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Ind-Sheaves

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I Ind-lim and ind-objects (cf. SGA 4)

$\mathcal{U}$ a universe, $\mathcal{C}$ a $\mathcal{U}$-category

$\mathcal{C}^\vee$ = the category of contravariant functors $\mathcal{C} \to \underline{\mathrm{Set}}$

$h^\vee : \mathcal{C} \to \mathcal{C}^\vee, \quad X \mapsto \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$.

$h^\vee$ is fully faithful

I : small filtrant category, $\alpha : I \to \mathcal{C}$ a functor

Def $\text{“}\varinjlim\text{”}\, \alpha \in \mathcal{C}^\vee$ is given by:

$$\text{“}\varinjlim\text{”}\, \alpha(Y) = \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(Y, \alpha(i))$$

Def $\mathrm{Ind}(\mathcal{C})$ is the full subcategory of $\mathcal{C}^\vee$ defined by:

$$A \in \mathrm{Ind}(\mathcal{C}) \Longleftrightarrow \exists\, \alpha : I \to \mathcal{C}, \ I \text{ small, filtrant}$$
$$\text{s.t. } A \simeq \text{“}\varinjlim\text{”}\, \alpha$$

Hence $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C}) \hookrightarrow \mathcal{C}^\vee$

Theorem Assume $\mathcal{C}$ is abelian.

(i) $\mathrm{Ind}(\mathcal{C})$ is abelian

(ii) $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ is exact and thick

(iii) $\mathrm{Ind}(\mathcal{C})$ admits small $\varinjlim$ and $\varprojlim$ and $\varinjlim$ over a filtrant category is exact

(iv) Assume $\mathcal{C}$ has enough injectives. Then

$$D^b(\mathcal{C}) \simeq D^b_{\mathcal{C}}(\mathrm{Ind}(\mathcal{C})).$$

II Ind-sheaves

X: topological space, Hausdorff, locally compact, countable at ∞

k: a field

$\mathrm{Mod}(k_X)$: abelian category of sheaves on X of k-vector spaces

$\mathrm{Mod}^c(k_X)$: subcateg. of compactly supported sh.

$$I^c(k_X) = \mathrm{Ind}(\mathrm{Mod}^c(k_X))$$

Z: $U \mapsto I^c(k_U)$ is a stack, not $U \mapsto I(k_U)$.

$$\mathrm{Mod}(k_X) \xrightarrow[i_X]{} I^c(k_X) \xrightarrow[\alpha_X]{} \mathrm{Mod}(k_X)$$

$$i_X(F) = \text{``}\varinjlim_{U \subset\subset X}\text{''}\, F_U$$

$$\alpha_X(\text{``}\varinjlim_i\text{''}\, F_i) = \varinjlim_i F_i$$

$$\mathrm{Hom}_{k_X}(\alpha_X(G), F) \simeq \mathrm{Hom}_{I^c(k_X)}(G, i_X(F)).$$

Restriction: $U \subset X$ open

$$F = \text{``}\varinjlim_i\text{''}\, F_i \in I^c(k_X)$$

$$F|_U = \text{``}\varinjlim_{i,\, V \subset\subset U}\text{''}\, (F_i)_V \in I^c(k_U)$$

<u>Lemma/Def</u>: the presheaf on X

$U \mapsto \mathrm{Hom}_{I^c(k_U)}(F|_U, G|_U)$ is a sheaf

denoted $\mathcal{H}om(F, G)$.

$\mathcal{H}om : I^c(k_X)^{op} \times I^c(k_X) \longrightarrow \mathrm{Mod}(k_X)$

<u>Th</u> α_X admits a left adjoint β_X

$$\mathrm{Hom}_{I^c(k_X)}(\beta_X(F), G) \simeq \mathrm{Hom}_{k_X}(F, \alpha_X(G))$$

$$\beta_X(F) : \mathrm{Mod}^c(k_X) \longrightarrow \underline{Ab}$$
$$G \longmapsto \Gamma(X; F \otimes \mathcal{H}om_{k_X}(G, k_X)).$$

<u>Notation</u> $Z \subset X$ locally closed

$$\beta_X(k_Z) = \widetilde{k}_Z$$

<u>Ex</u> U open in X

$$\widetilde{k}_U = \text{"}\varinjlim_{V \subset\subset U}\text{"}\, k_V \quad V \text{ open}$$

S closed in X

$$\widetilde{k}_S = \text{"}\varinjlim_{U}\text{"}\, k_{\bar{U}}, \quad S \subset U,\ U \text{ open}$$

<u>Tens</u> $I^c(k_X) \times I^c(k_X) \longrightarrow I^c(k_X)$

$$\text{"}\varinjlim_i\text{"}\, F_i \otimes \text{"}\varinjlim_j\text{"}\, G_j := \text{"}\varinjlim_{i,j}\text{"}(F_i \otimes G_j)$$

<u>IHom</u> : $I^c(k_X)^{op} \times I^c(k_X) \to I^c(k_X)$

$$\mathrm{IHom}\left(\text{``}\varinjlim_i\text{''} F_i, \text{``}\varinjlim_j\text{''} G_j\right) := \varprojlim_i \text{``}\varinjlim_j\text{''} \mathcal{H}om_{k_X}(F_i, G_j)$$

<u>Some formulas</u>

$$i_X(F \otimes G) \simeq i_X F \otimes i_X G$$

$$i_X \mathcal{H}om_{k_X}(F, G) \simeq \mathrm{I}\mathcal{H}om(i_X F, i_X G)$$

$$\alpha_X \circ \mathrm{I}\mathcal{H}om(F, G) = \mathcal{H}om(F, G)$$

$$\mathrm{I}\mathcal{H}om(F \otimes G, H) \simeq \mathrm{I}\mathcal{H}om(F, \mathrm{I}\mathcal{H}om(G, H))$$

$$\beta_X(F \otimes G) \simeq \beta_X F \otimes \beta_X G$$

$$\beta_X F \otimes \mathrm{I}\mathcal{H}om(G, K) \simeq \mathrm{I}\mathcal{H}om(G, \beta_X(F) \otimes K)$$

<u>External operations and derived category</u> $\left.\begin{array}{l} f^{-1}, f_*, f_{!!}, f^! \\ D^b(I^c(k_X)) \end{array}\right\}$ not here

III Construction of ind-sheaves

X: real analytic mf, $k = \mathbb{C}$
$\mathbb{R}C^c(k_X)$: $\mathbb{R}$-constructible sheaves
with compact support
$I\mathbb{R}^c(k_X) := \mathrm{Ind}(\mathbb{R}\text{-}C^c(k_X))$

$$\begin{array}{ccccc} \mathbb{R}C^c(k_X) & \hookrightarrow & \mathrm{Mod}^c(k_X) & \hookrightarrow & \mathrm{Mod}(k_X) \\ \downarrow & & \downarrow & & \\ I\mathbb{R}^c(k_X) & \hookrightarrow & I^c(k_X) & & \end{array}$$

Let $\mathcal{T} \subset \mathrm{Op}(X)$ be the family of open subanalytic relatively compact sets

Th Let $\alpha : \mathcal{T}^{op} \to \mathcal{A}b$ s.t.
$\alpha(\emptyset) = \{0\}$
$0 \to \alpha(U \cup V) \to \alpha(U) \oplus \alpha(V) \to \alpha(U \cap V)$
is exact $\forall U, V \in \mathcal{T}$.
Then $\exists!$ $A \in I\mathbb{R}^c(k_X)$ s.t.
$\mathrm{Hom}(k_U, A) = \alpha(U)$ $\forall U \in \mathcal{T}$.

Example $\mathcal{E}_X^{\infty t} \in I\mathbb{R}^c(k_X)$

associated to $\alpha(U) = \mathcal{T}hom(\mathbb{C}_U, \mathcal{E}_X^{\infty})$
(see K-S, Moderate and formal cohomology)

Let X be a complex manifold

<u>Def</u> $\mathcal{O}_X^t = R\mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, \mathcal{E}_{X_{\mathbb{R}}}^{\infty t})$

Z: $\mathcal{O}_X^t \in D^b(I^c(k_X))$ is not concentrated in degree 0

<u>Def.</u> M real analytic manif.
X a complexification

$$\mathcal{D}b_M^t = R\mathcal{H}om(D'_X \mathbb{C}_M, \mathcal{O}_X^t)$$

$$\mathcal{D}b_M = \alpha_M(\mathcal{D}b_M^t)$$

$\mathcal{D}b_M$ is also called the sheaf of distributions on M by some authors.

Let $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$

$$R\mathcal{H}om(F, \mathcal{O}^t) = T\mathcal{H}om(F, \mathcal{O}_X)$$

($T\mathcal{H}om$ is Kashiwara's functor of temperate cohomology).