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Dynamical representations of substituted Sturmian sequences

Shin-ichi YASUTOMI (安富真一)

Suzuka National College of Technology

1 Introduction

We announce some theorems about Sturmian words in this report. The proofs and details will be published elsewhere. We need some notations. Let L be an alphabet, i.e., a non-empty finite set of letters. Now, we set $L=\{0,1\}$. Let $W=\bigcup_{n=0}^{\infty}\{0,1\}^{n}$, $W^{*}=\bigcup_{n=0}^{\infty}\{0,1\}^{n}\cup\{0,1\}^{\mathbb{N}}$. For $x,y\in[0,1]$ we define $G(x,y),\hat{G}(x,y)\in W^{*}$ by

$$G(x,y)=G_{0}(x,y)G_{1}(x,y)\ldots,$$
$$\hat{G}(x,y)=\hat{G}_{0}(x,y)\hat{G}_{1}(x,y)\ldots,$$

where $G_{j}(x,y)=\lfloor(j+1)x+y\rfloor-\lfloor jx+y\rfloor$, $\hat{G}_{j}(x,y)=\lceil(j+1)x+y\rceil-\lceil jx+y\rceil$ for each integer j
and $\lfloor u\rfloor$ is an integral part of u and $\lceil u\rceil=-\lfloor-u\rfloor$ for each $u\in\mathbf{R}$.
Examples
$x=\frac{1}{3}$

$$G(x,0)=001001\ldots=(001)_{\infty}$$

$x=\sqrt{2}-1, y=\frac{1}{2}$

$$G(x,y)=0101001010\ldots$$

For $w\in L^{\mathbb{N}}$, we define $Sub(w)$ by

$$Sub(w)=\{u\in W\mid u \text{ is a subword of } w\}.$$

A Sturmian word is defined to be a word $w\in L^{\mathbb{N}}$ satisfying

$$||A|_{1}-|B|_{1}|\leq 1$$

for any $A,B\in Sub(w)$ with $|A|=|B|$, where for $u\in W$ $|u|$ is a length of u and $|u|_{1}$ is a number of the occurences of the letter 1 in u. In this lecture we consider only non periodic Sturmian word.

Theorem. (Morse and Hedlund [2];Coven and Hedlund[3]) *w is Sturmian if and only if w is equal to $G(x,y)$ or $\hat{G}(x,y)$ for some $x,y\in[0,1]$.*

A transformation f on W^{*} is called substitution on W^{*}, if f satisfies following conditions:

1. $f(0), f(1) \in W$,
2. for any $a \in W$ and $b \in W^*$, f(ab)=f(a)f(b).

Example

Let f be a substitution on W^* defined by

$$f : \begin{cases} 0 \longrightarrow 01 \\ 1 \longrightarrow 010 \end{cases} .$$

Then,

$$f(00101) = f(0)f(0)f(1)f(0)f(1) = 010101001010.$$

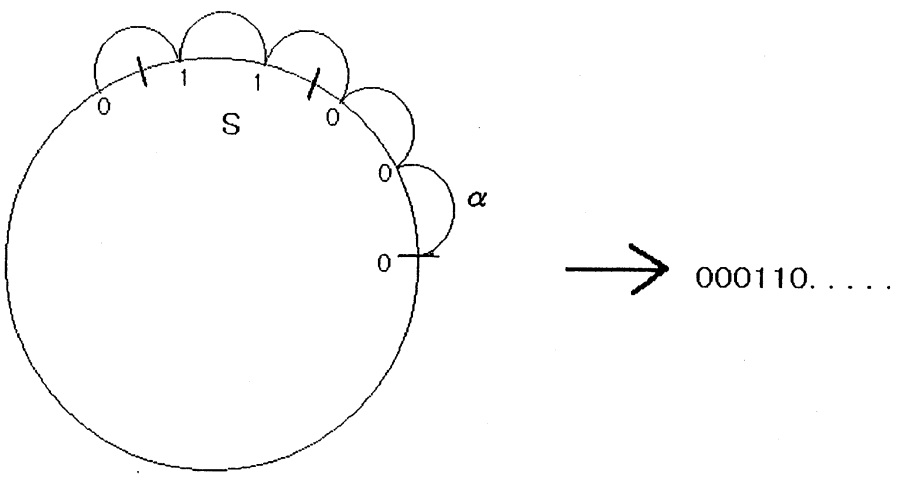
Let α be a real number. Define right infinite word $\chi(S, \alpha) \in W^*$ for a set S in interval $[0, 1]$ and α by

$$\chi(S, \alpha) = \lambda(S, \alpha; 0)\lambda(S, \alpha; 1) \cdots ,$$

where

$$\lambda(S, \alpha; n) = \begin{cases} 1 & \text{if } \langle n\alpha \rangle \in S, \\ 0 & \text{if } \langle n\alpha \rangle \notin S, \end{cases}$$

where $\langle x \rangle$ is a fractional part of x.



Example of $\chi(S, \alpha)$

We define a mod 1 semiclosed interval $[x, y)^{\sim}$ for $0 \le x, y \le 1$ by

$$[x, y)^{\sim} = \begin{cases} [x, y) & \text{if } 0 \le x \le y, \\ [0, y) \cup [x, 1) & \text{if } 0 \le y < x. \end{cases}$$

We can define a mod 1 semiclosed interval $(x, y]^{\sim}$ in the same manner as above.
Our main result is as follows.

Theorem 1. *Let S be a Sturmian sequence. Let F be a substitution with $GCD(|F(0)|, |F(1)|) = 1$. Then, there exist $x, y \in \mathbb{R}$ and integers $m_1, \dots, m_k$ and $n_1, \dots, n_k$ such that x is irrational and $0 < x < 1$ and*

$$\chi(I, x) = F(S),$$

where

$$I = \bigcup_{i=1}^{k} [\langle m_i x - y\rangle, \langle n_i x - y\rangle)^{\sim},$$

or

$$I = \bigcup_{i=1}^{k} (\langle m_i x - y\rangle, \langle n_i x - y\rangle]^{\sim}.$$

The converse is also true.

2 An algorithm on inhomogeneous Diophantine approximation

We introduce the following algorithm on inhomogeneous Diophantine approximation to prove main Theorem. Let us define functions t_0, t_1, t_2 on $\mathbb{R}^2$ by

$$\begin{aligned} t_0(x, y) &= \left(\frac{x}{1+x}, \frac{y}{1+x}\right), \\ t_1(x, y) &= \left(\frac{1}{2-x}, \frac{y}{2-x}\right), \\ t_2(x, y) &= (1-x, 1-y). \end{aligned}$$

Let us a domain X by

$$X = \{(x, y) | 0 \le x, y \le 1 \text{ and } y \ne mx + n \text{ for any integers } m, n\}.$$

We define domains S_i^0 $(i = 0, \dots, 5)$ by

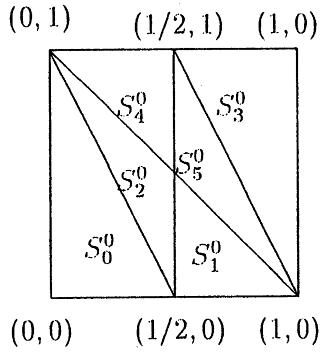


Figure of X

We define transformation T_0 on X as follows:

$$T_0(x, y) = \begin{cases} t_0^{-1}(x, y) & \text{if } (x, y) \in S_0^0, \\ t_1^{-1}(x, y) & \text{if } (x, y) \in S_1^0, \\ t_2^{-1} \circ t_0^{-1}(x, y) & \text{if } (x, y) \in S_2^0. \\ t_2^{-1} \circ t_0^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_3^0, \\ t_2^{-1} \circ t_1^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_4^0, \\ t_0^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_5^0. \end{cases}$$

We define domains S_i^1 $(i = 0, \ldots, 5)$ as follows:

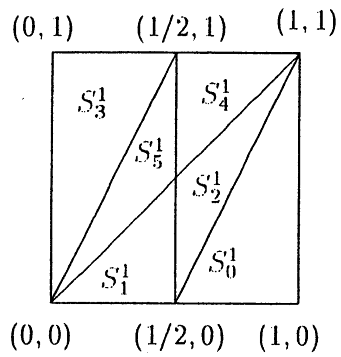
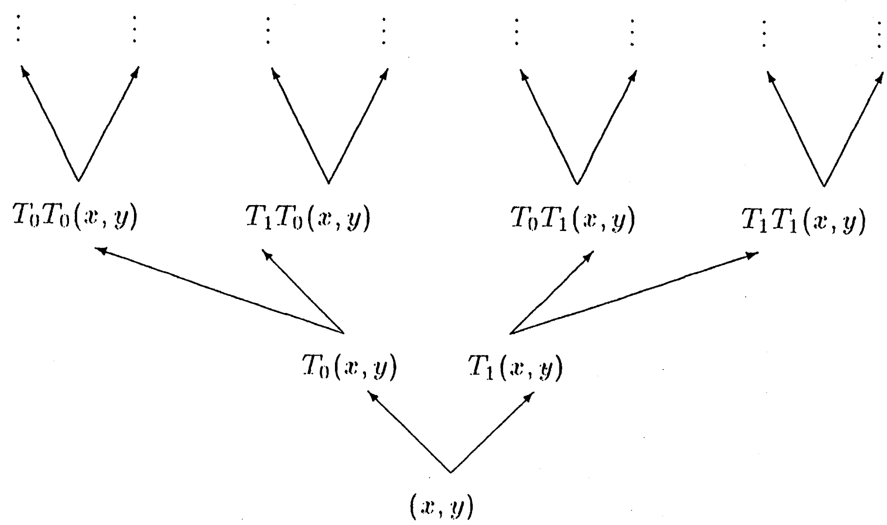


Figure of X

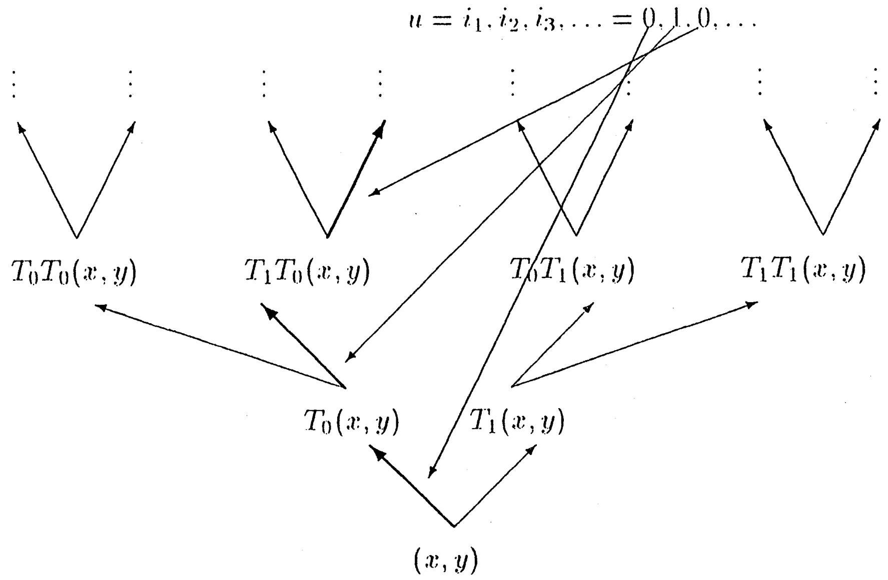
We define transformation T_1 on X as follows:

$$T_1(x,y) = \begin{cases} t_1^{-1}(x,y) & \text{if } (x,y) \in S_0^1, \\ t_0^{-1}(x,y) & \text{if } (x,y) \in S_1^1, \\ t_2^{-1} \circ t_1^{-1}(x,y) & \text{if } (x,y) \in S_2^1. \\ t_2^{-1} \circ t_1^{-1} \circ t_2^{-1}(x,y) & \text{if } (x,y) \in S_3^1, \\ t_2^{-1} \circ t_0^{-1} \circ t_2^{-1}(x,y) & \text{if } (x,y) \in S_4^1, \\ t_1^{-1} \circ t_2^{-1}(x,y) & \text{if } (x,y) \in S_5^1. \end{cases}$$

For $(x, y) \in X$ we consider the following binary tree:



We associate $u = \{i_1, i_2, \dots\} \in \{0,1\}^{\mathbb{N}}$ with a path in the tree like the following example:

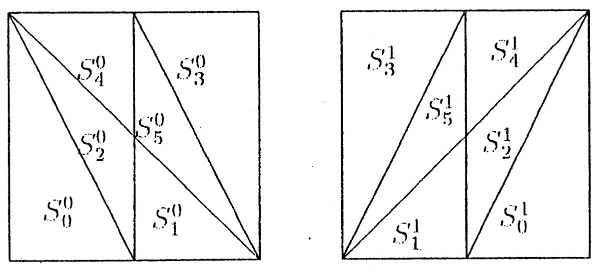


For $u = \{i_1, i_2, \dots\} \in \{0,1\}^{\mathbb{N}}$ and a positive integer n, we define $g(u, n, (x,y)) \in X$ by

$$g(u, n, (x,y)) = T_{i_n} \cdots T_{i_1}(x,y).$$

We define a sequence $S(u,(x,y)) = \{j_n\}_{n=1}^{\infty} \in \{0,1,2,3,4,5\}^{\mathbb{N}}$ which is called the name of (x,y) related to $u \in \{0,1\}^{\mathbb{N}}$ as follows: for $n = 1, 2, \dots$

$$g(u, n-1, (x,y)) \in S_{j_n}^{i_n}.$$



Figures of S_j^i $(0 \leqq j \leqq 5,\ i \in \{0,1\})$

$u = \{i_1, i_2, \dots\} \in \{0,1\}^{\mathbb{N}}$ is called good related to (x,y) if there exists infinitely many positive integers k such that $i_k = i_{k+1}$ and j_k and j_{k+1} satisfy one of following two conditons (1) and (2):

(1) $j_k = 1$ and $j_{k+1} \in \{0, 2\}$,

(2) $j_k = 4$ and $j_{k+1} \in \{3, 5\}$,.

where $\{j_1, j_2, \dots\}$ is the name of (x, y) related to u.

We define substitutions s_0, s_1, e on L by

$$s_0 : \begin{cases} 0 \to 0 \\ 1 \to 01 \end{cases}, \qquad s_1 : \begin{cases} 0 \to 01 \\ 1 \to 1 \end{cases}, \qquad e : \begin{cases} 0 \to 1 \\ 1 \to 0 \end{cases}.$$

For $(i, k) \in \{0, 1\} \times \{0, 1, 2, 3, 4, 5\}$, we define substitutions $\phi(i, k)$ as follows:

$$\phi(i,j) = \begin{cases} s_0 & \text{if } (i,j) = (0,0), \\ s_1 & \text{if } (i,j) = (0,1), \\ s_0 e & \text{if } (i,j) = (0,2). \\ e s_0 e & \text{if } (i,j) = (0,3), \\ e s_1 e & \text{if } (i,j) = (0,4), \\ e s_0 & \text{if } (i,j) = (0,5). \\ s_1 & \text{if } (i,j) = (1,0), \\ s_0 & \text{if } (i,j) = (1,1), \\ s_1 e & \text{if } (i,j) = (1,2). \\ e s_1 e & \text{if } (i,j) = (1,3), \\ e s_0 e & \text{if } (i,j) = (1,4), \\ e s_1 & \text{if } (i,j) = (1,5). \end{cases}$$

By the theory [1] we have the following important Lemma:

Lemma

$$\phi(i_1, j_1) \cdots \phi(i_n, j_n) G(g(u, n, (x, y))) = G(x, y).$$

For substitutions f and g on W^* we say that f is equivalent to f, if for any $w \in W$ $|f(w)| = |g(w)|$.

We have the following Theorem 2.

Theorem 2 *Let $(x, y) \in X$. Let $u = \{i_1, i_2, \dots\} \in \{0, 1\}^{\mathbb{N}}$ be a good path in the previous tree related to (x, y). Let I be a finite union of intervals $[\langle m_1\alpha - \beta\rangle, \langle m_2\alpha - \beta\rangle)^\sim$ $(m_1, m_2 \in \mathbb{Z})$. Then, there exist a integer $k \geq 0$ and a substitution ψ on W^* which is equivalent to $\phi(i_1, j_1) \cdots \phi(i_k, j_k)$ such that*

$$\chi(I, x) = \psi(G(g(u, k, (x, y)))),$$

where $\{j_1, j_2, \dots\}$ is the name of (x, y) related to u. The converse also holds.

From Theorem 2 and considering homogeneous cases $(y = mx + n)$, we get Theorem 1.

References

[1] S.Ito and S.Yasutomi, On continued fraction, substitution and characteristic sequences $[nx + y] - [(n-1)x + y]$,Japan. J. Math.16(1990),287-306.

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[3] E.M.Coven and G.A.Hedlund, Sequences with minimal block growth. Math. systems theory 7(1973),138-153

Present Address:
GENERAL EDUCATION, SUZUKA NATIONAL COLLEGE OF TECHNOLOGY,
SHIROKO, SUZUKA, MIE, 510-0294, JAPAN
E-mail:yasutomi@genl.suzuka-ct.ac.jp