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Integral expressions of harmonic polynomials on the single orbit: In the case of real rank 1

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Introduction.

It is known that classical harmonic polynomials on $\mathbf{C}^p$ are represented by integral formulas on any SO(p)-orbits except {0} (see, for example, [2],[5],[7],[13]). According to the formulation in [4], these classical harmonic polynomials on $\mathbf{C}^p$ can be canonically identified with the harmonic polynomials on $\mathfrak{p}$, where $\mathfrak{so}(p,1) = \mathfrak{k}_{\mathbf{R}} + \mathfrak{p}_{\mathbf{R}}$ is the Cartan decomposition of $\mathfrak{so}(p,1)$ and $\mathfrak{p}$ is the complexification of $\mathfrak{p}_{\mathbf{R}}$. In this situation, any SO(p)-orbit in $\mathbf{C}^p$ corresponds to a $K_{\mathbf{R}}$-orbit in $\mathfrak{p}$, where $K_{\mathbf{R}} = \exp \operatorname{ad} \mathfrak{k}_{\mathbf{R}}$. Therefore, from the classical integral formulas of classical harmonic polynomials, we can obtain the integral representation formulas of harmonic functions on some $K_{\mathbf{R}}$-orbits explicitly (see, for example, [10] Appendix).

Next we consider the cases $\mathfrak{su}(p,1)$ and $\mathfrak{sp}(p,1)$, which are classical real rank one cases except $\mathfrak{so}(p,1)$. Here we also write $\mathfrak{k}_{\mathbf{R}} + \mathfrak{p}_{\mathbf{R}}$ the Cartan decomposition of $\mathfrak{su}(p,1)$ or $\mathfrak{sp}(p,1)$ and $K_{\mathbf{R}} = \exp \operatorname{ad} \mathfrak{k}_{\mathbf{R}}$. As an extension of the above fact, in our previous papers [10],[11] we obtained integral representation formulas of harmonic polynomials for these two cases. But these formulas are expressed in the form of double integrals on some family of $K_{\mathbf{R}}$-orbits and they are not so simple.

In this paper we obtain the integral formulas on a single nilpotent $K_{\mathbf{R}}$-orbit for these cases. We here give only one example.

Let $\mathcal{H}_n$ be the space of homogeneous harmonic polynomials on $\mathfrak{p}$ of degree n and let $\mathcal{H}_n = \bigoplus_{k=0}^{N} \mathcal{H}_{n,k}$ be the irreducible decomposition as a $K_{\mathbf{R}}$-module, where

$$N = \begin{cases} 0 & (\mathfrak{g}_{\mathbf{R}} = \mathfrak{so}(p,1)), \\ n & (\mathfrak{g}_{\mathbf{R}} = \mathfrak{su}(p,1)), \\ [n/2] & (\mathfrak{g}_{\mathbf{R}} = \mathfrak{sp}(p,1)). \end{cases}$$

Then for any $f \in \mathcal{H}_{n,k}$ and any $X \in \mathfrak{p}$ we have the following integral formula

$$\begin{aligned} &\delta_{n,m}\, \delta_{k,\ell}\, f(X) \\ &= \dim \mathcal{H}_{n,k} \int_{K_{\mathbf{R}}} f(gE_0)\tilde{K}_{m,\ell}(X, gE_0)\, dg, \end{aligned}$$

where $E_0 \in \mathfrak{p}$ is

$$E_0 = \begin{pmatrix} 0 & e_1+ie_2 \\ e_1+ie_2 & 0 \end{pmatrix} \qquad (\mathfrak{g}_{\mathbf{R}} = \mathfrak{so}(p,1)),$$

$$E_0 = \begin{pmatrix} 0 & e_1 \\ {}^t e_2 & 0 \end{pmatrix} \qquad (\mathfrak{g}_{\mathbf{R}} = \mathfrak{su}(p,1)),$$

$$E_0 = \begin{pmatrix} 0 & e_1 & 0 & e_2 \\ 0 & 0 & {}^t e_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -{}^t e_1 & 0 \end{pmatrix} \qquad (\mathfrak{g}_{\mathbf{R}} = \mathfrak{sp}(p,1)),$$

and $\tilde{K}_{n,k}(\ ,Y) \in \mathcal{H}_{n,k}$ is some reproducing kernel. For the precise definitions of irreducible subspaces and reproducing kernels, see §2, §3.

§1. Preliminaries.

Let $\mathfrak{g}_{\mathbf{R}}$ be a classical real semi-simple Lie algebra with real rank one and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the complexification of the Cartan decomposition $\mathfrak{g}_{\mathbf{R}} = \mathfrak{k}_{\mathbf{R}} + \mathfrak{p}_{\mathbf{R}}$. Let G be the adjoint group of $\mathfrak{g}$.

Now we define the harmonic polynomials on $\mathfrak{p}$. We put K = exp ad $\mathfrak{k}$. For a polynomial f on $\mathfrak{p}$ and $g \in K$, gf is defined by $(gf)(X) = f(g^{-1}X)$ $(X \in \mathfrak{p})$. J denotes the ring of K-invariant polynomials on $\mathfrak{p}$. Let B(X,Y) $(X,Y \in \mathfrak{p})$ be the Killing form of $\mathfrak{g}$. In the case of real rank one, the generator P(X) of J is B(X,X), and according to the definition in [4], a polynomial f on $\mathfrak{p}$ is harmonic if and only if $(\partial P)f = 0$. S_n denotes the space of homogeneous polynomials on $\mathfrak{p}$ of degree n and $\mathcal{H}_n$ denotes the space of homogeneous harmonic polynomials on $\mathfrak{p}$ of degree n. We put $\mathfrak{N} = \{X \in \mathfrak{p};\ P(X)= 0\}$ and $h(\ ,Y) = Tr({}^t X\bar{Y})$ for $X,Y \in \mathfrak{p}$. Then it is known that $\mathcal{H}_n$ is generated by $\{h(\ ,Z)^n;\ Z \in \mathfrak{N}\}$ (cf. [4]). For general theory of harmonic polynomials on $\mathfrak{p}$, see [1], [4].

Let $K_{\mathbf{R}}$ be the adjoint group of $\mathfrak{k}_{\mathbf{R}}$. Then we have $K_{\mathbf{R}} \subset K$. We put $\Sigma = \left\{X \in \mathfrak{p}_{\mathbf{R}};\ P(X) = 1\right\}$. The set Σ consists of one $K_{\mathbf{R}}$-orbit. In the case $\mathfrak{g}_{\mathbf{R}} = \mathfrak{so}(p,1)$, it corresponds to the unit sphere in $\mathbf{R}^p$ and there exist well-known integral formulas for harmonic polynomials on Σ (see, for example [5],[6],[7]). In [10],[11] we gave an integral representaion formulas of harmonic polynomials on Σ in the cases $\mathfrak{g}_{\mathbf{R}} = \mathfrak{su}(p,1)$ and $\mathfrak{sp}(p,1)$.

§2. Integral formulas of harmonic polynomials on the $K_{\mathbf{R}}$-orbit in $\mathfrak{su}(p,1)$ case.

From now we put $\mathfrak{g} = \mathfrak{sl}(p+1,\mathbf{C})$ and $\mathfrak{g}_{\mathbf{R}} = \mathfrak{su}(p,1)$, where p is an integer satisfying $p \geq 2$. Then we have

$$\mathfrak{k}_{\mathbf{R}} = \left\{\begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix};\ A \in \mathfrak{u}(p),\ \alpha \in \mathfrak{u}(1)\ ,\ \mathrm{Tr}\,A + \alpha = 0\right\},$$

$$\mathfrak{p}_{\mathbf{R}} = \left\{\begin{pmatrix} 0 & x \\ {}^t\bar{x} & 0 \end{pmatrix};\ x \in \mathbf{C}^p\right\},$$

$$\mathfrak{k} = \left\{\begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix};\ A \in M(p,\mathbf{C}),\ \mathrm{Tr}\,A + \alpha = 0\right\},$$

$$\mathfrak{p} = \left\{\begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix};\ x,\ y \in \mathbf{C}^p\right\},$$

and $G = \mathrm{Ad}\,SL(p+1,\mathbf{C})$, $K_{\mathbf{R}} = \mathrm{Ad}\,S(U(p)\times U(1)) = \left\{\mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix};\ A \in U(p)\right\}$. For $X = \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix} \in \mathfrak{p}$, $P(X) = x\cdot y$ is the generator of J, where $z\cdot w = {}^t zw$ for $z,\ w \in \mathbf{C}^p$ and $\partial P = \sum_{j=1}^{p} \frac{\partial^2}{\partial x_j \partial y_j}$. We put $\mathfrak{N} = \{X \in \mathfrak{p};\ x\cdot y = 0\}$. It is known that $\dim \mathcal{H}_n = \frac{2(n+p-1)(n+2p-3)!}{n!\ (2p-2)!}$. For $X = \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix} \in \mathfrak{p}$ and $g = \mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in K_{\mathbf{R}}$ $(A \in U(p))$ we have $gX = \begin{pmatrix} 0 & Ax \\ {}^t(\bar{A}y) & 0 \end{pmatrix}$.

For $X = \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & \alpha \\ {}^t \beta & 0 \end{pmatrix} \in \mathfrak{p}$ we put $\tilde{K}_{n,k}(X,Y) = (x\cdot\bar{\alpha})^k (y\cdot\bar{\beta})^{n-k}$. Let $\mathcal{H}_{n,k}$ be the space which is spanned by the elements $\tilde{K}_{n,k}(\ ,Y)$ $(Y \in \mathfrak{N})$. From [8] Theorem 14.4 we can easily see that $\mathcal{H}_n = \bigoplus_{k=0}^{n} \mathcal{H}_{n,k}$ gives the irreducible decomposition as a $K_{\mathbf{R}}$-module and $\dim \mathcal{H}_{n,k} = \frac{(p+n-1)(k+p-2)!(n-k+p-2)!\Gamma(p-1)^{-2}}{(p-1)\ k!(n-k)!}$.

Let $\tilde{E}_r = \begin{pmatrix} 0 & re_1 \\ (1-r^2)^{1/2}\ {}^t e_2 & 0 \end{pmatrix} \in \mathfrak{N}$ and $E_1 = \begin{pmatrix} 0 & e_1 \\ {}^t e_1 & 0 \end{pmatrix} \in \Sigma$, where $e_1 = {}^t(1\ 0\ \cdots 0)$, and $e_2 = {}^t(0\ 1\ \cdots 0)$. Then we have $K_{\mathbf{R}}E_1 = \Sigma \simeq S^{2p-1}$ and $\mathfrak{N} = \bigcup_{q\ge 0} \bigcup_{0\le r\le 1} K_{\mathbf{R}}(q\tilde{E}_r)$.

In this section we give integral formulas of harmonic polynomials on each orbit in $\mathfrak{N}$. Now, our main result in this section is the following

Theorem 2.1. Assume $r \in (0,1)$.

(i) For any $f \in \mathcal{H}_{n,k}$ and any $X \in \mathfrak{p}$ we have

$$(2.1) \qquad \delta_{k,\ell}\,\delta_{m,n}\, f(X) = \dim \mathcal{H}_{n,k}\, C_{n,k}(r)\int_{K_{\mathbf{R}}} f(g\tilde{E}_r)\tilde{K}_{m,\ell}(X, g\tilde{E}_r)dg,$$

where $C_{n,k}(r) = \{r^{2k}(1-r^2)^{n-k}\}^{-1}$ and dg is the normalized Haar measure on $K_{\mathbf{R}}$.

In particular, we have

$$(2.2) \qquad \delta_{k,\ell}\,\delta_{m,n}\, f(X) = \dim \mathcal{H}_{n,k} \int_{K_{\mathbf{R}}} f(gE_0)\tilde{K}_{m,\ell}(X, gE_0)\, dg,$$

where $E_0 = \begin{pmatrix} 0 & e_1 \\ {}^t e_2 & 0 \end{pmatrix}$.

(ii) For any $f \in \mathcal{H}_{n,k}$ and any $h \in \mathcal{H}_{m,\ell}$ we have

$$(2.3) \qquad \int_{K_{\mathbf{R}}} f(g\tilde{E}_r)\overline{h(g\tilde{E}_r)}\, dg = 0 \qquad \text{if } (n,k) \neq (m,\ell),$$

$$(2.4) \qquad \int_{K_{\mathbf{R}}} f(gE_1)\overline{h(gE_1)}\, dg$$

$$= C(r)^{-1}\delta_{k,\ell}\frac{(p-1)!k!(n-k)!}{(n+p-1)!}\dim \mathcal{H}_{n,k}\int_{K_{\mathbf{R}}} f(g\tilde{E}_r)\overline{h(g\tilde{E}_r)}dg.$$

Remark. It is clear that Theorem 2.1 does not hold for $r = 0,1$ because $\mathcal{H}_{n,k}|_{K_{\mathbf{R}}\tilde{E}_1} = \{0\}$ $(k \neq n)$ and $\mathcal{H}_{n,k}|_{K_{\mathbf{R}}\tilde{E}_0} = \{0\}$ $(k \neq 0)$.

We can prove Theorem 3.1 by using the next lemma.

Lemma 2.2 (cf. [10]). (i) For any $f \in \mathcal{H}_n$ and any $X \in \mathfrak{p}$ we have

$$(2.5) \qquad \delta_{m,n}\, f(X) = \dim \mathcal{H}_n \int_0^1 \rho(t)\left(\int_{K_{\mathbf{R}}} f(g\tilde{E}_t)\langle X, g\tilde{E}_t\rangle^m dg\right) dt,$$

where we put $\langle X, Y\rangle = \mathrm{Tr}({}^t X\bar{Y})$ $(X, Y \in \mathfrak{p})$ and

$$\rho(t) = 2^{2p-2}\frac{\Gamma(p - 1/2)}{\pi^{1/2}\,\Gamma(p-1)}\, t^{2p-3}(1-t^2)^{p-2} \quad (0 \le t \le 1).$$

(ii) For any $f \in \mathcal{H}_m$ and any $h \in \mathcal{H}_n$ we have

$$(2.6) \quad \int_{K_{\mathbf{R}}} f(gE_1)\overline{h(gE_1)}dg = \frac{n!\Gamma(p)}{\Gamma(n+p)} \dim \mathcal{H}_n \int_0^1 \rho(t)\int_{K_{\mathbf{R}}} f(g\tilde{E}_t)\overline{h(g\tilde{E}_t)}dgdt.$$

For the proof, see [10] Theorem 2.2.

Corollary 2.3. (i) For any $X, Y \in \mathfrak{p}$, we put

$$R_{n,p}(X,Y) = \sum_{k=0}^{n} \dim \mathcal{H}^{k,n-k}\, \tilde{K}_{n,k}(X,Y).$$

Then for any $f \in \mathcal{H}_n$ we have

$$(2.7) \qquad f(X) = \int_{K_{\mathbf{R}}} f(gE_0)R_{n,p}(X, gE_0)\, dg.$$

(ii) For any $X = \begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix} \in \mathfrak{p}$ $(x\cdot\bar{x} < 1,\ y\cdot\bar{y} < 1)$ and any $Z = \begin{pmatrix} 0 & z \\ {}^t w & 0 \end{pmatrix} \in K_{\mathbf{R}}E_0$, we put

$$H_{n,p}(X,Z) = \frac{n+p-1}{p-1}\{(1-x\cdot\bar{z})(1-y\cdot\bar{w})\}^{1-p}.$$

Then for any $f \in \mathcal{H}_n$ it is valid

$$(2.8) \qquad f(X) = \int_{K_{\mathbf{R}}} f(gE_0)H_{n,p}(X, gE_0)\, dg.$$

§3. Integral formulas of harmonic polynomials on the $K_{\mathbf{R}}$-orbit in $\mathfrak{sp}(p,1)$ case.

In this section we consider the $\mathfrak{sp}(p,1)$ case ($p \in \mathbf{N}$, $p \geq 2$). From now we put $\mathfrak{g} = \mathfrak{sp}(p+1,\mathbf{C})$, $\mathfrak{g}_{\mathbf{R}} = \mathfrak{sp}(p,1)$,

$$\mathfrak{k}_{\mathbf{R}} = \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & a & 0 & b \\ -\bar{B} & 0 & \bar{A} & 0 \\ 0 & -\bar{b} & 0 & \bar{a} \end{pmatrix} ; \begin{array}{l} A \in \mathfrak{u}(p),\ a \in \mathfrak{u}(1),\ b \in \mathbf{C} \\ B \text{ is } p \times p \text{ symmetric} \end{array} \right\},$$

$$\mathfrak{p}_{\mathbf{R}} = \left\{ \begin{pmatrix} 0 & x & 0 & y \\ {}^t\bar{x} & 0 & {}^ty & 0 \\ 0 & \bar{y} & 0 & -\bar{x} \\ {}^t\bar{y} & 0 & -{}^tx & 0 \end{pmatrix} ; x,y \in \mathbf{C}^p \right\}.$$

Then we have

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & \alpha & 0 & \beta \\ C & 0 & -{}^tA & 0 \\ 0 & \gamma & 0 & -\alpha \end{pmatrix} ; \begin{array}{c} A,B,C \in M(p,\mathbf{C}) \\ {}^tB = B,\ {}^tC = C \\ \alpha,\beta,\gamma \in \mathbf{C} \end{array} \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x & 0 & w \\ {}^ty & 0 & {}^tw & 0 \\ 0 & z & 0 & -y \\ {}^tz & 0 & -{}^tx & 0 \end{pmatrix} ; x,y,z,w \in \mathbf{C}^p \right\},$$

$G = \mathrm{Ad}\, Sp(p+1,\mathbf{C})$, and

$$K_{\mathbf{R}} = \left\{ \mathrm{Ad}\begin{pmatrix} A & 0 & B & 0 \\ 0 & \alpha & 0 & \beta \\ -\bar{B} & 0 & \bar{A} & 0 \\ 0 & -\bar{\beta} & 0 & \bar{\alpha} \end{pmatrix} \in \mathrm{Ad}\, U(2p+2) ; \begin{array}{l} {}^tA\bar{A} + {}^t\bar{B}B = I_p, \\ {}^tA\bar{B} = {}^t\bar{B}A, \\ \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \end{array} \right\}.$$

For $X = \begin{pmatrix} 0 & x & 0 & w \\ {}^ty & 0 & {}^tw & 0 \\ 0 & z & 0 & -y \\ {}^tz & 0 & -{}^tx & 0 \end{pmatrix} \in \mathfrak{p}$, $P(X) = x\cdot y + z\cdot w = \frac{1}{4}\mathrm{Tr}\, X^2$ is the genarator of J and $\mathcal{H}_n = \{f \in S_n;\ \sum_{j=1}^{p}\left(\frac{\partial^2}{\partial x_j \partial y_j} + \frac{\partial^2}{\partial z_j \partial w_j}\right) f(X) = 0\}$.

It is known that $\dim \mathcal{H}_n = \frac{2(n+2p-1)(n+4p-3)!}{n!(4p-2)!}$. We put $\mathfrak{N} = \{X \in \mathfrak{p};$

$P(X) = 0\}$ and $\Sigma = \{X \in \mathfrak{p}_{\mathbf{R}};\ P(X) = 1\} \simeq S^{4p-1}$. Let $g = \mathrm{Ad}\begin{pmatrix} A & 0 & B & 0 \\ 0 & \alpha & 0 & \beta \\ -\bar{B} & 0 & \bar{A} & 0 \\ 0 & -\bar{\beta} & 0 & \bar{\alpha} \end{pmatrix} \in K_{\mathbf{R}}$. If we put $\Phi(X) = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbf{C}^{4p}$ we have

$$(3.1) \qquad \Phi(gX) = \begin{pmatrix} A(\bar{\alpha}x + \bar{\beta}w) + B(\bar{\alpha}z - \bar{\beta}y) \\ \bar{B}(-\beta x + \alpha w) + \bar{A}(\alpha y + \beta z) \\ -\bar{B}(\bar{\alpha}x + \bar{\beta}w) + \bar{A}(\bar{\alpha}z - \bar{\beta}y) \\ A(-\beta x + \alpha w) - B(\alpha y + \beta z) \end{pmatrix}.$$

Let $\tilde{E}_r = \Phi^{-1}\begin{pmatrix} re_1 \\ 0 \\ 0 \\ (1-r^2)^{1/2}e_2 \end{pmatrix} \in \mathfrak{N}$ $(0 \leq r \leq 1)$. Then we have $\Phi(g\tilde{E}_r) = \begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix}$, where

$$(3.2) \qquad \begin{aligned} x' &= \bar{\alpha}ra_1 + \bar{\beta}(1-r^2)^{1/2}\, a_2, \\ y' &= -\beta r\bar{b}_1 + \alpha(1-r^2)^{1/2}\, \bar{b}_2, \\ z' &= -\bar{\alpha}r\bar{b}_1 - \bar{\beta}(1-r^2)^{1/2}\, \bar{b}_2, \\ w' &= -\beta ra_1 + \alpha(1-r^2)^{1/2} a_2, \end{aligned}$$

and $a_j = Ae_j$, $b_j = Be_j$ $(j = 1,2)$. From (3.2) we can see that $\mathfrak{N} = \bigcup_{q \geq 0} \bigcup_{\sqrt{1/2} \leq r \leq 1} K_{\mathbf{R}}(q\tilde{E}_r)$. Let

$$H_1 = \left\{ \mathrm{Ad}\begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ -\bar{B} & 0 & \bar{A} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in K_{\mathbf{R}} \right\}$$

and

$$H_2 = \left\{ \mathrm{Ad}\begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & I_p & 0 \\ 0 & -\bar{\beta} & 0 & \bar{\alpha} \end{pmatrix} \in K_{\mathbf{R}} \right\}.$$

Then H_1 and H_2 are subgroups of $K_{\mathbf{R}}$, and for any $g \in K_{\mathbf{R}}$ there are unique $h_j \in H_j$ $(j=1,2)$ such that $g = h_1h_2$. Furthermore, if $g_j \in H_j (j = 1,2)$, we have $g_1g_2 = g_2g_1$. dh_j is the normalized Haar measure on H_j $(j = 1,2)$. For $h \in H_1$ and $\tilde{E}_1 \in \mathfrak{N}$ consider the mapping φ: $H_1\tilde{E}_1 \longrightarrow S^{4p-1}$ defined by

$$\varphi(h\tilde{E}_1) = \varphi(\Phi^{-1}\begin{pmatrix} a_1 \\ 0 \\ -\bar{b}_1 \\ 0 \end{pmatrix}) = \begin{pmatrix} \mathrm{Re}\ a_1 \\ \mathrm{Im}\ a_1 \\ \mathrm{Re}(-\bar{b}_1) \\ \mathrm{Im}(-\bar{b}_1) \end{pmatrix}.$$

From the condition on H_1 we see that φ is bijective and $\int_{H_1} f(h\tilde{E}_1)dh_1 = \int_{S^{4p-1}} f\circ\varphi^{-1}(s)ds$, where ds is the normalized $O(4p)$-invariant measure on S^{4p-1}. For $X = \Phi^{-1}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$, $X' = \Phi^{-1}\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix}$ $\in \mathfrak{p}$ we define

$$\langle X, X'\rangle = (1/2)\mathrm{Tr}({}^tX\overline{X'}) = \Phi(X)\cdot\overline{\Phi(X')},$$

$$K_2(X,X') = (x\cdot\bar{x}'+z\cdot\bar{z}')(y\cdot\bar{y}'+w\cdot\bar{w}') + (x\cdot\bar{w}'-z\cdot\bar{y}')(y\cdot\bar{z}'-w\cdot\bar{x}'),$$

$$\tilde{K}_m(X,X') = \frac{\Gamma(2p+m)}{m!\Gamma(2p)}\int_{K_{\mathbf{R}}} \langle g\tilde{E}_1, X'\rangle^m \langle X, g\tilde{E}_1\rangle^m\, dg,$$

$$\tilde{K}_{n,k}(X,X') = \tilde{K}_{n-2k}(X,X')\{K_2(X,X')\}^k,$$

$(m = 0,1,2,\cdots,\ k = 0,1,\cdots,[n/2])$. Remark that for any X, $X' \in \mathfrak{p}$, $g \in K_{\mathbf{R}}$

$$(3.3)\qquad \tilde{K}_{n,k}(X,X') = \overline{\tilde{K}_{n,k}(X',X)},$$

$$(3.4)\qquad \tilde{K}_{n,k}(X,X') = \tilde{K}_{n,k}(gX,gX').$$

Next we put $K_{n,k}(X,X') = (n-2k+1)^{-1}\langle X,X'\rangle^{n-2k}\{K_2(X,X')\}^k$ $(X, X' \in \mathfrak{p})$. If $X' \in \mathfrak{N}$ we can see that $K_{n,k}(\ ,X') \in \mathcal{H}_n$. Then the following proposition is valid.

Proposition 3.1. (i) We have the following formula:

$$(3.5)\qquad \tilde{K}_m(X,X') = (m+1)^{-1} \sum_{m_1+m_2+2m_3=m} \frac{(m_1+m_3)!(m_2+m_3)!}{m_1!\,m_2!(m_3!)^2}$$
$$\times\ (x\cdot\bar{x}'+z\cdot\bar{z}')^{m_1}(y\cdot\bar{y}'+w\cdot\bar{w}')^{m_2}(y\cdot\bar{z}'-w\cdot\bar{x}')^{m_3}(z\cdot\bar{y}'-x\cdot\bar{w}')^{m_3}.$$

(ii) There exist $a_{m,q} \in \mathbf{R}$ ($q = 1,2,\cdots,[m/2]$) such that

$$(3.6)\qquad \langle X,X'\rangle^m = (m+1)\tilde{K}_m(X,X') + \sum_{q=1}^{[m/2]} a_{m,q}\, K_{m,q}(X,X')\quad (X,X'\in \mathfrak{p}).$$

(iii) There exist $b_{m,q} \in \mathbf{R}$ ($q = 1,2,\cdot\cdot,[m/2]$) such that

$$(3.7)\qquad \langle X,X'\rangle^m = (m+1)\tilde{K}_m(X,X') + \sum_{q=1}^{[m/2]} b_{m,q}\, \tilde{K}_{m,q}(X,X')\quad (X,X'\in \mathfrak{p}).$$

Proof. When $\alpha,\ \beta \in \mathbf{C}^{4p}$ and $\alpha\cdot\alpha = \beta\cdot\beta = 0$, the following formula is well known (see [7]):

$$\int_{S^{4p-1}} (s\cdot\alpha)^m\, \overline{(s\cdot\beta)}^m\, ds = \frac{2^{-m}\, m!\,\Gamma(2p)}{\Gamma(m+2p)}\, (\alpha\cdot\bar{\beta})^m.$$

From this formula and (3.2) we have

$$\tilde{K}_m(X,X') = \frac{\Gamma(2p+m)}{m!\,\Gamma(2p)} \int_{H_2}\left(\int_{H_1} \{a_1\cdot(\bar{\alpha}\bar{x}'-\beta\bar{w}')-\bar{b}_1\cdot(\bar{\alpha}\bar{z}'+\beta\bar{y}')\}^m\right.$$
$$\left.\times\{\bar{a}_1\cdot(\alpha x-\bar{\beta}w)-b_1\cdot(\alpha z+\bar{\beta}y)\}^m dh_1\right)dh_2$$
$$= \int_{H_2} \{(\bar{\alpha}\bar{x}'-\beta\bar{w}')\cdot(\alpha x-\bar{\beta}w)+(\bar{\alpha}\bar{z}'+\beta\bar{y}')\cdot(\alpha z+\bar{\beta}y)\}^m\, dh_2$$
$$= \sum_{|m|=m} \left(\int_{H_2} |\alpha|^{2m_1}|\beta|^{2m_2}(\alpha\beta)^{m_3}(\bar{\alpha}\bar{\beta})^{m_4}\, dh_2\right) m!\,(x\cdot\bar{x}'+z\cdot\bar{z}')^{m_1}$$
$$\times\ \frac{(y\cdot\bar{y}'+w\cdot\bar{w}')^{m_2}(z\cdot\bar{y}'-x\cdot\bar{w}')^{m_3}(y\cdot\bar{z}'-w\cdot\bar{x}')^{m_4}}{m_1!\ m_2!\ m_3!\ m_4!},$$

where $|m| = \sum_{j=1}^{4} m_j$. This gives (3.5) because

$$\int_{H_2} |\alpha|^{2m_1}|\beta|^{2m_2}(\alpha\beta)^{m_3}(\bar{\alpha}\bar{\beta})^{m_4}\, dh_2 = \delta_{m_3,m_4}\frac{(m_1+m_3)!(m_2+m_3)!}{(m+1)!}.$$

(ii) We can prove (3.6) by induction.

(iii) Using (3.6) we can prove (3.7) easily. q.e.d.

From (3.6) we can see that $\tilde{K}_{n,k}(\ \ ,X') \in \mathcal{H}_n$ if $X'\in \mathfrak{N}$. Now

$\mathcal{H}_{n,k}$ denotes the subspace of $\mathcal{H}_n$ which is spanned by $\{\widetilde{K}_{n,k}(\ ,Z);\ Z \in \mathfrak{N}\}$. From (3.4) and (3.7) it is clear that $\mathcal{H}_{n,k}$ is the $K_{\mathbf{R}}$-invarint space and $\mathcal{H}_n = \sum_{k=0}^{[n/2]} \mathcal{H}_{n,k}$.

From now we put $E_0 = \Phi^{-1}\begin{pmatrix} e_1 \\ 0 \\ 0 \\ e_2 \end{pmatrix} \in \mathfrak{N}$ and $E_1 = \Phi^{-1}\begin{pmatrix} e_1 \\ e_1 \\ 0 \\ 0 \end{pmatrix} \in \Sigma$. We already showed $K_{\mathbf{R}}E_1 = \Sigma$ and integral formula of harmonic polynomials on $K_{\mathbf{R}}E_1$ ([11]). Our purpose of this section is getting the reproducing kernel of $\mathcal{H}_{n,k}$ on each orbit in $\mathfrak{N}$.

Our main theorem in this section is the following

Theorem 3.2. Assume $\sqrt{1/2} \leq r < 1$.

(i) For any $f \in \mathcal{H}_{n,k}$ and $X \in \mathfrak{p}$ we have

(3.8) $$C_{n,k}(r)\delta_{n,m}\delta_{k,\ell}f(X) = \dim \mathcal{H}_{n,k} \int_{K_{\mathbf{R}}} f(g\widetilde{E}_r)\widetilde{K}_{m,\ell}(X, g\widetilde{E}_r)dg,$$

where $C_{n,k}(r) = \int_0^1 \{(2r^2-1)t + 1-r^2\}^{n-2k}\, r^{2k}(1-r^2)^k dt$.

In particular, we have

(3.9) $$\delta_{n,m}\delta_{k,\ell}f(X) = \dim \mathcal{H}_{n,k} \int_{K_{\mathbf{R}}} f(gE_0)\widetilde{K}_{m,\ell}(X, gE_0)dg.$$

(ii) For any $f \in \mathcal{H}_{n,k}$, $h \in \mathcal{H}_{m,\ell}$ and $X \in \mathfrak{p}$ we have

(3.10) $$\int_{K_{\mathbf{R}}} f(g\widetilde{E}_r)\overline{h(g\widetilde{E}_r)}dg = 0 \quad \text{if } (n,k) \neq (m,\ell),$$

(3.11) $$\int_{K_{\mathbf{R}}} f(gE_1)\overline{h(gE_1)}dg = C_{n,k} \int_{K_{\mathbf{R}}} f(gE_0)\overline{h(gE_0)}dg,$$

where $C_{n,k} = \{\dim \mathcal{H}_{n,k}\, (n-k+1)!k!\Gamma(2p)\}/\{\Gamma(n+2p)(n-2k+1)^2\}$.

To prove Theorem 3.2, we need some lemmas.

Lemma 3.3 ([11]). For any $f \in \mathcal{H}_n$, $h \in \mathcal{H}_m$ and any $X \in \mathfrak{p}$ we have

$$(3.12) \qquad \int_0^1 \rho(t)\int_{K_{\mathbf{R}}} f(g\tilde{E}_t)\langle X, g\tilde{E}_t\rangle^m \, dg\, dt = (\dim \mathcal{H}_n)^{-1}\delta_{n,m} f(X),$$

where for $t \in [0,1]$ we define

$$\rho(t) = 2^{4p-3} \frac{\Gamma(2p - \frac{1}{2})}{\pi^{1/2}\ \Gamma(2p-2)} t^{4p-5} (1-t^2)^{2p-3}(2t^2-1)^2.$$

$$(3.13) \qquad \int_{K_{\mathbf{R}}} f(gE_1)\overline{h(gE_1)}dg$$

$$= \frac{n!\Gamma(2p)}{\Gamma(n+2p)} \dim \mathcal{H}_n \int_0^1\int_{K_{\mathbf{R}}} f(g\tilde{E}_t)\overline{h(g\tilde{E}_t)}\rho(t)dg\, dt.$$

For the proof see [11] Theorem 2.2.

Lemma 3.4. We put $h_{n,k}(X) = \langle X, \tilde{E}_1\rangle^{n-2k}\{K_2(X, E_0)\}^k$ $(X \in \mathfrak{p})$.

(i) For any $X \in \mathfrak{p}$ we have

$$(3.14) \qquad \int_{K_{\mathbf{R}}} h_{n,k}(gE_0)K_{n,k}(X, gE_0)dg = A_{n,k}h_{n,k}(X),$$

where

$$A_{n,k} = \frac{(n-k+1)!\ k!\ \Gamma(2p-2)\ \Gamma(2p)(n-2k+1)^{-2}}{\Gamma(2p+n-k-1)\Gamma(2p+k-2)(2p+n-1)}.$$

(ii) For any $X \in \mathfrak{p}$ we have

$$(3.15) \qquad \int_{K_{\mathbf{R}}} h_{n,k}(gE_0)K_{n,\ell}(X, gE_0)dg = 0 \quad \text{if } k < \ell,$$

$$(3.16) \qquad \int_{K_{\mathbf{R}}} h_{n,k}(gE_0)\tilde{K}_{n,k}(X, gE_0)dg = A_{n,k}h_{n,k}(X).$$

Lemma 3.5. (i) If $(n,k) \neq (m,\ell)$ we have for any $X, Y \in \mathfrak{p}$

$$(3.17) \qquad \int_{K_{\mathbf{R}}} \tilde{K}_{m,\ell}(gE_0, Y)\tilde{K}_{n,k}(X, gE_0)dg = 0.$$

(ii) For any $g_0 \in K_{\mathbf{R}}$, $X \in \mathfrak{p}$ and $r \in [0,1]$ we have

$$(3.18) \qquad \int_{K_{\mathbf{R}}} \tilde{K}_{n,k}(gE_0, g_0\tilde{E}_r)\tilde{K}_{n,k}(x, gE_0)dg = A_{n,k}\tilde{K}_{n,k}(X, g_0\tilde{E}_r)$$

Lemma 3.6. For any $f \in \mathcal{H}_{n,k}$ we have

$$\int_{K_{\mathbf{R}}} f(gE_0)\tilde{K}_{m,\ell}(X,gE_0)dg = \delta_{n,m}\delta_{k,\ell}\, A_{n,k}\, f(X) \quad (X \in \mathfrak{p}). \tag{3.19}$$

Proposition 3.7. (i) The following formula is valid:

$$A_{n,k} = (\dim \mathcal{H}_{n,k})^{-1}. \tag{3.20}$$

(ii) $\mathcal{H}_n = \bigoplus_{q=0}^{[n/2]} \mathcal{H}_{n,q}$ gives the irreducible decomposition as a $K_{\mathbf{R}}$-module.

Proof. (i) For $f,h \in \mathcal{H}_{n,k}$ we define the inner product $(\ ,\ )$ as follows:

$$(f,g) = \int_{K_{\mathbf{R}}} f(gE_0)\overline{h(gE_0)}dg.$$

Assume that $N = \dim \mathcal{H}_{n,k}$ and that $\{f_j; 1 \le j \le N\}$ is any orthonormal basis of $\mathcal{H}_{n,k}$ with respect to $(\ ,\ )$. We put for any X, $Y \in \mathfrak{p}$

$$P(X,Y) = \sum_{j=1}^{N} \overline{f_j(Y)}f_j(X).$$

For any $X \in \mathfrak{N}$ there exist $c_j(X) \in \mathbf{C}$ $(j = 1,2,\cdots,N)$ such that

$$\tilde{K}_{n,k}(\ ,X) = \sum_{j=1}^{N} c_j(X)f_j.$$

Since for $Y \in \mathfrak{N}$ $P(\ ,Y) \in \mathcal{H}_{n,k}$, we have from (3.19)

$$\begin{aligned} P(X,Y) &= A_{n,k}^{-1} \int_{K_{\mathbf{R}}} P(gE_0,Y)\tilde{K}_{n,k}(X,gE_0)dg \\ &= A_{n,k}^{-1} \sum_{j=1}^{N} \overline{c_j(X)} \sum_{\ell=1}^{N} \int_{K_{\mathbf{R}}} \overline{f_\ell(Y)}f_\ell(gE_0)\overline{f_j(gE_0)}\, dg \\ &= A_{n,k}^{-1} \sum_{j=1}^{N} \overline{c_j(X)f_j(Y)} = A_{n,k}^{-1}\, \tilde{K}_{n,k}(X,Y) \quad (X,Y \in \mathfrak{N}). \end{aligned}$$

Therefore, we get

$$N = \sum_{j=1}^{N}\int_{K_{\mathbf{R}}} \overline{f_j(gE_0)}f_j(gE_0)dg = \int_{K_{\mathbf{R}}} P(gE_0,gE_0)dg$$

$$= A_{n,k}^{-1} \int_{K_{\mathbf{R}}} \widetilde{K}_{n,k}(gE_0, gE_0)dg = A_{n,k}^{-1} \int_{K_{\mathbf{R}}} \widetilde{K}_{n,k}(E_0, E_0)dg.$$

Since $\widetilde{K}_{n,k}(E_0, E_0) = 1$, we obtain (3.20).

(ii) We can see $\mathcal{H}_n = \bigoplus_{k=0}^{[n/2]} \mathcal{H}_{n,k}$ from (3.19). From [3] we can see that the number of the $K_{\mathbf{R}}$-irreducible subspaces of S_n is $1 + 2 + \cdots + ([n/2]+1)$. Therefore the number of irreducible subspaces of $\mathcal{H}_n$ is $[n/2]+1$, because $S_n = \bigoplus_{k=0}^{n} \mathcal{H}_k J_{n-k}$ (cf. [4]), where J_m is the space of the K-invariant homogeneous polynomial of degree m. Therefore we get (ii). q.e.d.

Remark. Propositon 3.7 may be known, but the author does not know the references.

Proof of Theorem 3.2. We obtain (3.9) by (3.19) and (3.20). By the same method of the proof of Lemma 3.4 we have

$$(3.21) \qquad \int_{K_{\mathbf{R}}} h_{n,k}(g\widetilde{E}_r)\widetilde{K}_{n,k}(X, g\widetilde{E}_r)dg = A_{n,k}C_{n,k}(r)h_{n,k}(X),$$

where $C_{n,k}(r) = \int_{H_2} r^{2k}(1-r^2)^k\{|\alpha|^2 r^2 + |\beta|^2(1-r^2)\}^{n-2k}\, dh_2$.

(3.21) gives (3.8) because $h_{n,k}$ is the generator of $\mathcal{H}_{n,k}$.

By (3.19) we see that $f_{m,\ell}(X) = \int_{K_{\mathbf{R}}} h_{n,k}(g\widetilde{E}_r)\widetilde{K}_{m,\ell}(X, g\widetilde{E}_r)dg \in \mathcal{H}_{m,\ell}$. From (3.7),(3.12) and (3.19) we can see that there exist $C_{m,q} \in \mathbf{R}\setminus\{0\}$ $(q = 1,2,\cdots,[m/2])$ such that

$$\int_{K_{\mathbf{R}}} h_{n,k}(g\widetilde{E}_r)\langle X, g\widetilde{E}_r\rangle^m\, dg$$
$$= \sum_{q=0}^{[m/2]} C_{m,q} \int_{K_{\mathbf{R}}} h_{n,k}(g\widetilde{E}_r)\widetilde{K}_{m,q}(X, g\widetilde{E}_r)dg = \sum_{q=0}^{[m/2]} C_{m,q}\, f_{m,q}(X).$$

On the other hand, for some $C'_{n,k}(r)$, $C''_{n,k}(r) \in \mathbf{C}$ we have

$$\int_{K_{\mathbf{R}}} h_{n,k}(g\tilde{E}_r)\langle X, g\tilde{E}_r\rangle^m \, dg = \delta_{m,n} C'_{n,k}(r) \int_{K_{\mathbf{R}}} h_{n,k}(gE_0)\langle X, gE_0\rangle^n dg$$

$$= \delta_{n,m} C''_{n,k}(r) h_{n,k}(X) \in \mathscr{H}_{n,k}.$$

Therefore $f_{m,\ell} = 0$ if $(n,k) \neq (m,\ell)$, and this equation shows (3.10).

(3.21) implies

$$(3.22) \quad \int_0^1 \int_{K_{\mathbf{R}}} h_{n,k}(g\tilde{E}_t)\tilde{K}_{n,k}(X, g\tilde{E}_t)\rho(t) dg \, dt$$

$$= 2^{4p-3} \frac{\Gamma(2p-\frac{1}{2})\Gamma(2p+n-k-1)\Gamma(2p+k-2)}{\pi^{1/2} \, \Gamma(2p-2)\Gamma(4p+n-2)} \int_{K_{\mathbf{R}}} h_{n,k}(gE_0)\tilde{K}_{n,k}(X, gE_0) dg.$$

(3.11) follows from (3.13) and (3.22). q.e.d.

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