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Continuation of Holomorphic Functions from Subvarieties to Pseudoconvex Domains

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1. Introduction

Let D be a bounded pseudoconvex domain in $\mathbf{C^n}$ and V a subvariety of D. In the present paper, we give some recent results concerning holomorphic extensions from V to D in some function spaces. In 1965, Hörmander obtained L^2 estimates for the $\bar{\partial}$ problem in bounded pseudoconvex domains in $\mathbf{C^n}$. In 1970, Henkin, Grauert-Lieb and Lieb obtained the uniform estimates for the $\bar{\partial}$ problem in strictly pseudoconvex domains in $\mathbf{C}^n$ with smooth boundary. Corresponding to these results, extension problems were studied by two different methods. The one is the extension using the integral formula in the case where D is a bounded pseudoconvex domain with a support function (for example, bounded strictly pseudoconvex domains or bounded convex domains with smooth boundary). The other is the L^2 extension using the Hilbert space theory in the case where D is a general bounded pseudoconvex domain. The main purpose of the present paper is to introduce Berndtsson's another proof of the L^2 extension theorem of Ohsawa-Takegoshi in bounded pseudoconvex domains.

2. Some recent results

Definition. Let D be an open set in $\mathbf{C}^n$ and $\varphi \in C^\infty(D)$ a real function. We denote by $L^2(D,\varphi)$ the space of square-integrable functions in D with respect to the measure $e^{-\varphi}d\mu$, where $d\mu$ is the Lebesgue measure in $\mathbf{C}^n$. We denote by $L^2_{(p,q)}(D,\varphi)$ the space of (p,q)-forms with coefficients in $L^2(D,\varphi)$,

$$f = \sum_{|I|=p,|J|=q}{}' f_{I,J} dz^I \wedge d\bar{z}^J,$$

where $\sum'$ means that the summation is performed only over strictly increasing multi-indices. We set

$$|f|^2 = \sum_{I,J}{}' |f_{I,J}|^2, \quad \|f\| = \left(\int_D |f|^2 e^{-\varphi} d\mu\right)^{\frac{1}{2}}.$$

For $f, g \in L^2_{(p,q)}(D,\varphi)$ with $f = \sum_{I,J}' f_{I,J} dz^I \wedge d\bar{z}^J$, $g = \sum_{I,J}' g_{I,J} dz^I \wedge d\bar{z}^J$, we define the inner product in $L^2_{(p,q)}(D,\varphi)$ by

$$(f,g) = \sum_{I,J}' \int_D f_{I,J}\overline{g_{I,J}} e^{-\varphi} d\mu.$$

Then $L^2_{(p,q)}(D,\varphi)$ is a Hilbert space with this inner product.

Theorem 1.(Hörmander[14]) *Let D be a bounded pseudoconvex open set in $\mathbf{C}^n$, let δ be the diameter of D, and let ψ be a plurisubharmonic function in D. For every $f \in L^2_{p,q}(D,\varphi)$, $q > 0$, with $\bar{\partial} f = 0$, one can then find $u \in L^2_{(p,q-1)}(D,\varphi)$ such that $\bar{\partial} u = f$ and*

$$q \int_D |u|^2 e^{-\varphi} dV \leq e\delta^2 \int_D |f|^2 e^{-\varphi} dV$$

Theorem 2.(Henkin[10], Ramirez[17]) *Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. Then there exist a pseudoconvex domain $\tilde{D} \supset \overline{D}$ and functions $K(\zeta,z)$ and $\Phi(\zeta,z)$ defined for $\zeta \in \partial D$ and $z \in \tilde{D}$ such that*

(1) *$K(\zeta,z)$ and $\Phi(\zeta,z)$ are holomorphic in $z \in \tilde{D}$ and continuous in $\zeta \in \partial D$*

(2) *For every $\zeta \in \partial D$ the function $\Phi(\zeta,z)$ vanishes on the closure $\overline{D}$ only at the point $z = \zeta$.*

(3) *For any holomorphic function f in D that is continuous on $\overline{D}$ and any $z \in D$, the integral formula*

$$f(z) = \int_{\partial D} f(\zeta) \frac{K(\zeta,z)}{\Phi(\zeta,z)^n} d\sigma(\zeta)$$

holds, where $d\sigma$ is the (2n-1) dimensional Lebesgue measure on ∂D.

Definition. Let $f(x)$ be a function on D. Then we define

$$|f|_0 = \sup_{x \in D} |f(x)|.$$

Let f be a (0,q)-form with the coefficients $f_{i_1,\cdots,i_q}$. Then we define

$$|f|_0 = \max_{i_1,\cdots,i_q} |f_{i_1,\cdots,i_q}|_0.$$

Theorem 3.(Henkin[11], Grauert-Lieb[8], Lieb[15]) *Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. Then there exists a constant K such that if f is a $\bar{\partial}$ closed C^∞(0,q+1)-form on D, then there exists a C^∞(0,q)-form u on D with*

$$\bar{\partial} u = f \quad \textit{and} \quad |u|_0 \leq K|f|_0.$$

Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary and let $\tilde{M}$ be a submanifold in a neighborhood $\tilde{D}$ of $\overline{D}$ which meets ∂D transversally. We set $M = \tilde{M} \cap D$. Let Ω be a domain in some complex manifold. We denote by $H^\infty(\Omega)$ the space of all bounded holomorphic functions in Ω. We also denote by $A^\infty(\Omega)$ the space of all holomorphic functions in Ω that are C^∞ on $\overline{\Omega}$. In this setting, we have theorem 4 and 5.

Theorem 4.(Henkin[12]) *There exists a linear extension operator $E : H^\infty(M) \to H^\infty(D)$. Moreover, Ef is continuous on $\overline{D}$ if f is continuous on $\overline{M}$.*

Theorem 5.(Adachi[1], Elgueta[7]) *There exists a linear extension operator $E : A^\infty(M) \to A^\infty(D)$.*

Remark. *Amar[4] proved theorem 5 when D is pseudoconvex. Henkin-Leiterer[13] proved theorem 4 without assuming the transversality.*

Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. Let $\gamma : \partial D \times D \to \mathbf{C}^n$ be a smooth mapping such that

$$(\zeta - z, \gamma) = \sum_{j=1}^{n} (\zeta_j - z_j)\gamma_j(\zeta, z) \neq 0 \quad \text{on} \quad \partial D \times D.$$

Let $h_1, \cdots, h_m (m < n)$ be holomorphic functions in a neighborhood $\tilde{D}$ of $\overline{D}$. Define

$$\tilde{V} = \{z \in \tilde{D} | h_1(z) = \cdots = h_m(z) = 0\}, \quad V = \tilde{V} \cap D.$$

We say V intersects ∂D transversally if

$$d\rho \wedge \partial h_1 \wedge \cdots \wedge \partial h_m \neq 0 \quad \text{on} \quad \partial V.$$

In the above setting, we have the following:

Theorem 6.(Stout[19], Hatziafratis[9]) *There is a smooth form $K_V(\zeta, z)$ on $\partial V \times \overline{V}$ which is of type (0,0) in z and (n-m-1, n-m) in ζ such that if f is holomorphic in V and continuous on $\overline{V}$, then for $z \in V$*

$$f(z) = \int_{\zeta \in \partial V} f(\zeta) \frac{K_V(\zeta, z)}{(\zeta - z, \gamma(\zeta, z))^{n-m}}. \tag{1}$$

Moreover, $K_V(\zeta, z)$ is holomorphic in $z \in D$ provided that $\gamma(\zeta, z)$ is holomorphic in $z \in D$.

Let D be a bounded convex domain with a defining function ρ. Then we can choose

$$\gamma_i(\zeta, z) = \frac{\partial \rho}{\partial \zeta_i}(\zeta).$$

Let E(f)(z) be the right hand side of (1). Then we have

Theorem 7.(Adachi-Cho[3]) *Let D be a bounded convex domain in $\mathbf{C}^n$ with real analytic boundary and let V be a one dimensional subvariety of D defined above. Then we have*

(1) *Let $1 \leq p < \infty$. If $f \in H^p(V)$, then $E(f) \in H^p(D)$.*

(2) *Suppose that V has no singular points and $1 \leq p < \infty$. If $f \in \mathcal{O}(V) \cap L^p(V)$, then $E(f) \in \mathcal{O}(D) \cap L^p(D)$,*

where $\mathcal{O}(V)$(resp. $\mathcal{O}(D)$) denotes the space of all holomorphic functions in V(resp. D).

A bounded domain $\Omega \subset \mathbf{C}^n$ is an analytic polyhedron with defining functions ϕ_j if

$$\Omega = \{z \in \mathbf{C}^n | |\phi_j(z)| < 1, j = 1, \cdots, N\},$$

where the defining functions ϕ_j are holomorphic in some neighborhood $\tilde{\Omega}$ of $\overline{\Omega}$. We set $\sigma_I = \{z \in \overline{\Omega} | |\phi_j(z)| = 1, j \in I\}$. We say that Ω is non-degenerate if $\partial\phi_{i_1} \wedge \cdots \wedge \partial\phi_{i_k} \neq 0$ on σ_I for every multiindex $I = \{i_1, \cdots, i_k\}$ such that $|I| = k \leq n$. We say that Ω is strongly non-degenerate if $\partial\phi_{i_1} \wedge \cdots \wedge \partial\phi_{i_k} \neq 0$ on σ_I for all multiindices I. Let $\tilde{V}$ be a regular subvariety of $\tilde{\Omega}$ of codimension m given by

$$\tilde{V} = \{z \in \tilde{\Omega} | h_1(z) = \cdots = h_m(z) = 0\},$$

where $h_j \in \mathcal{O}(\tilde{\Omega})$, and $\partial h_1 \wedge \cdots \wedge h_m \neq 0$ on $\tilde{V}$. We set $V = \tilde{V} \cap \Omega$. We impose the transversal assumption that

$$\partial h_1 \wedge \cdots \wedge \partial h_m \wedge \partial\phi_{i_1} \wedge \cdots \wedge \partial\phi_{i_k} \neq 0 \quad \text{on} \quad \overline{V} \cap \sigma_I,$$

for every multiindex I such that $|I| = k \leq n-m$. For a strongly non-degenerate polyhedron Ω we can define the Hardy spaces

$$H^p(\Omega) = \left\{ f \in \mathcal{O}(\Omega) | \sup_{\varepsilon > 0} \|f\|_{L^p(\sigma_\varepsilon)} < \infty \right\}.$$

In the above setting, we have by applying the integral formula obtained by Berndtsson[5]:

Theorem 8.(Adachi-Andersson-Cho[3])

(1) *Let Ω be a non-degenerate analytic polyhedron. For each $f \in \mathcal{O}(V) \cap L^p(V), 1 \leq p < \infty$, there exists $F \in \mathcal{O}(\Omega) \cap L^p(\Omega)$ such that $F(z) = f(z)$ for $z \in V$ and $||F||_{L^p(\Omega)} \leq C||f||_{L^p(V)}$.*

(2) *Let Ω be a strongly non-degeneate analytic polyhedron. Then for all $f \in H^p(V)$, $1 < p \leq \infty$, there exists $F \in H^p(\Omega)$ such that $F(z) = f(z)$ for $z \in V$ and $||F||_{H^p(\Omega)} \leq c||f||_{H^p(V)}$.*

3. Outline of the proof of the theorem of Ohsawa-Takegoshi due to Berndtsson

In this section, we shall prove the extension theorem of Ohsawa-Takegoshi by following the Berndtsson's proof[6]. Using L^2 space techniques, Ohsawa and Takegoshi obtained the following:

Theorem 9.(Ohsawa-Takegoshi[16]) *Let D be a bounded pseudoconvex domain in C^n. We set $H = \{z \in \mathbf{C}^n | z_1 = 0\}$. Then there exists a constant C which depends only on the diameter of D such that, for any plurisubharmonic function φ on D, and for any holomorphic function f on $H \cap D$, there exists a holomorphic function F in D such that*

$$F|_{H\cap D} = f, \quad \int_D |F|^2 e^{-\varphi} d\mu \leq C \int_{H\cap D} |f|^2 e^{-\varphi} d\mu_1,$$

where $d\mu$ and $d\mu_1$ are Lebesgue measures in $\mathbf{C}^n$ and $\mathbf{C}^{n-1}$, respectively.

Lemma 1.(Hörmander[14]) Let D be a bounded open set in $\mathbf{C}^n$ with smooth boundary ∂D and let ρ be a smooth defining function for D. For $f = \sum_J{}' f_J d\bar{z}^J \in C^1_{(0,q)}(\overline{D})$ and $u = \sum_K{}' u_K d\bar{z}^K \in C^1_{(0,q-1)}(\overline{D})$, the following equality is valid

$$(\bar{\partial} u, f) = -\int_D \sum_K{}' \sum_{j=1}^n u_K \overline{\delta_j f_{jK}} e^{-\varphi} d\mu + \int_{\partial D} \sum_K{}' u_K \overline{\sum_{j=1}^n f_{jK} \frac{\partial \rho}{\partial z_j}} e^{-\varphi} \frac{dS}{|\partial \rho|}.$$

Definition. For $u \in C^1(D)$, define

$$\delta_j u = e^{\varphi} \frac{\partial}{\partial z_j}(u e^{-\varphi}) = \frac{\partial u}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} u, \quad \partial_k u = \frac{\partial u}{\partial z_k} \quad \bar{\partial}_k u = \frac{\partial u}{\partial \bar{z}_k}.$$

For $C^1(0,q)$-form $f = \sum_{|J|=q}{}' f_J d\bar{z}^J$, define $\bar{\partial}^* f = -\sum_K{}' \sum_{j=1}^n \delta_j f_{jK} d\bar{z}^K$. We define

$$f \in \mathrm{Def}(\bar{\partial}^*) \Longleftrightarrow \sum_{j=1}^n f_{jK} \frac{\partial \rho}{\partial z_j} = 0 \quad \text{on} \quad \partial D \quad \text{for all} \quad K.$$

We say f satisfies the boundary condition if $f \in \mathrm{Def}(\bar{\partial}^*)$. When f satisfies the boundary condition, we have from lemma 1

$$(\bar{\partial}u, f) = (u, \bar{\partial}^* f).$$

Lemma 2.(Hörmander[14]) Let $\alpha = \sum_{|J|=q}{}' \alpha_J d\bar{z}^J$ be a smooth (0,q)-form in $\overline{D}$ and $\alpha \in \mathrm{Def}(\bar{\partial}^*)$. For $\varphi \in C^\infty(\overline{D})$ we have

$$\begin{aligned} \|\bar{\partial}^*\alpha\| + \|\bar{\partial}\alpha\|^2 &= \sum_K{}' \sum_{j,k=1}^n \int_D \alpha_{jK}\overline{\alpha_{kK}} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} e^{-\varphi} d\mu + \sum_J{}' \sum_{j=1}^n \int_D \left| \frac{\partial \alpha_J}{\partial \bar{z}_j} \right|^2 e^{-\varphi} d\mu \\ &+ \sum_K{}' \sum_{j,k=1}^n \int_{\partial D} \alpha_{jK}\overline{\alpha_{kK}} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} e^{-\varphi} \frac{dS}{|\partial \rho|}. \end{aligned}$$

We assume that φ is a smooth function in $\overline{D}$ from lemma 3 to lemma 7. Thus $f \in L^2(D, \varphi)$ means $f \in L^2(D)$. We omit the proof of lemma 3, since the detailed proof of lemma 3 is given in [6].

Lemma 3. Let w be a real valued smooth function in $\overline{D}$. $\alpha = \sum_{j=1}^n \alpha_j d\bar{z}_j$ is a smooth $(0,1)$-form in $\overline{D}$ satisfying the boundary condition. Then we have

$$\begin{aligned} &\int_D w \sum_{j,k=1}^n \varphi_{j\bar{k}} \alpha_j \overline{\alpha_k} e^{-\varphi} d\mu - \int_D w_{j\bar{k}} \alpha_j \overline{\alpha_k} e^{-\varphi} d\mu \\ &+ \int_D w |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \int_D w \sum_{j,k=1}^n \left| \frac{\partial \alpha_k}{\partial \bar{z}_j} \right|^2 e^{-\varphi} d\mu + \int_{\partial D} w \sum_{j,k=1}^n \rho_{j\bar{k}} \alpha_j \overline{\alpha_k} e^{-\varphi} \frac{dS}{|\partial \rho|} \\ &= 2\mathrm{Re} \int_D w \bar{\partial} \bar{\partial}^* \alpha \cdot \bar{\alpha} e^{-\varphi} d\mu + \int_D w |\bar{\partial}\alpha|^2 e^{-\varphi} d\mu. \end{aligned}$$

Definition. Let $\psi \in C^\infty(\overline{D})$ and $\alpha = \sum_{j=1}^n \alpha_j d\bar{z}_j \in C^\infty_{(0,1)}(\overline{D})$. We define the inner product of $g = \psi \bar{\partial} \left(\frac{1}{z_1} \right)$ and α by

$$< g, \alpha >= \sum_{j=1}^n < \psi \frac{\partial}{\partial \bar{z}_j} \left(\frac{1}{z_1} \right), \alpha_j >= \lim_{\varepsilon \to 0} \int_D \psi(z) \frac{\partial}{\partial \bar{z}_1} \left(\frac{\bar{z}_1}{|z_1|^2 + \varepsilon} \right) \overline{\alpha_1(z)} e^{-\varphi(z)} d\mu(z).$$

Moreover, if we define

$$h_\varepsilon(z) = \psi(z)\bar\partial\left(\frac{\overline{z_1}}{|z_1|^2+\varepsilon}\right),$$

then we obtain

$$< g, \alpha >= \lim_{\varepsilon\to 0}(h_\varepsilon, \alpha). \tag{2}$$

In view of lemma 6, the right hand side of (2) exists. For $u \in L^1(D)$ and a $(0,1)$-form α in D with compact support, we define

$$< \bar\partial u, \alpha >= (u, \bar\partial^*\alpha).$$

Then we have the following:

Lemma 4. Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. Let f be a holomorphic function in $\overline{D}$ and $g = f\bar\partial\left(\frac{1}{z_1}\right)$. Let $u \in L^1(D)$. If the equality

$$< g, \alpha >= \int_D u\overline{\bar\partial^*\alpha}e^{-\varphi}d\mu$$

holds for any $\bar\partial$ closed $\alpha \in C^\infty_{(0,1)}(\overline{D})$which satisfies the boundary condition, then $g = \bar\partial u$ in the sense of distribution.

Proof. Let α be a C^∞(0,1)-form in D with compact support. We define

$$\mathrm{Def}(\bar\partial) = \{g \in L^2_{(0,q)}(D,\varphi)|\bar\partial g \in L^2_{(0,q+1)}(D,\varphi)\}.$$

For Laplace-Beltrami operator $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : L^2_{(0,1)}(D,\varphi) \to L^2_{(0,1)}(D,\varphi)$, define

$$\mathrm{Def}(\Box) = \{\alpha \in L^2_{(0,1)}(D,\varphi)|\alpha \in \mathrm{Def}(\bar\partial), \bar\partial\alpha \in \mathrm{Def}(\bar\partial^*), \alpha \in \mathrm{Def}(\bar\partial^*), \bar\partial^*\alpha \in \mathrm{Def}(\bar\partial)\},$$

$$\mathcal{H} = \{\alpha \in \mathrm{Def}(\Box)| \quad \Box\alpha = 0\}.$$

Then $\mathcal{H}$ is a closed subspace of the Hilbert space $L^2_{(0,1)}(D,\varphi)$. Let $H : L^2_{(0,1)}(D,\varphi) \to \mathcal{H}$ be the orthogonal projection. From the theory of the $\bar\partial$ Neumann problem, there exists Neumann operator $\mathcal{N} : L^2_{(0,1)}(D,\varphi) \to \mathrm{Def}(\Box)$ such that

$$\alpha = \bar\partial\bar\partial^*\mathcal{N}\alpha + \bar\partial^*\bar\partial\mathcal{N}\alpha + H\alpha.$$

For $\beta \in \mathcal{H}$, we have

$$0 = (\Box\beta, \beta) = (\bar\partial\bar\partial^*\beta, \beta) + (\bar\partial^*\bar\partial\beta, \beta) = \|\bar\partial^*\beta\|^2 + \|\bar\partial\beta\|^2.$$

Hence we obtain $\bar{\partial}\beta = \bar{\partial}^*\beta = 0$. From lemma 2, it holds that

$$\begin{aligned} 0 = \|\bar{\partial}\beta\|^2 + \|\bar{\partial}^*\beta\|^2 &\geq \sum_{j,k=1}^{n} \|\frac{\partial \beta_j}{\partial \bar{z}_k}\|^2 + \int_{\partial D} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_k} \beta_j \bar{\beta}_k \frac{dS}{|\partial \rho|} \\ &\geq \sum_{j,k=1}^{n} \|\frac{\partial \beta_j}{\partial \bar{z}_k}\|^2 + c\int_{\partial D} |\beta|^2 \frac{dS}{|\partial \rho|}. \end{aligned}$$

Thus β_j is holomorphic in D and 0 in ∂D so that $\beta = 0$. Therefore $\mathcal{H} = 0$. We set

$$\alpha_1 = \bar{\partial}\bar{\partial}^* \mathcal{N}\alpha, \quad \alpha_2 = \bar{\partial}^* \bar{\partial} \mathcal{N}\alpha.$$

Since Neumann operator maps smooth $(0,1)$-forms to smooth $(0,1)$-forms in the strictly pseudoconvex domain D, α_1 and α_2 are both smooth $(0,1)$-forms in $\overline{D}$. Obviously, $\bar{\partial}\alpha_1 = 0$. If $\bar{\partial}\beta = 0$, then by lemma 1 $(\beta, \alpha_2) = (\bar{\partial}\beta, \bar{\partial}\mathcal{N}\alpha) = 0$. Hence $\alpha_2 \perp \mathrm{Ker}(\bar{\partial})$. On the other hand, from lemma 1, for any smooth function β on $\overline{D}$, we have

$$0 = (\bar{\partial}\beta, \alpha_2) = (\beta, \bar{\partial}^*\alpha_2) + \int_{\partial D} \beta \overline{\alpha_2 \cdot \partial \rho} e^{-\varphi} \frac{dS}{|\partial \rho|}.$$

Thus $\bar{\partial}^*\alpha_2 = 0$. Therefore α_2 satisfies the boundary condition. Hence, α_1 satisfies the boundary condition. If we set

$$h_\varepsilon(z) = f(z)\bar{\partial}\left(\frac{\overline{z_1}}{|z_1|^2 + \varepsilon}\right),$$

then we have

$$< g, \alpha_2 > = \lim_{\varepsilon \to 0}(h_\varepsilon, \alpha_2) = 0.$$

Thus we have

$$< g, \alpha > = < g, \alpha_1 > = \int_D u\overline{\bar{\partial}^*\alpha_1} e^{-\varphi} d\mu = \int_D u\overline{\bar{\partial}^*\alpha} e^{-\varphi} d\mu = (u, \bar{\partial}^*\alpha) = < \bar{\partial}u, \alpha >,$$

which means $g = \bar{\partial}u$.

Lemma 5. Let g be the same as in lemma 4. Let λ be a non-negative real valued function in D with the property that $\frac{1}{\lambda}$ is integrable. If the inequality

$$| < g, \alpha > |^2 \leq C \int_D |\bar{\partial}^*\alpha|^2 \frac{e^{-\varphi}}{\lambda} d\mu$$

holds for any $\bar{\partial}$ closed $\alpha \in C^\infty_{(0,1)}(\overline{D})$ which satisfies the boundary condition, then there exists $u \in L^1(D, \varphi)$ such that

$$\bar{\partial}u = g, \quad \int_D |u|^2 \lambda e^{-\varphi} d\mu \leq C.$$

Proof. Let $C_b^\infty(\overline{D})$ be the space of all $\bar\partial$ closed $C^\infty(0,1)$-forms in $\overline{D}$ which satisfies the boundary condition. We set

$$F = \{\bar\partial^* \alpha | \alpha \in C_b^\infty(\overline{D})\}, \quad \varphi_1 = \frac{e^{-\varphi}}{\lambda}.$$

Then, F is a vector subspace of $L^2(D, \varphi_1)$. For $w \in F$, there exists $\alpha \in C_b^\infty(\overline{D})$ such that $w = \bar\partial^* \alpha$. We define

$$\Phi(w) =< g, \alpha > .$$

Then $\Phi(w)$ is independent of the choice of α. Also, we have

$$|\Phi(w)|^2 \leq C\|w\|_{\varphi_1}^2, \quad \|\Phi\| \leq \sqrt{C}.$$

Thus Φ is a bounded anti-linear operator on F. From the Hahn-Banach theorem, Φ is extended to a bounded anti-linear operator on $L^2(D, \varphi_1)$. From the Riesz representation theorem, there exists $v \in L^2(D, \varphi_1)$ such that

$$\Phi(w) = (v, w)_{\varphi_1}, \quad \|v\|_{\varphi_1} = \|\Phi\| \leq \sqrt{C}.$$

Therefore we have

$$< g, \alpha >= \Phi(w) = (v, w)_{\varphi_1} = \int_D v\overline{\bar\partial^* \alpha} \frac{e^{-\varphi}}{\lambda}, \quad \int_D |v|^2 \frac{e^{-\varphi}}{\lambda} d\mu = \|v\|_{\varphi_1}^2 \leq C.$$

If we set $u = \dfrac{v}{\lambda}$, then

$$\int_D |u|^2 \lambda e^{-\varphi} d\mu \leq C, \quad < g, \alpha >= \int_D u\overline{\bar\partial^* \alpha} e^{-\varphi} d\mu.$$

On the other hand, we have

$$\int_D |u| e^{-\varphi} d\mu \leq \int_D \frac{|v|^2}{\lambda} e^{-\varphi} d\mu \int_D \frac{e^{-\varphi}}{\lambda} d\mu \leq C \int_D \frac{e^{-\varphi}}{\lambda} d\mu < \infty.$$

Thus, $u \in L^1(D, \varphi)$. From lemma 4, we obtain $\bar\partial u = g$.

Lemma 6. For $\varphi \in C^\infty(\overline{D})$, it holds that

$$\lim_{\varepsilon \to 0} \int_D \frac{\varepsilon}{(|z_1|^2 + \varepsilon)^2} \varphi(z) d\mu(z) = \pi \int_{\{z_1 = 0\} \cap D} \varphi(z) d\mu_1(z),$$

where $d\mu$ and $d\mu_1$ are Lebesgue measures in $\mathbf{C}^n$ and $\mathbf{C}^{n-1}$, respectively.

Lemma 7. Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary and $D \subset \{z||z_1| \leq 1\}$. Let φ be a smooth plurisubharmonic function in $\overline{D}$ and let α be a $\bar{\partial}$ closed smooth $(0,1)$-form in $\overline{D}$ which satisfies the boundary condition. Then, for $0 < \delta < 1$, we have

$$\int_{\{z_1=0\}\cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 \leq \frac{2}{\pi}\left(1+\frac{1}{\delta^2}\right)\int_D \frac{|\bar{\partial}^*\alpha|^2}{|z_1|^{2\delta}} e^{-\varphi} d\mu.$$

Proof. For $0 < \delta < 1$, we set

$$w^\delta = 1 - |z_1|^{2\delta} = 1 - (z_1\bar{z}_1)^\delta.$$

From lemma 3, we have

$$\begin{aligned}
&\int_D w^\delta \sum_{j,k=1}^n \varphi_{j\bar{k}}\alpha_j\overline{\alpha_k} e^{-\varphi} d\mu + \delta^2 \int_D |z_1|^{2\delta-2}|\alpha_1|^2 e^{-\varphi} d\mu + \int_D w^\delta |\bar{\partial}^*\alpha|^2 e^{-\varphi} d\mu \\
&+ \int_D w^\delta \sum_{j,k=1}^n \left|\frac{\partial \alpha_j}{\partial \bar{z}_k}\right|^2 e^{-\varphi} d\mu + \int_{\partial D} w^\delta \sum_{j,k=1}^n \rho_{jk}\alpha_j\overline{\alpha_k} e^{-\varphi}\frac{dS}{|\partial\rho|} \\
&= 2\mathrm{Re}\int_D w^\delta \bar{\partial}\bar{\partial}^*\alpha \cdot \bar{\alpha} e^{-\varphi} d\mu.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\delta^2 \int_D |z_1|^{2\delta-2}|\alpha_1|^2 e^{-\varphi} d\mu + \int_D w^\delta |\partial^*\alpha|^2 e^{-\varphi} d\mu \leq 2\mathrm{Re}\int_D w^\delta \bar{\partial}\bar{\partial}^*\alpha \cdot \bar{\alpha} e^{-\varphi} d\mu \\
&= 2\mathrm{Re}(\bar{\partial}^*\alpha, \bar{\partial}^*(w^\delta\alpha)) = 2\mathrm{Re}(\bar{\partial}^*\alpha, w^\delta\bar{\partial}^*\alpha - \sum_{j=1}^n \frac{\partial w^\delta}{\partial z_j}\alpha_j) \\
&= 2\int_D w^\delta |\bar{\partial}^*\alpha|^2 e^{-\varphi} d\mu - 2\mathrm{Re}\int_D \bar{\partial}^*\alpha \overline{\frac{\partial w^\delta}{\partial z_1}\alpha_1} e^{-\varphi} d\mu \\
&\leq 2\int_D w^\delta |\bar{\partial}^*\alpha|^2 e^{-\varphi} d\mu + 2\int_D |\bar{\partial}^*\alpha|\delta|z_1|^{2\delta-1}|\alpha_1| e^{-\varphi} d\mu \\
&\leq 2\int_D w^\delta |\bar{\partial}^*\alpha|^2 e^{-\varphi} d\mu + 2\int_D |\bar{\partial}^*\alpha|^2|z_1|^{2\delta} e^{-\varphi} d\mu + \frac{1}{2}\int_D \delta^2|\alpha_1|^2|z_1|^{2\delta-2} e^{-\varphi} d\mu.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{1}{2}\delta^2 \int_D |z_1|^{2\delta-2}|\alpha_1|^2 e^{-\varphi} d\mu &\leq \int_D (1-|z_1|^{2\delta})|\bar{\partial}^*\alpha|^2 e^{-\varphi} d\mu + 2\int_D |\bar{\partial}^*\alpha|^2|z_1|^{2\delta} e^{-\varphi} d\mu \\
&= \int_D |\bar{\partial}^*\alpha|^2 e^{-\varphi} d\mu + \int_D |\bar{\partial}^*\alpha|^2|z_1|^{2\delta} e^{-\varphi} d\mu \leq 2\int_D |\bar{\partial}^*\alpha|^2 e^{-\varphi} d\mu.
\end{aligned}$$

Therefore, for $0 < \delta < 1$, we obtain

$$(3) \qquad \delta^2 \int_D |z_1|^{2\delta-2} |\alpha_1|^2 e^{-\varphi} d\mu \leq 4 \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu.$$

On the other hand, we set

$$w_\varepsilon = \frac{1}{\pi} \log \frac{1}{|z_1|^2 + \varepsilon}, \quad w = \frac{1}{\pi} \log \frac{1}{|z_1|^2}.$$

We apply lemma 3 to w_ε and let $\varepsilon \to 0$, then by lemma 6

$$\int_{\{z_1=0\}\cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 + \int_D w |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu \leq 2\mathrm{Re} \int_D w \bar{\partial} \bar{\partial}^* \alpha \cdot \bar{\alpha} e^{-\varphi} d\mu.$$

By the same calculation as the first part and applying (3) to $0 < \delta < 1$, we have

$$\int_{\{z_1=0\}\cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 \leq \frac{1}{\pi} \int_D \log \frac{1}{|z_1|^2} |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{2}{\pi} \int_D |\bar{\partial}^* \alpha| \frac{|\alpha_1|}{|z_1|} e^{-\varphi} d\mu$$

$$\leq \frac{1}{\pi} \int_D \log \frac{1}{|z_1|^2} |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{2}{\pi} \int_D \frac{|\bar{\partial}^* \alpha|^2}{|z_1|^{2\delta}} e^{-\varphi} d\mu + \frac{1}{2\pi} \int_D |z_1|^{2\delta-2} |\alpha_1|^2 e^{-\varphi} d\mu$$

$$\leq \frac{1}{\pi} \int_D \log \frac{1}{|z_1|^2} |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{2}{\pi} \int_D \frac{|\bar{\partial}^* \alpha|^2}{|z_1|^{2\delta}} e^{-\varphi} d\mu + \frac{2}{\pi\delta^2} \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu$$

$$\leq \frac{1}{\pi\delta^2} \int_D \log \frac{1}{|z_1|^{2\delta}} |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{2}{\pi} \int_D \frac{|\bar{\partial}^* \alpha|^2}{|z_1|^{2\delta}} e^{-\varphi} d\mu + \frac{2}{\pi\delta^2} \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu.$$

Using the fact that $x\left(\log \frac{1}{x} + 2\right) \leq 2$ for $0 < x \leq 1$, we have

$$\int_{\{z_1=0\}\cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 \leq \frac{2}{\pi} \int_D \frac{|\bar{\partial}^* \alpha|^2}{|z_1|^{2\delta}} e^{-\varphi} d\mu + \frac{1}{\pi\delta^2} \int_D \frac{2|\bar{\partial}^* \alpha|^2}{|z_1|^{2\delta}} e^{-\varphi} d\mu$$

$$= \frac{2}{\pi} \left(1 + \frac{1}{\delta^2}\right) \int_D \frac{|\bar{\partial}^* \alpha|^2}{|z_1|^{2\delta}} e^{-\varphi} d\mu,$$

which completes the proof.

Lemma 8. Let D be a pseudoconvex domain in $\mathbf{C}^n$ and $X = \{z \in D | z_1 = 0\}$. Let f be a holomorphic function in X. If H is locally integrable in D and satisfies $\bar{\partial} H = f \bar{\partial}\left(\frac{1}{z_1}\right)$, then there exists a holomorphic function $\tilde{H}$ in D such that $\tilde{H}(z) = z_1 H(z)$ a.e. and $\tilde{H}(z) = f(z)$ for $z \in X$.

Proof. There exists a neighborhood ω of X in D such that f can be extended to be holomorphic in ω. Let $\chi \in C^\infty(D)$ be a function such that $\chi = 1$ in a neighborhood of X in ω, $\mathrm{supp}(\chi) \subset \omega$ and $0 \leq \chi \leq 1$ in D. We set

$$\omega = \frac{f\bar{\partial}\chi}{z_1}.$$

Then ω satisfies that $\omega \in C^\infty_{(0,1)}(D), \bar{\partial}\omega = 0$. Define

$$G = \frac{\chi f}{z_1} - H,$$

then G is locally integrable. Since we have

$$\bar{\partial}G = \bar{\partial}(\chi f)\frac{1}{z_1} + \chi f\bar{\partial}\left(\frac{1}{z_1}\right) - \bar{\partial}H = f\bar{\partial}\chi\frac{1}{z_1} + \chi f\bar{\partial}\left(\frac{1}{z_1}\right) - \bar{\partial}H = \bar{\partial}\chi\frac{f}{z_1} = \omega,$$

there exists a smooth function $\tilde{G}$ in D such that $\tilde{G} = G$ a.e.. We set

$$\chi(z)f(z) - z_1\tilde{G}(z) = \tilde{H}(z),$$

then we have $z_1H(z) = \tilde{H}(z)$ a.e. and $\tilde{H}(z) = f(z)$ for $z \in X$. Moreover we have

$$\bar{\partial}\tilde{H}(z) = (\bar{\partial}\chi(z))f(z) - z_1\bar{\partial}\tilde{G}(z) = (\bar{\partial}\chi(z))f(z) - z_1\omega(z) = 0.$$

Hence $\tilde{H}(z)$ is holomorphic in D.

Lemma 9. Let D be an open set in $\mathbf{C}^n$ and let $K \subset D$ be a compact set. Then there exists a constant C such that for any holomorphic function f in D and any neighborhood ω of K

$$\sup_K |f| \leq C\|f\|_{L^1(\omega)}.$$

Lemma 10. Let $\{u_k\}$ be a sequence of holomorphic functions in D which are uniformly bounded on any compact subset of D. Then there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that $\{u_{k_j}\}$ converges uniformly on any compact subset of D to a holomorphic function in D .

Theorem 10.(Berndtsson[6]) Let D be a bounded pseudoconvex domain in C^n and let φ be a plurisubharmonic function in D. We set $X = \{z \in D | z_1 = 0\}$. Suppose that

$D \subset \{z \in \mathbf{C}^n \mid |z_1| \leq A\}$. If f is holomorphic in X, then there exists a holomorphic function F in D such that

$$F|_X = f, \quad \int_D |F|^2 e^{-\varphi} d\mu \leq 4A^2\pi \int_X |f|^2 e^{-\varphi} d\mu_1,$$

where $d\mu$ and $d\mu_1$ are Lebesgue measures in $\mathbf{C}^n$ and $\mathbf{C}^{n-1}$, respectively.

Proof. Without loss of generality, we may assume that $A = 1$. There exists an increasing sequence of bounded strictly pseudoconvex domains in $\mathbf{C}^n$ with smooth boundary such that $\overline{D_n} \subset\subset D$ and $\bigcup_{n=1}^{\infty} D_n = D$. Let $\{\varphi_n\}$ be a sequence of C^∞ plurisubharmonic functions in $\overline{D_n}$ such that $\varphi_n \downarrow \varphi$. We set $g = f\bar{\partial}\left(\frac{1}{z_1}\right)$. Let α be a $\bar{\partial}$ closed (0,1)-form which satisfies the boundary condition on ∂D_n. From lemma 7, we have

$$\begin{aligned}
&|< g, \alpha >_{\varphi_n}|^2 = \left|\lim_{\varepsilon \to 0} \int_{D_n} f \frac{\varepsilon}{(|z_1|^2 + \varepsilon)^2} \overline{\alpha_1} e^{-\varphi_n} d\mu\right|^2 = \left|\int_{\{z_1=0\}\cap D_n} \pi f \overline{\alpha_1} e^{-\varphi_n} d\mu_1\right|^2 \\
&\leq \pi^2 \int_{\{z_1=0\}\cap D_n} |f|^2 e^{-\varphi_n} d\mu_1 \int_{\{z_1=0\}\cap D_n} |\alpha_1|^2 e^{-\varphi_n} d\mu_1 \\
&\leq 2\pi \left(1 + \frac{1}{\delta^2}\right) \int_{\{z_1=0\}\cap D_n} |f|^2 e^{-\varphi_n} d\mu_1 \int_{D_n} \frac{|\bar{\partial}^* \alpha|^2}{|z_1|^{2\delta}} e^{-\varphi_n} d\mu.
\end{aligned}$$

From lemma 5, there exist integrable functions u_δ^n in D_n such that

$$\bar{\partial} u_\delta^n = g, \quad \int_{D_n} |u_\delta^n|^2 |z_1|^{2\delta} e^{-\varphi_n} d\mu \leq 2\pi \left(1 + \frac{1}{\delta^2}\right) \int_{\{z_1=0\}\cap D_n} |f|^2 e^{-\varphi_n} d\mu_1.$$

We set $F_\delta^n = u_\delta^n z_1$. Then, from lemma 8, F_δ^n are holomorphic in D_n and satisfy $F_\delta^n|_{\{z_1=0\}\cap D_n} = f|_{\{z_1=0\}\cap D_n}$. Suppose that

$$\int_X |f|^2 e^{-\varphi} d\mu_1 = C < \infty,$$

then it holds that

$$\begin{aligned}
\int_{D_n} |F_\delta^n|^2 e^{-\varphi_n} d\mu &= \int_{D_n} |u_\delta^n|^2 |z_1|^2 e^{-\varphi_n} d\mu \leq \int_{D_n} |u_\delta^n|^2 |z_1|^{2\delta} e^{-\varphi_n} d\mu \\
&\leq 2\pi \left(1 + \frac{1}{\delta^2}\right) \int_{\{z_1=0\}\cap D_n} |f|^2 e^{-\varphi_n} d\mu_1 \leq 2\pi \left(1 + \frac{1}{\delta^2}\right) C.
\end{aligned}$$

Therefore, for some fixed n, there exists a constant C_1 such that

$$\int_{D_n} |F_\delta^n|^2 d\mu \leq C_1.$$

From lemma 9,10, there exists a sequence $\{\delta_j\}$ with $\delta_j \to 1$ such that $F^n_{\delta_j}$ converges uniformly on any compact subset of D_n to F^n. Then F^n are holomorphic in D_n and satisfy $F^n|_{\{z_1=0\}\cap D_n} = f|_{\{z_1=0\}\cap D_n}$. Moreover, we have

$$\int_{D_n} |F^n|^2 e^{-\varphi_n} d\mu \leq 4\pi C.$$

Let K be a compact subset of D. There exists a natural number N such that $K \subset D_n$, $(n \geq N)$. If we set

$$M_n = \min_{\overline{D_n}} e^{-\varphi_n},$$

then, for $n \geq N$, there exist a constant C_2 such that

$$4\pi C \geq \int_{D_n} |F^n|^2 e^{-\varphi_n} d\mu \geq M_N \int_{D_N} |F^n|^2 d\mu \geq C_2 \sup_K |F^n|^2.$$

Thus $\{F^n\}$ are uniformly bounded on any compact subset of D. Then we can find a subsequence $\{F^{k_n}\}$ of $\{F^n\}$ which converges uniformly on any compact subset of D. We set $\lim_{n\to\infty} F^{k_n} = F$. Then F is holomorphic in D and satisfies $F|_X = f$. For any compact subset K of D, we have

$$\int_K |F|^2 e^{-\varphi} d\mu = \lim_{n\to\infty} \int_K |F^{k_n}|^2 e^{-\varphi_{k_n}} d\mu \leq 4\pi C,$$

which completes the proof.

Remark. Siu[18] also obtained another proof of the theorem of Ohsawa-Takegoshi in which the constant $C = \frac{64}{9}\pi A^2 \left(1 + \frac{1}{4e}\right)^{1/2}$ provided $D \subset \{z||z| \leq A\}$.

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