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The application of the combination of circle method and sieve methed

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§1. Circle method

In 1920's, Hardy and Littlewood introduced a new method in their serial papers in the name **Some problems of "Partitio Numerorum"**.

Let $\{a_m\}$ denote a strictly increasing sequence of non-negative integers and

$$F(z)=\sum_{m=1}^{\infty} z^{a_m}, \qquad |z|<1.$$

Then

$$F^s(z)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_s=1}^{\infty} z^{a_{m_1}+\cdots+a_{m_s}}=\sum_{n=0}^{\infty} R_s(n)z^n,$$

where $R_s(n)$ is the number of solutions of the equation

$$n=a_{m_1}+\cdots+a_{m_s}. \tag{1}$$

By Cauchy's integral theorem,

$$R_s(n)=\frac{1}{2\pi i}\int_C F^s(z)z^{-n-1}\,dz,$$

where C is a circle of radius ρ $(0<\rho<1)$, the centre of which is at origin. So this method is called circle method.

In 1928, I. M. Vinogradov refined the circle method. He replaced the infinite series $F(z)$ by a finite sum. In 1937, he was successful to use circle method in solving Goldbach Conjecture on odd integer. His result is called three primes theorem.

§2. Three primes theorem

Vinogradov used the relation

$$\int_0^1 e(m\alpha)\,d\alpha=\begin{cases}1, & m=0\\ 0, & m\neq 0\end{cases}$$

to express $T(N)$ which is the number of solutions of the equation

$$N = p_1 + p_2 + p_3 \tag{2}$$

as the integral

$$T(N) = \int_0^1 \Big(\sum_{p\le N} e(p\alpha)\Big)^3 e(-N\alpha)\,d\alpha, \tag{3}$$

where N is an odd integer, p_i $(i = 1,2,3)$ is the prime number, $e(x) = e^{2\pi i x}$.

Then he divided the interval $[0,1]$ into two parts

$$E_1 = \{\alpha : |\alpha - \frac{a}{q}| \le N^{-1}\log^B N,\ (a,q) = 1,\ q \le \log^B N\},$$
$$E_2 = [0,1] - E_1,$$

where B is a large positive constant.

If $\alpha \in E_1$, he applied the prime number theorem in the arithmetic progression to get an asymptotic formula for the trigonometric sum. Hence,

$$\int_{E_1} \Big(\sum_{p\le N} e(p\alpha)\Big)^3 e(-N\alpha)\,d\alpha \sim C(N)\cdot\frac{N^2}{\log^3 N}, \tag{4}$$

where $C(N)$ is greater than a positive constant.

If $\alpha \in E_2$, then $|\alpha - \frac{a}{q}| \le \frac{1}{q^2}$, where $q > \log^B N$. Using his original idea, Vinogradov transformed the trigonometric sum on prime variable

$$\sum_p e(p\alpha) \tag{5}$$

into bilinear form

$$\sum_m \sum_l a(m)b(l)e(ml\alpha)$$

so that he could get a non-trivial estimate

$$\sum_{p\le N} e(p\alpha) \ll \frac{N}{\log^8 N}.$$

By this estimate, we have

$$\int_{E_2} \Big(\sum_{p\le N} e(p\alpha)\Big)^3 e(-N\alpha)\,d\alpha \ll \frac{N}{\log^8 N}\int_0^1 \Big|\sum_{p\le N} e(p\alpha)\Big|^2 d\alpha \ll \frac{N^2}{\log^8 N}. \tag{6}$$

Hence,

$$T(N) \sim C(N)\cdot\frac{N^2}{\log^3 N},$$

which means that for sufficiently large odd integer N, three primes theorem holds .

§3. Sieve method

If we use circle method to study Goldbach Conjecture on even integer, we should deal with the integral

$$T_1(N) = \sum_{N=p_1+p_2} 1 = \int_0^1 \Big(\sum_{p\le N} e(p\alpha)\Big)^2 e(-N\alpha)\,d\alpha.$$

By the discussion in the formula (6), it is easy to see that we can not get a good estimate for the integral on E_2. So we have to find other way. Now the most powerful method to explore binary problems is sieve method.

Sieve method can be traced back to at least two thousands years ago. But only after 1920's, it was remade to be a powerful theoretic method. Brun, Rosser and Selberg made great contribution to sieve method.

Let $\mathcal{A} = \{a \le N : a = N - p_1,\ p_1 \text{ is the prime number}\}$. We define the sieve function

$$S(\mathcal{A}, z) = \#\{n \in \mathcal{A},\ \text{the prime factors of n} > z\}.$$

It is easy to see

$$T_1(N) = S(\mathcal{A}, N^{\frac{1}{2}}) + O(N^{\frac{1}{2}}).$$

Our purpose is to prove

$$S(\mathcal{A}, N^{\frac{1}{2}}) > c_0 \cdot \frac{N}{\log^2 N}, \tag{7}$$

where c_0 is a positive constant.

By Buchstab's identity,

$$S(\mathcal{A}, N^{\frac{1}{2}}) = S(\mathcal{A}, z) - \sum_{z<p\le N^{\frac{1}{2}}} S(\mathcal{A}_p, p). \tag{8}$$

We shall apply two kinds of estimate.

One is the estimate for the lower bound

$$S(\mathcal{A}, z) \ge f(\frac{\log D}{\log z}) \cdot C_1(N) \cdot \frac{N}{\log^2 N} - R_{-}(D), \tag{9}$$

where $f(u)$ is a function having good property, $C_1(N)$ is greater than a positive constant, $R_{-}(D)$ is the error term.

The key point in the sieve method is to estimate the error term. In the present case, the error term depends on the distribution of prime numbers in the arithmetic progression on average. So by the estimate for the zero density of L function, we can get an estimate for the error term.

Similarly we have the estimate for the upper bound

$$S(\mathcal{A}, z) \le F(\frac{\log D}{\log z}) \cdot C_1(N) \cdot \frac{N}{\log^2 N} + R_{+}(D). \tag{10}$$

We apply the above procedure repeatedly, but we can not collect enough numerals to produce a positive constant c_0 in the formula (7). So we have to consider a weeker problem on the expression

$$N = P_2 + p,$$

where P_2 has at most two prime factors, p is a prime. Jingrun Chen was successful to solve this problem.

§4. **Combination of circle method and sieve method**

On the modern development of three primes theorem, one of most important problems is three primes theorem in the short interval. It states that the equation

$$\begin{gathered} N = p_1 + p_2 + p_3, \\ \frac{N}{3} - A < p_i \leq \frac{N}{3} + A, \quad i = 1, 2, 3 \end{gathered} \tag{11}$$

has solutions, where N is an odd integer, $A = N^{\varphi+\varepsilon}$.

We hope that for small exponent φ, the equation (11) has solutions. Wolke showed that on Generalized Riemann Hypothesis, for $\varphi = \frac{1}{2}$, the equation (11) has solutions.

Early in 1950's and 1960's, Haselgrove, C. D. Pan and Jingrun Chen went in for studying this problem. Chen got the exponent $\varphi = \frac{2}{3}$. But in the works of Pan and Chen, there is a fatal error. Up to 1987, C. D. Pan and C. B. Pan corrected this error and got $\varphi = \frac{91}{96}$.

Now in this problem, we should divide the interval $[0, 1]$ into three parts

$$\begin{aligned} E_1 &= \{\alpha : |\alpha - \frac{a}{q}| \leq A^{-1} \log^B N, \ (a, q) = 1, \ q \leq \log^B N\}, \\ E_3 &= \{\alpha : A^{-1} \log^B N < |\alpha - \frac{a}{q}| \leq A^{-2} N, \ (a, q) = 1, \ q \leq \log^B N\}, \\ E_2 &= [0, 1] - E_1 - E_3, \end{aligned} \tag{12}$$

where B is a large positive constant.

On E_1, we still apply the prime number theorem in the arithmetic progression to get an asymptotic formula. On E_2, we use Vinogradov's estimate for the trigonometric sum. On E_3, we should transform the trigonometric sum on prime variable into the zero density of L function in the short interval. The error of Pan and Chen is to transform trigonometric sum into the zero density of L function in the long interval.

In 1988, by the new estimate for the zero density of L function in the short interval, I got $\varphi = \frac{2}{3}$. So we can recover Chen's result. But limited by only application of Vinogradov's estimate for trigonometric sum in the circle method, we can only achieve $\varphi = \frac{2}{3}$. Therefore we should find new way to decrease the exponent φ.

One way is to apply the sieve method to the variable p_1 in the eqation (11). We can define the corresponding set $\mathcal{A}$ and the sieve function. Similar to the binary problem, we can deal with the error term of sieve method by the estimate for the distribution of prime numbers in the arithmetic progression on average.

Now we have extra variable so that we can proceed in some average form. We can get a better estimate for the error term in the sieve method. On the binary problem, when applying Buchstab's identity (8), we can only get upper bound and lower bound. On the ternary problem, we can get asymptotic formula for the sieve function in some range. This advantage results in a positive lower bound similar to that in the formula (7). I achieved the exponent $\varphi = 0.636$.

The other way was found by Zhan. He employed only cicle method. But in the estimate on E_2, he replaced Vinogradov's method by an analytic method. By this analytic method, one could apply the estimate for the fourth power of mean value of L function. Now Jutila's result plays an important role, which is a generalization of Iwaniec's result

$$\int_T^{T+T^{\frac{2}{3}}} |\zeta(\frac{1}{2}+it)|^4\,dt \ll T^{\frac{2}{3}+\varepsilon} \tag{13}$$

to L function. This result depends on the theory of modular form. Zhan's exponent is $\varphi = \frac{5}{8}$.

Then I combined the methods of Zhan and mine. In my observation, on the estimate for the error term in the sieve method, we should deal with such equations as

$$\begin{gathered} N = m_1 m_2 l + p_2 + p_3, \\ \frac{N}{3} - A < m_1 m_2 l \le \frac{N}{3} + A, \ \frac{N}{3} - A < p_i \le \frac{N}{3} + A, \quad i = 2, 3, \end{gathered} \tag{14}$$

where m_1, m_2 satisfy some conditions.

Now we still apply the circle method. The division (12) is kept. We shall estimate the trigonometric sum

$$\sum_{m_1, m_2, l} a(m_1) b(m_2) e(m_1 m_2 l \alpha)$$

which is more flexible than the trigonometric sum on prime variable. We can deal with it by the estimates for the mean value of Dirichlet's polynomials and for the fourth power of mean value of L function. Of course these estimates are in the short interval. Jutila's result still works.

We can get some asymptotic formulas in the sieve method, which depend on the estimate for the weighted zero density of L function in the short interval. Some ideas of Heath-Brown, Iwaniec and Pintz on the distribution of prime numbers in the short interval $(x, x+y)$ are employed. Repeating Buchstab's identity, we can get a positive lower bound.

At last we achieve the combination of circle method and sieve method. If only using the circle method, we can get an asymptotic formula for the number of solutions. If the sieve method is involved, the asymptotic formula can not be obtained. But the application of sieve method can evade some difficult parts on the estimate although the method gets complicated. In this way, I got the exponent $\varphi = 0.6$.

Afterwards I refined the method to get $\varphi = \frac{23}{39}$ in 1991 and $\varphi = \frac{7}{12}$ in 1994. In a manuscript, Mikawa also proved $\varphi = \frac{7}{12}$. In 1998, Baker and Harman proved $\varphi = \frac{4}{7}$.

There are other examples for the application of the combination of circle method and sieve method.

§5. Exceptional set of Goldbach numbers in the short interval

An even integer which can be written as a sum of two primes is called a Goldbach number. In 1938, Hua and other people used Vinogradov's method on three primes theorem to prove that almost all even integers are Goldbach numbers. Here 'almost all' means that for $2n \leq N$, the exceptional numbers are $o(N)$.

The modern development is the study on the exceptional set of Goldbach numbers in the short interval. In 1973, Ramachandra proved that in the short interval $(N, N + N^{0.6+\varepsilon})$, almost all even integers are Goldbach numbers. He used circle method and the estimate for the zero density of L function. The zero density of L function is similar to that of ζ function which is used by Montgomery in the problem of the distribution of prime numbers in the short interval $(x, x + x^{0.6})$. Using Huxley's estimate for the zero density of L function, we can get the exponent $\frac{7}{12}$.

In 1991, by the application of circle method and sieve method, I got the exponent $\frac{23}{42}$. The frame of sieve method here was adopted from the work of Iwaniec and Pintz on the distribution of prime numbers in the short interval $(x, x + x^{\frac{23}{42}})$.

In 1993, Perelli and Pintz made great progress on circle method. They got the exponent $\frac{7}{36}$. They used the circle method and the estimate for the zero density of L function. But now the zero density of L function is similar to that of ζ function which is used in the problem of the distribution of prime numbers in almost all short interval $(x, x + x^{\frac{1}{6}})$.

Mikawa got the exponent $\frac{7}{48}$ by the combination of the circle method and sieve method. In 1994, I got the exponent $\frac{7}{78}$. I used the combination of sieve method with circle method again. Then there were some improvements such as the exponent $\frac{7}{81}$ of Li and $\frac{7}{84}$ of mine. In 1996, I got the exponent $\frac{7}{108}$ which corresponds to the exponent $\frac{1}{18}$ in the distribution of prime numbers in almost all short interval. On the later problem the last exponent is $\frac{1}{20}$. The reason is that the results on L functions are not so good as that on ζ function.

§6. Piatetski-Shapiro–Vinogradov theorem

One interesting problem is to ask whether three primes theorem is still true or not for the primes belonging to a thin set. An important thin subset of prime numbers is of form $p = [n^c]$, where $c > 1$, n is a positive integer and $[x]$ denotes the integral part of x. Correspondingly, there is prime number theorem

$$\sum_{\substack{p \le x \\ p=[n^c]}} 1 \sim \frac{x^{\frac{1}{c}}}{\log x} \tag{15}$$

which holds for suitable range of c.

In 1992, Balog and Friedlander proved for $1 \le c < \frac{21}{20}$, there is an expression

$$N = p_1 + p_2 + p_3, \quad p_i = [n_i^c], \quad i = 1, 2, 3, \tag{16}$$

where N is a sufficiently large odd integer. This means that three primes theorem holds for the thin set.

They used the circle method. Then Rivat extended the range of c to $1 \le c < \frac{199}{188}$. In 1995, I used the combination of circle method and sieve method to extend the range of c to $1 \le c < \frac{16}{15}$.