数理解析研究所講究録
1159 巻 2000 年 127-134

On a class of convolution systems in one variable

Yasunori Okada (Chiba University) (岡田靖則)

We study a system of convolution equations

$$\mu_j * u = 0, \qquad (j = 1, 2, \ldots, \ell), \tag{E}$$

where each μ_j is a hyperfunction with compact support defined on $\mathbb{R}$ and u an unknown function on $\mathbb{R}$.

Let $\mathcal{E}(\mathbb{R})$ be the space of differentiable functions on $\mathbb{R}$, $\mathcal{E}'(\mathbb{R})$ the space of distributions with compact support, $\mathcal{B}(\mathbb{R})$ the space of hyperfunctions, and $\mathcal{B}_c(\mathbb{R})$ the space of hyperfunctions with compact support. We regard $\mathcal{B}_c(\mathbb{R})$ as a commutative ring with respect to the convolution product, and $\mathcal{E}'(\mathbb{R})$ as its subring.

Generally, an ideal of $\mathcal{E}'(\mathbb{R})$ can not be characterized by its common zeros of the Fourier images. But every closed ideal can be characterized by its common zeros (with multiplicities) of the Fourier images. Thus when every kernel μ_j is a distribution, the space of differentiable solutions to (E) has the dense subspace consisting of all exponential polynomial solutions.

When we study hyperfunction solutions to (E), we can not apply these topological methods directly even if the kernels are distributions.

We compare the two spaces: the space of differentiable solutions, that of hyperfunction solutions. For this purpose, we construct a suitable infinite-order elliptic differential operator with given growth order. Using this operator, we show that any hyperfunction solution to (E) is a "infinite-order derivative" of a differentiable solution to (E).

1 Main result

Theorem 1.1. *Let $u \in \mathcal{B}(\mathbb{R})$ be a solution to the system (E). Then there exist a hyperfunction $\nu \in \mathcal{B}_{\{0\}}(\mathbb{R})$ supported in the origin and a differentiable solution $v \in \mathcal{E}(\mathbb{R})$ to the system (E) which satisfy $u = \nu * v$.*

Remark 1.2. A hyperfunction ν supported in the origin is written as $\nu = J(D)\delta$ for a suitable differential operator $J(D)$ of infinite-order with constant coefficients. Thus we can say "any hyperfunction solution u to (E) is an infinite-order derivative $J(D)v$ of a differentiable solution v" in some sense.

Corollary 1.3. *Assume that every kernel μ_j is a distribution and that the set*

$$\{\zeta \in \mathbb{C} \mid \forall j, \hat{\mu}_j(\zeta) = 0\}$$

is a finite set. Then every hyperfunction solution to (E) is an exponential polynomial solution.

Now we give the sketch of the proof. We follow the idea due to A. Kaneko [2].

First we explain the construction of a suitable differential operator $J(D)$ of infinite-order with constant coefficients.

Let $\varphi(t)$ be an increasing function defined on $t \geq 1$ with $\lim_{k\to\infty} \varphi(k) = \infty$ and $\varphi(1) > 1$, and $\{\theta(k)\}_{k=1,2,\ldots}$ a bounded sequence of real numbers with $\sup_k |\theta(k)| \leq \pi/12$. We define

$$\zeta_k := \sqrt{-1}k\varphi(k)e^{\sqrt{-1}\theta(k)}, \quad k = 1, 2, \ldots$$

and

$$J_{\varphi,\theta}(\zeta) := \prod_{k=1}^{\infty} \left(1 + \left(\frac{\zeta}{k\varphi(k)e^{\sqrt{-1}\theta(k)}}\right)^2\right).$$

Note that this infinite product converges and that $J_{\varphi,\theta}$ becomes an entire function of infra-exponential growth with

$$V(J_{\varphi,\theta}) := \{\zeta \in \mathbb{C} \mid J_{\varphi,\theta}(\zeta) = 0\} = \{\pm\zeta_k\}_{k=1,2,\ldots}.$$

Moreover we give

Lemma 1.4. $J = J_{\varphi,\theta}$ *satisfies that for any* $r, s \in \mathbb{R}$, $r > 0$, $|s| \le \pi/12$

$$\left|J(re^{\sqrt{-1}s})\right| \ge C \exp \frac{cr}{\varphi(r+1)},$$

for some positive constants C *and* c, *and that for any* $\zeta \in V(J)$

$$|J'(\zeta)| \ge \frac{1}{|\zeta|}.$$

Proof. For r, take N with $N\varphi(N) \le r < (N+1)\varphi(N+1)$. Then we have

$$\left|J(re^{\sqrt{-1}s})\right| = \prod_{k=1}^{\infty} \left|1 + \frac{r^2}{(k\varphi(k))^2} e^{2\sqrt{-1}(s-\theta(k))}\right|$$

from the estimate $|s - \theta(k)| \le \pi/6$,

$$\ge \prod_{k=1}^{N} \left(1 + \frac{r^2}{2(k\varphi(k))^2}\right) \prod_{k=N+1}^{\infty} \left(1 + \frac{r^2}{2(k\varphi(k))^2}\right)$$

$$\ge \left(1 + \frac{1}{2}\right)^N \ge \frac{2}{3} \exp\left((N+1)\log(3/2)\right)$$

using $N + 1 > r/\varphi(N+1) \ge r/\varphi(r+1)$,

$$\ge \frac{2}{3} \exp \frac{r \log(3/2)}{\varphi(r+1)}.$$

About the second estimate, we use

$$J'(\pm\zeta_j) = \mp\frac{2}{\zeta_j} \prod_{k \ne j} \left(1 + \left(\frac{\pm\zeta_k}{k\varphi(k)e^{\sqrt{-1}\theta(k)}}\right)^2\right)$$

We have

$$|J'(\pm\zeta_j)| = \frac{2}{|\zeta_j|} \prod_{k \ne j} \left|1 - \left(\frac{j\varphi(j)}{k\varphi(k)}\right)^2 e^{2\sqrt{-1}(\theta(j)-\theta(k))}\right|$$

$$\ge \frac{2}{|\zeta_j|} \prod_{k=1}^{j-1} \left(\frac{j^2}{k^2}\frac{\varphi(j)^2}{\varphi(k)^2} - 1\right) \prod_{k=j+1}^{\infty} \left(1 - \frac{j^2}{k^2}\frac{\varphi(j)^2}{\varphi(k)^2}\right)$$

$$\ge \frac{2}{|\zeta_j|} \prod_{k=1}^{j-1} \left(\frac{j^2}{k^2} - 1\right) \prod_{k=j+1}^{\infty} \left(1 - \frac{j^2}{k^2}\right)$$

Substituting $z = j$ into the formula

$$\frac{\sin(\pi z)}{\pi z(1 - z^2/j^2)} = \prod_{k \neq j} (1 - z^2/k^2)$$

we have the desired result. $\square$

Proposition 1.5. *Let $J = J_{\varphi,\theta}$ be as above. Then any hyperfunction solution $u \in \mathcal{B}(\mathbb{R})$ to $J(D)u = 0$ is real analytic.*

This was prooved in Kawai [4]. Or in this case we can regard $J(D)$ as a holomorphic microlocal operator with constant coefficients defined in a neighborhood of $\dot{T}^*_{\mathbb{R}}\mathbb{C}$ with the inverse $1/J(D)$.

We denote by A the Fourier image of $\mathcal{B}_c(\mathbb{R})$, which consists of entire functions of exponential type with infra-exponential growth on the real axis.

Lemma 1.6. *Let $u \in \mathcal{B}(\mathbb{R})$ be a hyperfunction and $F_j(\zeta) \in A$, $(j = 1, 2, \ldots, \ell)$. Assume that every F_j is not identically zero. Then we can find $\varphi(t)$ and $\theta(k)$ as above such that $J = J_{\varphi,\theta}$ satisfies*

$$\exists v \in \mathcal{E}(\mathbb{R}), J(D)v = u, \tag{1}$$

$$\exists q_j, r_j \in A,\ q_j F_j + r_j J \equiv 1, \quad j = 1, 2, \ldots, \ell. \tag{2}$$

Proof. For (1), we may choose $\varphi(t)$ with sufficiently slow growth. We note that the functions of the form $\exp(cr/\varphi(r+1))$ in Lemma 1.4 themselves are infra-exponential and that any function of infra-exponential growth can be estimated from above by these functions. See Kaneko [3].

To prove the second formula, we put $F(\zeta) = \prod_j F_j(\zeta) F_j(-\zeta)$, and we may construct $q, r \in A$ with $qF + rJ \equiv 1$. Note that F and J are even. First we construct q with $q(\zeta)F(\zeta) = 1$ for any $\zeta \in V(J)$. Using the minimum modulus estimate of entire functions of exponential type, we can choose $\{\theta(k)\}_k$ with

$$|F(\pm\zeta_k)| = \left|F(\pm\sqrt{-1}k\varphi(k)e^{\sqrt{-1}\theta(k)})\right| \geq C\exp(-ck\varphi(k))$$

with some positive constants C and c. Thus we must solve the interpolation problem

$$\forall\zeta \in V(J),\ q(\zeta) = 1/F(\zeta),$$
$$q(\zeta) \le C' \exp(c' |\zeta|),$$

with the estimate $1/F(\zeta) \le C' \exp(c|\zeta|)$ on $V(J)$. It is possible if we remark the estimate $J'(\zeta) \ge 1/|\zeta|$ on $V(J)$ in Lemma 1.4 and Theorem 4 in Berenstein-Taylor [1]. Note that when we apply the theorem to our problem, we must take a suitable weight subharmonic function $p(\zeta)$ satisfying several properties. $p(\zeta) = |\zeta|$ is not good in order to make sure $q \in A$. But we omit the details here.

Once we can construct q, we define r by $r = (1 - qF)/J$. Note that since J is of infra-exponential growth, the exponential behavior of the quotient r is the same as that of the dividend $1 - qF$. Thus we have $r \in A$. □

Now we give

Proof of the Theorem 1.1. We define $F_j(\zeta) = \hat{\mu}_j(-\zeta)$, apply Lemma 1.6 to u and F_j's, and get J, v, q_j's, r_j's. Note that $F_j(D) = \mu_j *$. We put

$$w = \prod_{j=1}^{\ell}(1 - q_j(D)F_j(D))v = \prod_{j=1}^{\ell}(r_j(D)J(D))v.$$

Then we have

$$w = v + \sum_{k=1}^{\ell}(-1)^k \sum_{j_1<\cdots<j_k} \prod_{i=1}^{k} q_{j_i}(D)F_{j_i}(D)v,$$

and

$$Jw = u + \sum_{k=1}^{\ell}(-1)^k \sum_{j_1<\cdots<j_k} \prod_{i=1}^{k} q_{j_i}F_{j_i}u = u,$$
$$F_k(D)w = \left(\prod_{j=1}^{\ell-1}(r_jJ)\right) r_\ell(D)F_k(D)J(D)v = 0.$$

Moreover since $J(w - v) = u - u = 0$, we have $w - v$ is real analytic by virtue of Proposition 1.5. Thus we have $w \in \mathcal{E}(\mathbb{R})$. □

2 An example

In this section we give a very easy example.

Question 2.1. Let $u \in \mathcal{B}(\mathbb{R})$ be a hyperfuntion with two period independent over $\mathbb{Q}$. Is u a constant?

In other words: We consider the following convolution system

$$\begin{cases} (\delta(\cdot - \alpha) - \delta(\cdot + \alpha)) * u = 0, \\ (\delta(\cdot - \beta) - \delta(\cdot + \beta)) * u = 0, \end{cases} \tag{$E_{\alpha,\beta}$}$$

for positive constants α and β with $\beta/\alpha \notin \mathbb{Q}$, and the equation

$$\delta' * u = 0. \tag{E_∂}$$

Does a hyperfuntion solution $u \in \mathcal{B}(\mathbb{R})$ to $(E_{\alpha,\beta})$ satisfies (E_∂)?

We can give the affirmative answer to this question:

Answer 2.2. For any α, β with $\beta/\alpha \notin \mathbb{Q}$, every hyperfunction solution u to $(E_{\alpha,\beta})$ is a solution to (E_∂).

Proof. Since the Fourier images of the kernel functions in $(E_{\alpha,\beta})$ are $\sin \alpha\zeta$ and $\sin \beta\zeta$, and that in (E_∂) is ζ up to non-zero constants, the common zeros of each system is equal to $\{0\}$. By corollary 1.3, we can show that every hyperfunction solution to each system is a constant. $\square$

Remark 2.3. In this situation, we also want to know whether the system $(E_{\alpha,\beta})$ is always "isomorphic" to (E_∂), i.e., whether the module $M_{\alpha,\beta} := \mathcal{B}_c(\mathbb{R})/(\mathcal{B}_c(\mathbb{R})*\sin(\alpha D)+\mathcal{B}_c(\mathbb{R})*\sin(\beta D))$ is isomorphic to $M_\partial := \mathcal{B}_c(\mathbb{R})/\mathcal{B}_c(\mathbb{R})* D$ as $\mathcal{B}_c(\mathbb{R})$ modules.

Answer 2.4. Sometimes, for example if β/α is algebraic, the system $(E_{\alpha,\beta})$ is "isomorphic" to the equation (E_∂). But there is a pair (α, β), where $(E_{\alpha,\beta})$ is not "isomorphic" to (E_∂).

From now on, we discuss about the A modules

$$\hat{M}_{\alpha,\beta} := A/(A\sin(\alpha\zeta) + A\sin(\beta\zeta)), \quad \hat{M}_\partial := A/A\zeta,$$

instead of $M_{\alpha,\beta}$ and M_∂.

Remark 2.5. The Fourier image of $\mathcal{E}'(\mathbb{R})$ is denoted by A^f . In Berenstein-Taylor [1], the authors discussed about A^f modules

$$\hat{M}^f_{\alpha,\beta} := A^f/(A^f \sin(\alpha\zeta) + A^f \sin(\beta\zeta)), \quad \hat{M}^f_\partial := A^f/A^f\zeta,$$

and gave a similar result. We follow their idea.

Now we give a brief explanation of this answer. We can show, from an easy discussion about unitary commutative rings, that there exists an isomorphism as A modules between two modules, both of which have a single generator, if and only if the annihilator ideals are the same. The inclusion $A\sin(\alpha\zeta) + A\sin(\beta\zeta) \subset A\zeta$ is clear. Thus we want to know when ζ belongs to $A\sin(\alpha\zeta) + A\sin(\beta\zeta)$.

Lemma 2.6. *The necessary and sufficient condition for $\zeta \in A\sin(\alpha\zeta) + A\sin(\beta\zeta)$ is that the number β/α has the following estimate:*

$$\forall\varepsilon > 0, \exists C_\varepsilon, \forall n > 0, \ \mathrm{dis}(\frac{\beta}{\alpha}, \frac{1}{n}\mathbb{Z}) > C_\varepsilon e^{-\varepsilon n}. \tag{3}$$

The necessity is clear. We can show the sufficiency using the same interpolation theorem (Theorem 4 in Berenstein-Taylor [1]).

Proof of the Answer 2.4. By virtue of the famous Roth's theorem, if β/α is algebraic, $\mathrm{dis}(\beta/\alpha, (1/n)\mathbb{Z})$ can be estimated from below by $C(1/n)^c$ with some constants C and c. On the other hand, we can construct a transcendental number β/α which does not satisfy (3). For exapmle, we may take an increasing sequence $\{n_j\}_j \subset \mathbb{Z}_+$ satisfying

$$\frac{2}{n_{j+1}!} \leq \exp(-n_j!), \quad \forall j$$

and set $\beta/\alpha := \sum_j 1/n_j!$. Then β/α is not rational and

$$\mathrm{dis}(\frac{\beta}{\alpha}, \frac{1}{n_j!}\mathbb{Z}) \leq \sum_{k\geq j+1} \frac{1}{n_k!} < \frac{2}{n_{j+1}!} \leq \exp(-n_j!).$$

This estimate shows that β/α does not satisfy (3). ☐

Remark 2.7. The modules $\hat{M}_{\alpha,\beta}$ and $\hat{M}^f_{\alpha,\beta}$ have resolutions

$$0 \to A^f \xrightarrow{\phi_2} (A^f)^2 \xrightarrow{\phi_1} A^f \to 0, \qquad (C^{f\cdot})$$

$$0 \to A \xrightarrow{\phi_2} (A)^2 \xrightarrow{\phi_1} A \to 0 \qquad (C^{\cdot})$$

respectively, where,

$$\phi_2 = \cdot(\sin(\beta\zeta)/\zeta, -\sin(\alpha\zeta)/\zeta), \quad \phi_1 = \cdot \begin{pmatrix} \sin(\alpha\zeta) \\ \sin(\beta\zeta) \end{pmatrix}.$$

In Berenstein-Taylor [1], the authors also showed that if $\hat{M}^f_{\alpha,\beta}$ is not isomorphic to $\hat{M}^f_{\vartheta}$, the "extension" $H^1(Hom(C^{f\cdot}, \mathcal{E}(\mathbb{R})))$ does not vanish. Thus we also expect the similar result about $H^1(Hom(C^{\cdot}, \mathcal{B}(\mathbb{R})))$, but I do not have the answer yet.

References

[1] Berenstein, C. A., and Taylor, B. A., *A New Look at Interpolation Theory for Entire Functions of One Variable*, Adv. Math., **33** (1979) 109–143.

[2] Kaneko, A., *Representation of hyperfunctions by measures and some of its applications*, J. Math. Fac. Sci. 1972.

[3] Kaneko, A., *Introduction to hyperfunctions*, Kluwer Acad. Publ., (1988).

[4] Kawai, T., *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **17** (1970) 467–517.