda_j t}\varphi_j(x) \quad \text{on } D, \tag{9.5}$$

$$A\varphi_j(x) = -\lambda_j \varphi_j(x) \quad \text{on } D, \tag{9.6}$$

and

$$\varphi_j (j = 0, 1, \cdots) \text{ satisfies the boundary condition (9.2).} \tag{9.7}$$

Then,

$$U(t,x,y) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\varphi_j(x)\varphi_j(y) \tag{9.8}$$

converges uniformly on $[\delta,\infty) \times \bar{D} \times \bar{D}$ for any fixed $\delta > 0$. For any $f \in L_2(D)$ and for

$$f(x) = \sum_{j=0}^{\infty} C_j \varphi_j(x), \tag{9.9}$$

$$(U_t f)(x) = \sum_{j=0}^{\infty} C_j e^{-\lambda_j t} \varphi_j(x) \tag{9.10}$$

converges uniformly on $[\delta, \infty) \times \bar{D}$ for any $\delta > 0$. Of course, (9.10) represents a "general" solution of (9.1) satisfying the boundary condition (9.2) and the initial condition

$$\lim_{t \to +0} \|(U_t f)(x) - f(x)\|_{L_2(D,dx)} = 0.$$

For these properties, see, for example, [17]. By using the fact that (9.10) converges uniformly on $[\delta, \infty) \times \bar{D}$ for any fixed $\delta > 0$, we can give .

Theorem 9.1 *([45]).* *$\{C_j\}_{j=0}^{\infty}$ and so, f and $(U_t f)(x)$ on $\{t > 0\} \times D$ can be determined and represented by the observation*

$$(U_t f)(x) \quad (t > \tau,\ x \in E) \tag{9.11}$$

for any fixed large positive constant τ and for a very small set E around any fixed point $x^* \in \bar{D}$.

Furthermore, a general corresponding result for the solutions of hyperbolic type is derived by using the Reznitskaya transform. These results may be called the "*principle of telethoscope*".

Acknowledgments

This research was partially supported by the Japanese Ministry of Education, Science, Sports and Culture; Grant–in–Aid Scientific Research, Kiban Kenkyuu (A)(1), 10304009.

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