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L^p-boundedness of wave operators for two dimensional Schrödinger operators

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1 Introduction, Theorems

This lecture is concerned with the boundedness in L^p or Sobolev spaces of the wave operators for the Schrödinger operators. Let $H_0 = -\triangle$ be the free Schrödinger operator on $\mathbf{R}^d$, $d \geq 1$, and $H = H_0 + V$ be its perturbation by a multiplication operator with a real valued function V. It is well known in the spectral and scattering theory for Schrödinger operators([1], [4], [5], [6]) that if V is short range, viz. $V(x)$ decays at infinity like $\sim C|x|^{-1-\epsilon}$, $\epsilon > 0$, then:

1. The operator H is selfadjoint in the Hilber space $L^2(\mathbf{R}^d)$ with the domain $H^2(\mathbf{R}^d)$, the Sobolev space of order 2.

2. The spectrum of H consists of non-positive eigenvalues and the absolutely continuous spectrum $[0, \infty)$. The singular continuous spectrum of H is absent.

3. The wave operators defined by the limits $W_\pm u = \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} u$ exist. The wave operators $W_\pm$ are unitary from $L^2(\mathbf{R}^d)$ onto the absolutely continuous spectral subspace $L^2_{\mathrm{ac}}(H)$ for H and intertwine H_0 and the absolutely continuous part HP_{ac} of H: $W_\pm H_0 W_\pm^* = HP_{\mathrm{ac}}$, where P_{ac} is the projection onto $L^2_{\mathrm{ac}}(H)$.

It follows from the peoperty (3) that $f(H)P_{\rm ac} = W_\pm f(H_0)W_\pm^*$ for any Borel function f on $\mathbf{R}^1$ and the mapping properties of $f(H)P_{\rm ac}$ between L^p spaces or Sobolev spaces $W^{k,p}(\mathbf{R}^d)$ may be derived from those of $f(H_0)$ if $W_\pm$ and $W_\pm^*$ are bounded in the L^p or Sobolev spaces. Note that $f(H_0)$ is the convolution operator by the Fourier transform $K(x)$ of the function $f(\xi^2)$ and the $L^p - L^q$ continuity of $f(H)$ may be derived by studying the function $K(x)$. In particular, we can obtain the following $L^p - L^q$ estimates for the propagator $e^{-itH}P_c(H)$ of the time dependent Schrödinger equations $i\partial_t u = Hu$ on the continuos spectral subspace and for the propagator $\dfrac{\sin t\sqrt{H}}{\sqrt{H}}P_c(H)$ of the wave equation with potential $\partial_t^2 u + Hu = 0$ by applying our theorems to the well-known estimates for free equations:

$$\|e^{-itH}P_c(H)u\|_{L^p} \leq C|t|^{-d(1/2-1/p)}\|u\|_{L^{p'}};$$

$$\left\|\frac{\sin t\sqrt{H}}{\sqrt{H}}P_c(H)u\right\|_p \leq C|t|^{-(d-1)(1/2-1/p)}\|u\|_{W^{(d-1)/2-(d+1)/p,p'}},$$

both for $2 \leq p$ and $1/p + 1/p' = 1$.

When the spatial dimensions $d \geq 3$, we have shown in our previous papers ([15], [16]) that the wave operators $W_\pm$ are bounded in $L^p(\mathbf{R}^d)$ for all $1 \leq p \leq \infty$ under suitable conditions. For small potentials, the following theorem([15]) covers rather general potentials and when $d = 3$, the wave operator is bounded in L^p for any $1 \leq p \leq \infty$ when $\|\langle x\rangle^\sigma V\|_{L^2}$ is small for some $\sigma > 1$, $\langle x\rangle = (1+x^2)^{1/2}$. We write $d_* = (d-1)/(d-2)$.

Theorem 1.1 *Let $d \geq 3$. Suppose that $\mathcal{F}(\langle x\rangle^\sigma)V \in L^{d_*}(\mathbf{R}^d)$ for some $\sigma > 2/d_*$ and $\|\mathcal{F}(\langle x\rangle^\sigma)V\|_{L^{d_*}}$ is sufficiently small. Then $W_\pm$ are bounded in $L^p(\mathbf{R}^d)$ for all $1 \leq p \leq \infty$.*

For larger potentials, we need an additional spectral condition for $W_\pm$ to be bounded in L^p and we obtain the following theorem([16]).

Theorem 1.2 *Let $d \geq 3$. Suppose that there exists a constant $C > 0$ such that, for some $\rho > d/2$, V satisfies*

$$\|V\|_{L^\rho(|x-y|<1)} \leq C(1+|x|)^{-(3d+2+\epsilon)/2}.$$

Suppose, in addition, that zero is neither eigenvalue nor resonance of the operator H. Then, the wave operators $W_\pm$ are bounded in $L^p(\mathbf{R}^d)$ for all $1 \leq p \leq \infty$.

Here, zero is said to be a resonance of H if the equation $-\Delta u + V(x)u(x) = 0$ has a solution u such that $\langle x\rangle^{-1-\epsilon}u \in L^2(\mathbf{R}^d)$ but $u \notin L^2(\mathbf{R}^d)$. It is well know that 0 is not a resonace for H. If 0 is a resonance or eigenvalue of H, itis known that the wave operators cannot be bounded in L^p for all $1 < p < \infty$.

Remark 1.3 *If $D^\alpha V$, $|\alpha| = 0, 1, \ldots, \ell$, satisfy the conditions of* Theorem 1.1 *or* Theorem 1.2, *then the wave operators $W_\pm$ are bounded in the Sobolev space $W^{k,p}(\mathbf{R}^d)$ for all $1 \leq p \leq \infty$ and $k = 0, 1, \ldots, \ell$. See* [15] *for the details.*

In the lower dimensions $d = 1$ and $d = 2$, however, the high singularities of the resolvent $(H_0 - z)^{-1}$ at $z = 0$ prevents us to apply the analysis valid for higher dimensions. The one dimensional case, however, can be treated as follows by writing the wave operators in terms of the generalized eigenfunctions in the form

$$W_\pm u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_\pm(x,k)\hat{u}(k)dk.$$

We estimate the integral kernel

$$K(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iky}\phi_\pm(x,k)dk$$

by using the information on the eigenfunctions obtained in the ODE theory, and then appply standard L^p estimates for the singular integral operators on the line. In this way, we can prove the following theorem([3]). We denote by $f_\pm(x,k)$ the solution of $-f''+Vf=k^2f$ which satisfies $|f_\pm(x,k)-e^{\pm ixk}|\to 0$ as $x\to\pm\infty$. We say that V is of generic type if $[f_+(x,0),f_-(x,0)]\neq 0$, and of exceptional type if $[f_+(x,0),f_-(x,0)]=0$, where $[u,v]=u'v-uv'$ is the Wronskian.

Theorem 1.4 *Suppose $\langle x\rangle^3V\in L^1(\mathbf{R}^1)$ if V is of generic type and $\langle x\rangle^4V\in L^1(\mathbf{R}^1)$ if V is of exceptional type. Then, the wave operators $W_\pm$ are bounded in $L^p(\mathbf{R}^d)$ for all $1<p<\infty$.*

Remark 1.5 *The decay conditions on the potential has been relaxed by Weder [14] to $\langle x\rangle^2V\in L^1(\mathbf{R}^1)$ or to $\langle x\rangle^3V\in L^1(\mathbf{R}^1)$ in respective cases. Moreover, $W_\pm$ are bounded in the Hardy space $H^1(\mathbf{R}^d)$ and BMO space. See, [14] for the details.*

The purspose of this lecture is to extend these results to two dimensions. We assume that V is bounded and satisfies the following decay condition.

Assumption 1.6 *The potential $V(x)$ satisfies $|V(x)|\le C\langle x\rangle^{-\delta}$, $x\in\mathbf{R}^2$ for some $\delta>6$.*

For stating the main result, we need some notation which we introduce now. For $s\in\mathbf{R}$ and integral $k\ge 0$, $H^{k,s}(\mathbf{R}^2)=\{f:\sum_{|\alpha|\le k}\|\langle x\rangle^sD^\alpha f\|_2<\infty\}$ is the weighted Sobolev space, and $L^{2,s}(\mathbf{R}^2)=H^{0,s}(\mathbf{R}^2)$. For Banach spaces X and Y, $B(X,Y)$ is the space of bounded operators from X to Y, $B(X)=B(X,X)$. We denote the boundary values on the positive reals of the resolvents $R_0(z)$ and $R(z)=(H-z)^{-1}$ by

$$R_0^\pm(\lambda)\equiv\lim_{\epsilon\to+0}R_0(\lambda\pm i\epsilon),\qquad R^\pm(\lambda)\equiv\lim_{\epsilon\to+0}R(\lambda\pm i\epsilon).$$

These limits exist in $B(L^{2,\sigma}(\mathbf{R}^2), H^{2,-\sigma}(\mathbf{R}^2))$, $\sigma > 1/2$ and they are locally Hölder continuous with respect to $\lambda \in (0,\infty)$ (cf. [1]). In two dimensions, $R_0^{\pm}(k^2)$ has the logarithmic singularities at $k = 0$ and has the following asymptotic expansion as a $B(L^{2,s}(\mathbf{R}^2), H^{2,-s}(\mathbf{R}^2))$-valued function, $s > 3$:

$$R_0^{\pm}(k^2) = c^{\pm}(k)P_0 + G_0 + O(k^2 \log k), \tag{1.1}$$

where $c^{\pm}(k) = 1 \pm i\frac{2}{\pi}\gamma \pm i\frac{2}{\pi}\log\frac{k^2}{2}$, γ is the Euler number, P_0 is the rank one operator defined by

$$P_0 u(x) = \int_{\mathbf{R}^2} u(x)dx$$

and G_0 is the minimal Green function of $-\triangle$:

$$G_0 u(x) = \frac{-1}{2\pi}\int_{\mathbf{R}^2}(\log|x-y|)u(y)dy$$

We write $c_0 = \int V(x)dx$ and set $V_0(x) = c_0^{-1}V(x)$, $P = P_0V_0$ and $Q = 1 - P$. We have $P^2 = P$ and $Q^2 = Q$. We assume

Assumption 1.7 $c_0 \neq 0$ *and* $1 + QG_0VQ$ *is invertible in* $L^{2,-s}(\mathbf{R}^2)$ *for some* $1 < s < \delta - 1$.

The main theorem in this lecture may be stated as follows:

Theorem 1.8 *Suppose that* Assumption 1.6 *and* Assumption 1.7 *are satisfied. Then, for any* $1 < p < \infty$, *there exists a constant* $C > 0$ *such that*

$$||W_{\pm}u||_p \leq C_p||u||_p, \qquad u \in L^2(\mathbf{R}^2) \cap L^p(\mathbf{R}^2)$$

where the constant $C > 0$ *is independent of* u.

Some remarks are in order:

Remark 1.9 *If* Assumption 1.7 *is satisfied, then* $1 + QG_0VQ$ *is invertible in* $L^{2,-s}(\mathbf{R}^2)$ *for all* $1 < s < \delta - 1$ (cf. [7]). Assumption 1.7 *is satisfied if and only if there are no non-trivial solutions* $u \in H^2_{\rm loc}(\mathbf{R}^2)$ *of* $-\Delta u + V(x)u = 0$ *which satisfy the asymptotic behaviour at infinity*

$$\frac{\partial^\alpha}{\partial x^\alpha}\left(u - a - \frac{b_1 x_1 + b_2 x_2}{|x|^2}\right) = O(|x|^{-1-|\alpha|-\epsilon}), \quad |\alpha| \le 1 \qquad (1.2)$$

for some $\epsilon > 0$, *where* a, b_1 *and* b_2 *are constants. If at least one of the constants* a, b_1 *and* b_2 *does not vanish, then* u *is called a resonant solution or a half bound state and* 0 *is the resonance of* H. *If all these constants vanish, then* u *is an eigenfunction of* H *and* 0 *is an eigenvalue of* H.

Indeed, if $u \in L^{2,-s}$ satisfies $u + QG_0Vu = 0$, then $u = Qu$ and $-\Delta u + Vu = 0$ since $-\Delta Q = -\Delta$. Moreover, $u \in L^{2,-s}(\mathbf{R}^2)$ for any $s > 1$ and letting $|x| \to \infty$ in the integral expression $G_0Vu(x) = \dfrac{-1}{2\pi}\displaystyle\int \log|x-y|V(y)u(y)dy$ and using $Pu = \displaystyle\int V_0(x)u(x)dx = 0$, we see that u satisfies (1.2)(cf. [2]). On the other hand if u satisfies $-\Delta u + V(x)u = 0$ and (1.2), then, by comparing the singularities at $\xi = 0$ of the Fourier transforms $\mathcal{F}(Vu)(\xi)$ and $\xi^2\mathcal{F}u(\xi)$, we have $\mathcal{F}(Vu)(0) = 0$ or $Qu = u$. And, in virtue of (1.2), the limit as $R \to \infty$ of the boundary integral in the right hand side of

$$\begin{aligned}
&\lim_{R\to\infty} \frac{-1}{2\pi}\int_{|y|\le R} (-\Delta u)(y)\log|x-y|dy \\
&\quad = u(x) + \lim_{R\to\infty}\frac{1}{2\pi}\int_{|y|=R}\left(\frac{\partial u}{\partial n}(y)\log|x-y| - u(y)\frac{\partial \log|x-y|}{\partial n}\right)dy
\end{aligned}$$

converges to the constant $-a$. It follows that $G_0Vu = -u(x) + a$ and $QG_0VQu + u = 0$, since $Qa = 0$.

Remark 1.10 *As in the higher dimensional case, we can prove by applying the commutator method of* [15] *that* $W_\pm$ *are bounded in the Sobolev space*

$W^{k,p}(\mathbf{R}^2)$ *for any* $1 < p < \infty$ *and* $k = 0, \ldots, \ell$ *if* V *satisfies* $|D^\alpha V(x)| \le C_\alpha \langle x \rangle^{-\delta}$ *for* $|\alpha| \le \ell$ *and* Assumption 1.7.

Remark 1.11 *Likewise, if* $z = 0$ *is a resonance or an eigenvalue of* H, $W_\pm$ *cannot be bounded in* $L^p(\mathbf{R}^2)$ *for all* $1 < p < \infty$. *Indeed Murata* [7] *has shown that* $e^{-itH} P_{\mathrm{ac}}$ *in this case satisfies*

$$\lim_{t \to \infty} \|(\log t) e^{-itH} P_{\mathrm{ac}} f - C_0 f\|_{L^{2,-s}} = 0, \quad s > 3, \tag{1.3}$$

where $C_0 \ne 0$ *is an explicitly computable finite rank operator. This clearly contradicts with the* L^p *boundedness of* $W_\pm$ *because the latter would imply*

$$\begin{aligned} \|(\log t) e^{-itH} P_{\mathrm{ac}} f\|_{L^{2,-s}} &\le \|(\log t) W_+ e^{-itH_0} W_+^* f\|_{L^p} \\ &\le C_p \|f\|_{L^{p'}} (\log t) t^{-2(1/2 - 1/p)} \to 0 \quad (t \to \infty) \end{aligned}$$

for sufficiently large $p > 2$ *and* $p' = p/(p-1)$ *and because* $L^{2,-s} \cap L^p$ *is dense in* $L^{2,-s}$.

In what follows we deal with W_+ only. W_- may be treated similarly. We use the following notation and convention. $D_j = -i\partial/\partial x_j$, $j = 1, 2$, and we use the vector notation $D = (D_1, D_2)$, $\langle D \rangle = (1 + D^2)^{1/2}$. $\|u\|_p$ is the L^p norm of u, $1 \le p \le \infty$. Σ is the unit circle $S^1 \subset \mathbf{R}^2$ and $d\omega$ denotes the standard line element of Σ. $\mathcal{F}u(\xi) = \hat{u}(\xi) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbf{R}^2} e^{-ix\cdot\xi} u(x) dx$ is the Fourier transform of u, Various constants are denoted by the same letter C if their specific values are not important, and these constants may differ from one place to another. We take and fix throughout this paper the cut-off functions $\chi(t) \in C_0^\infty(\mathbf{R}^1)$ and $\tilde{\chi}(t) \in C^\infty(\mathbf{R}^1)$, $\chi(t) + \tilde{\chi}(t) \equiv 1$, such that $\chi(t) = \chi(-t)$, $0 \le \chi(t), \tilde{\chi}(t) \le 1$, $\chi(t) = 1$ for $|t| \le c$ and $\chi(t) = 0$ for $|t| \ge 2c$, where $0 < c < 1$ is the sufficiently small constant to be specified below. We note that $\chi(H_0)$ is the convolution operator with the Fourier transform of $\chi(\xi^2) \in C_0^\infty(\mathbf{R}^2)$ and $\chi(H_0)$ and $\tilde{\chi}(H_0)$ are bounded operators in $L^p(\mathbf{R}^2)$ for any $1 \le p \le \infty$. For f and g in suitable function spaces, $\langle f, g\rangle = \int f(x)\overline{g(x)}dx$.

2 Outline of the Proof

We outline the proof of Theorem 1.8. The basic strategy is similar to the one employed in [15] and [16] for proving the corresponding property in higher dimensions $d \ge 3$: We start from the stationary representation formula ([6]):

$$W_+ u = u - \frac{1}{\pi i}\int_0^\infty R^-(k^2)V\{R_0^+(k^2) - R_0^-(k^2)\}kudk \tag{2.4}$$

and expand W_+ into the sum of a few Born terms and the remainder

$$W_+ = \sum_{j=0}^{\ell} W_+^{(j)} + W_{\ell+1}$$

by successively replacing $R^-(k^2)$ by $R^-(k^2) = R_0^-(k^2) - R_0^-(k^2)VR^-(k^2)$ in the right of (2.4): $W_+^{(0)} = I$ is the identity operator and for $j = 1, \dots, \ell$,

$$W^{(j)}u = \frac{(-1)^j}{\pi i}\int_0^\infty R_0^-(k^2)V(R_0^-(k^2)V)^{j-1}\{R_0^+(k^2) - R_0^-(k^2)\}kudk, \tag{2.5}$$

$$W_{\ell+1}u = \frac{(-1)^{\ell+1}}{\pi i}\int_0^\infty R_0^-(k^2)VF_\ell(k^2)\{R_0^+(k^2) - R_0^-(k^2)\}kudk, \tag{2.6}$$

where $F_\ell(k^2) = (R_0^-(k^2)V)^{\ell-1}R^-(k^2)V$. We prove that the Born terms $W^{(j)}$ are bounded in $L^p(\mathbf{R}^2)$ for all $1 < p < \infty$ by showing that they are superpositions of compositions of essentially one dimensional convolution operators; the remainder term $W_{\ell+1}$ has the integral kernel $K(x,y)$ which satisfies the condition of Schur's lemma

$$\sup_{x\in\mathbf{R}^2}\int_{\mathbf{R}^2}|K(x,y)|dy < \infty, \quad \sup_{y\in\mathbf{R}^2}\int_{\mathbf{R}^2}|K(x,y)|dx < \infty$$

and, therefore $W_{\ell+1}$ is bounded in $L^p(\mathbf{R}^2)$ for all $1 \le p \le \infty$. We explain here the difficulties which we encounter in this approach, in two dimensions in particular, and the ideas how to overcome these difficulties. As the difficulties are of different kinds in the low energy part and the high energy part, we split W_+ into the high $W_+\tilde{\chi}(H_0)$ and the low energy parts $W_+\chi(H_0)$ by using the cut-off functions introduced above.

First, we prove that the first two Born terms $W^{(1)}$ and $W^{(2)}\tilde{\chi}(H_0)$ are bounded in $L^p(\mathbf{R}^2)$ for any $1 < p < \infty$. We write $W^{(1)} = W^{(1)}(V)$ when we want to make the dependence on V explicit.

Lemma 2.12 *The operators $W^{(1)}$ and $W^{(2)}$ may be written in the form*

$$W^{(1)}u(x) = \frac{i}{4\pi}\int_\Sigma d\omega \int_0^\infty K(t + 2x\omega, \omega)u(x + t\omega)dt; \tag{2.7}$$

$$W^{(2)}u(x) = C\int_{\Sigma^2} d\Omega \int_{[0,\infty)^2} \hat{K}_2(t_1, t_2 + 2x\omega_2, \omega_1, \omega_2)u(x + t_1\omega_1 + t_2\omega_2)dt_1dt_2, \tag{2.8}$$

where $C = (i/4\pi)^2$, $d\Omega = d\omega_1 d\omega_2$ and

$$K(t, \omega) = \int_0^\infty \hat{V}(r\omega)e^{itr/2}dr, \tag{2.9}$$

$$\hat{K}_2(t_1, t_2, \omega_1, \omega_2) = \int_{[0,\infty)^2} e^{i(t_1s_1+t_2s_2)/2}\hat{V}(s_1\omega_1)\hat{V}(s_2\omega_2 - s_1\omega_1)ds_1ds_2. \tag{2.10}$$

Proof. By writing $V(x) = (2\pi)^{-1}\int e^{ix\xi}\hat{V}(\xi)d\xi$ we have

$$(\mathcal{F}W^{(1)}u)(\xi) = -\int_0^\infty \frac{1}{2\pi}\left(\int \frac{\hat{V}(\eta)}{\xi^2 - \lambda + i0}\delta((\xi-\eta)^2 - \lambda)\hat{u}(\xi - \eta)d\eta\right)d\lambda.$$

Computing the Fourier inverse transform in

$$W^{(1)}u(x) = -\frac{1}{(2\pi)^2}\int\int \frac{e^{ix\xi}\hat{V}(\eta)\hat{u}(\xi-\eta)}{2\xi\cdot\eta - \eta^2 + i0}d\eta d\xi$$

we obtain (2.7). For obtaining (2.8), we repeat similar computations. See the proof of Proposition 2.2, Lemma 2.3 and Lemma 2.4 of [15] for the details. ∎

When $d \geq 3$, the similar computation produces expressions (2.7) and (2.8) for $W^{(1)}$ and $W^{(2)}$ with $K \in L^1(\mathbf{R}\times\Sigma)$ and $\hat{K}_2 \in L^1(\mathbf{R}^2\times\Sigma^2)$. Hence, the classical Minkowski inequality implies that $W^{(1)}$ and $W^{(2)}$ are bounded in $L^p(\mathbf{R}^2)$ for any $1 \leq p \leq \infty$ if $d \geq 3$. If $d = 2$, this is obviously not the case, however, we can show

$$\begin{gathered} K_1(t,\omega) = K(t,\omega) - 2\hat{V}(0)\tilde{\chi}(t)/it \in L^1(\mathbf{R}\times\Sigma) \\ \|K_1\|_{L^1} \leq C\|\langle x\rangle^s V\|_2, \qquad s > 1 \end{gathered}$$

and that the integral operator which arises when K is replaced by $\tilde{\chi}(t)/it$ in (2.7) is a superposition $\int_\Sigma F_\omega u(x)d\omega$ over $\omega \in \Sigma$ of

$$F_\omega u(x) = \int_0^\infty \frac{\tilde{\chi}(t+2x\omega)}{t+2x\omega} u(x+t\omega)dt. \tag{2.11}$$

After rotating the coordinates by ω, we estimate $F_{\mathbf{e}_1}u(x)$ as follows separately for $x_1 > 0$ and for $x_1 < 0$:

$$\begin{aligned} |F_{\mathbf{e}_1}u(x)| \leq \theta(x_1) & \int_0^\infty \frac{|u(t,x_2)|}{t+x_1}dt \\ +\theta(-x_1) & \int_{-\infty}^0 \frac{|u(t,x_2)|}{|t+x_1|}dt + \theta(-x_1)\left|\int_0^\infty \frac{u(t,x_2)}{t+x_1}dt\right|. \end{aligned}$$

We then apply L^p boundedness theorem for the one-dimensional Hardy-Littlewood operators on the half lines $(0,\pm\infty)$ to the first two integrals on the right and for one dimensional singular integral operator of the Calderon-Zygmund type to the third, and conclude that $\{F_\omega : \omega \in \Sigma\}$ is a family of uniformly bounded operators in L^p for any $1 < p < \infty$. In this way, we obtain the estimate

$$\|W^{(1)}(V)u\|_p \leq C_{ps}\|\langle x\rangle^s V\|_2\|u\|_p, \quad \text{for any } s > 1. \tag{2.12}$$

The proof of the L^p boundedness of $W^{(2)}\tilde{\chi}(H_0)$ is a bit more involved. We write $\hat{K}_2$ as a sum of three functions $K_{21}+K_{22}+K_{23}$;

$$K_{21} \in L^1(\mathbf{R}^2 \times \Sigma^2),$$

$$K_{22} = C(\tilde{\chi}(t_1)/t_1) \times K(t_2, \omega_2),$$

with $K(t,\omega)$ being defined by (2.9), and

$$K_{23} = (\tilde{\chi}(t_2)/t_2) \times K'(t_1, \omega_1), \qquad K' \in L^1(\mathbf{R}^1 \times \Sigma).$$

We show that the operators which are produced by replacing $\hat{K}_2$ in (2.8) by K_{2j} are bounded in L^p for any $1 < p < \infty$ as follows.

1. The operator arising from K_{21} can be estimated by using the Minkowski inequality.

2. If we denote by M the convolution operator with $\tilde{\chi}(|x|)/|x|^2$, then the operator arising from K_{22} is of the form $W^{(1)}M$. The operator $M\tilde{\chi}(H_0)$ is bounded in L^p by Calderon-Zygmund theory;

3. The operator arising from K_{23} may be written in the form

$$\int_\Sigma \int_0^\infty K'(t_1, \omega_1) \left(\int_\Sigma (F_{\omega_2} u)(x + t_1\omega_1) d\omega_2 \right) d\omega_1 dt_1$$

   and the estimate for (2.11) mentioned above and the Minkowski inequality imply that this also is bounded in L^p.

We then prove that the high energy part $W_3\tilde{\chi}(H_0)$ of the remainder W_3 is bounded in L^p for any $1 \le p \le \infty$ by showing that its integral kernel $T(x,y)$ is bounded by a constant times $\langle x\rangle^{-1/2}\langle y\rangle^{-1/2}\langle |x| - |y|\rangle^{-2}$. We write $F(k) = R_0^-(k^2)VR^-(k^2)$. Because $R_0^\pm(k^2)$ is the convolution operator with

$G^{\pm}(x,k) = (\pm i/4)H_0^{\pm}(k|x|)$, where $H_0^{\pm}(z) = H_0^{(j)}(z)$ is the 0-th order Hankel function of the j-th kind, $\pm$ corresponding to $(-1)^{j+1}$ (cf. [12]), $T(x,y)$ is given as $T(x,y) = T^{+}(x,y) - T^{-}(x,y)$:

$$T^{\pm}(x,y) = -\frac{1}{\pi i}\int_0^{\infty} \langle F(k)VG^{\pm}(y-\cdot,k), VG^{+}(x-\cdot,k)\rangle \tilde{\chi}(k^2)kdk. \quad (2.13)$$

In virtue of the classical estimate for Hankel functions $H_0^{\pm}(k|x|) \sim \dfrac{Ce^{\pm ik|x|}}{\sqrt{k|x|}}$ and the decay property of the resolvent at high energy

$$\|\langle x\rangle^{-\sigma-j}(d/dk)^j F(k)\langle x\rangle^{-\sigma-j}\|_{B(L^2)} \leq Ck^{-2}$$

for $j = 0,1,2$ and $\sigma > 1/2$, the integral (2.13) is absolutely convergent. However, a simple minded estimate by using these facts only would yield $|T^{\pm}(x,y)| \leq C\langle x\rangle^{-1/2}\langle y\rangle^{-1/2}$ which is far from being sufficient to conclude that $W_3\tilde{\chi}(H_0)$ is bounded in L^p for all $1 < p < \infty$. This difficulty can be resolved by exploiting the old method in [15] and [16]: We write $G^{\pm}(x-y,k) = e^{\pm ik|x|}G^{\pm}_{k,x}(y)$ so that

$$T^{\pm}(x,y) = -\frac{1}{\pi i}\int_0^{\infty} e^{-i(|x|\mp|y|)k}\langle F(k)VG^{\pm}_{y,k}, VG^{+}_{x,k}\rangle k\tilde{\chi}(k^2)dk, \quad (2.14)$$

and apply the integration by parts twice to the k-integral in the right by using the identity

$$\frac{1+i(|x|\mp|y|)(\partial/\partial k)}{1+(|x|\mp|y|)^2} e^{-i(|x|\mp|y|)k} = e^{-i(|x|\mp|y|)k}.$$

This yields the desired estimate

$$|T^{\pm}(x,y)| \leq C(|x|\mp|y|)^{-2}\langle x\rangle^{-1/2}\langle y\rangle^{-1/2}.$$

The estimate of the low energy part of the wave operator $W_{+}\chi(H_0)$ is a little more involved. Here we write

$$R^{-}(k^2)V = R_0^{-}(k^2)V(1+R_0^{-}(k^2)V)^{-1}$$

in (2.4) and investigate the low energy behavior of $(1+R_0^-(k^2)V)^{-1}$ following the argument in [7] and [2]. We find that, for $0 < k < 2c$, c being a sufficiently small constant, which is the constant to be used for defining the cut off χ, $(1 + R_0^-(k^2)V)^{-1}$ can be written as the sum

$$(1 + R_0^-(k^2)V)^{-1} = \sum_{j=0}^{4} d_j(k)K_j + N(k).$$

1. For $0 \le j \le 4$, K_j is an integral operator with the integral kernel $K_j(x,y)$ which satisfies for some $s > 1$

$$\int_{\mathbf{R}^2} \|\langle x\rangle^s V K_{jy}\|_2 dy < \infty, \qquad K_{jy}(x) = K_j(x, x-y). \tag{2.15}$$

2. $d_j(k)$ satisfies $|(\partial/\partial\xi)^\alpha d_j(|\xi|)| \le C_\alpha |\xi|^{-|\alpha|}$.

3. The remainder $N(k)$ is an operator valued function which satisfies the estimate for $j = 0, 1, 2$:

$$\|(d/dk)^j N(k)\|_{B(L^{2,-s})} \le C_j k^{2-j} |\log k|, \quad s > 3,$$

(Actually $d_0(k) = 1$ and K_j for $1 \le j \le 4$ are rank one operators.)

The operator which is produce by inserting $R_0^-(k^2)VN(k)\chi(k^2)$ in place of $R^-(k^2)V$ in (2.4) is an integral operator with the kernel $\tilde{T}^+(x,y) - \tilde{T}^-(x,y)$, $\tilde{T}^\pm(x,y)$ being given by the right hand side of (2.14) with $N(k)\chi(k^2)$ in place of $F(k)V\tilde{\chi}(k^2)$. The method employed for estimating $T^\pm(x,y)$ applies because $N(k)$ vanishes at $k =$ with derivative, and yields

$$|\tilde{T}^\pm(x,y)| \le C\langle |x| \mp |y|\rangle^{-2}\langle x\rangle^{-1/2}\langle y\rangle^{-1/2}$$

and the operator in question is bounded in L^p for any $1 \le p \le \infty$. The operator produced by inserting $R_0^-(k^2)Vd_j(k)K_j$ in place of $R^-(k^2)V$ in (2.4) may be written as

$$\frac{-1}{\pi i}\int_0^\infty R_0^-(k^2)VK_jd_j(k)\{R_0^+(k^2)-R_0^-(k^2)\}\chi(k^2)kudk. \tag{2.16}$$

Observing that

$$d_j(k)\{R_0^+(k^2)-R_0^-(k^2)\}=\{R_0^+(k^2)-R_0^-(k^2)\}d_j(|D|)$$

and that the integral operator may be written as

$$\int A(x,y)u(y)dy=\int A(x,x-y)u(x-y)dy=\int A_y(x)\tau_y u(x)dy,$$

viz. the superposition of the composition of the multiplication by $A_y(x) = A(x,x-y)$ and the translation τ_y by y, we rewrite (2.16) in the form

$$\int_{\mathbf{R}^2}\left(\frac{-1}{\pi i}\int_0^\infty R_0^-(k^2)VK_{jy}\{R_0^+(k^2)-R_0^-(k^2)\}kdk\right)d_j(|D|)\chi(H_0)\tau_y udy. \tag{2.17}$$

The operator in the parenthesis is nothing but $W^{(1)}(VK_{jy})$ and, in virtue of (2.12), the L^p-norm of (2.17) may be estimated as follows:

$$\begin{aligned}&\left\|\int_{\mathbf{R}^2}W^{(1)}(VK_{jy})d_j(|D|)\chi(H_0)\tau_y udy\right\|_p\\ &\qquad\le C\|u\|_p\|d_j(|D|)\chi(H_0)\|_{B(L^p)}\int_{\mathbf{R}^2}\|\langle x\rangle^s VK_{jy}\|_2dy.\end{aligned}$$

Because Fourier multipliers $d_j(|D|)\chi(H_0)$ are bounded in L^p by the well known theorem in the Fourier analysis and because (2.15) implies that the integral in the right is finite, the operators arising from $d_j(k)K_j$, $j=0,\dots,4$ are all bounded in L^p for any $1<p<\infty$. Combining these all, we completes the proof of Theorem 1.8.

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