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A unified viewpoint about geometric objects in hyperbolic space and the generalized tilt formula

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Abstract

ローレンツ空間内の任意の点に対してアフィン超平面を定義します。この超平面が二葉双曲面の上葉、即ち双曲空間と交わる時には、点や球面、測地的超平面、等距離超曲面、ホロ球面といった双曲空間の基本的な幾何学的対象物が、超平面を定義する点の位置に応じて生じます。

この対応を使う事により、J. R. Weeks が定義した tilt を重み付き単体に対して拡張し、その性質及び具体的な求め方を与えます。

Key words: tilt formula, canonical decomposition, convex hull construction, simplex, hyperbolic geometry.

1991 Mathematics Subject Classifications: Primary: 51M10; secondary: 51M09, 57Q15.

1 Introduction

This paper is a summary of [Us3].

D. B. A. Epstein and R. C. Penner gave in [EP] a method for decomposing any noncompact complete hyperbolic manifold of finite volume with *weight* at each cusp into ideal polyhedra. This decomposition is called the *Euclidean decomposition*, and defined via a *convex hull construction* in Lorentzian space. Each vertex of the hull is in the positive light cone and corresponds to a lift of a cusp, and each face of the hull corresponds to an ideal polyhedron in the

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manifold. Especially if all weights are equal, then the decomposition is called the *canonical decomposition.*

For a simplex in Lorentzian space whose vertices are in the positive light cone, J. R. Weeks defined in [We1] the *tilt* relative to each of its faces. It gives an efficient tool for deciding whether or not the dihedral angle between two simplices holding a face in common is convex. So it becomes a useful tool to determine whether or not a given decomposition of the manifold is obtained from the convex hull. He also provided an efficient formula, called the *tilt formula*, to obtain tilts from intrinsic geometry of the simplex when its dimension is two or three. Using this formula, he made the hyperbolic structures computation program "SnapPea" (cf. [We2]). Then M. Sakuma and J. R. Weeks generalized the tilt formula to general dimensions in [SW] .

S. Kojima gave in [Ko1, Ko2] a method for decomposing any complete hyperbolic manifold of finite volume with non-empty totally geodesic boundary into partially truncated polyhedra. In many cases each polyhedron is a partially truncated simplex. Since such a simplex is lifted to a simplex in Lorentzian space whose vertices may not be in the positive light cone, it is meaningful to generalize the concept of the tilt and to establish the tilt formula for the generalized tilt. The main purpose of the paper is to do it (see Theorem 4.4 and Corollary 4.5).

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2 Lorentzian space and hyperbolic geometry

2.1 Basic facts on Lorentzian space model

The $n+1$-*dimensional Lorentzian space* (or simply *Lorentzian* $n+1$-*space*) $\mathbf{E}^{1,n}$ is the real vector space $\mathbf{R}^{n+1}$ of dimension $n+1$ with the *Lorentzian inner product* $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n$, where $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_0, y_1, \ldots, y_n)$. Throughout this paper, we assume $n \geq 2$. The *Lorentzian norm* of $\boldsymbol{x}$ in $\mathbf{E}^{1,n}$ is defined to be the complex number $\sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. If the Lorentzian norm of $\boldsymbol{x}$ is zero (resp. positive, imaginary), then $\boldsymbol{x}$ is said to be *light-like* (resp. *space-like*, *time-like*). The coordinate x_0 of $\mathbf{E}^{1,n}$ is called the *height*. Now we define six connected subsets in $\mathbf{E}^{1,n}$ as follows: the set of time-like vectors with positive height is $T^+ := \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0 \text{ and } x_0 > 0 \}$, the set of time-like vectors with negative height is $T^- := \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0 \text{ and } x_0 < 0 \}$, the set of light-like vectors is $L := \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \}$, the set of light-like vectors with positive height is $L^+ := \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \text{ and } x_0 > 0 \}$ $(\subset L)$, the set of light-like vectors with negative height is $L^- := \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \text{ and } x_0 < 0 \}$ $(\subset L)$, and the set of space-like vectors is $S := \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0 \}$. Then $\mathbf{E}^{1,n}$ is disjointly divided as follows: $\mathbf{E}^{1,n} = T^+ \sqcup T^- \sqcup L^+ \sqcup \{\boldsymbol{o}\} \sqcup L^- \sqcup S$, where $\boldsymbol{o}$ is the origin $(0, 0, \ldots, 0)$ of $\mathbf{E}^{1,n}$, and $\cdot \sqcup \cdot$ means the disjoint union of sets. We call T^+ the *future cone*, T^- the *past cone*, L the *light cone*, L^+ the *positive light cone*, L^- the *negative light cone*, and S the *side cone*. For any $\boldsymbol{x} \in \mathbf{E}^{1,n}$ with $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \neq 0$, we denote by $n(\boldsymbol{x})$ its normalized vector, that is, $n(\boldsymbol{x}) := \frac{\boldsymbol{x}}{\sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}}$.

Let $H_T^+ := \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1 \text{ and } x_0 > 0 \}$ be the upper sheet of the (standard) hyperboloid of two sheets. The restriction of the quadratic form induced by $\langle \cdot, \cdot \rangle$ on $\mathbf{E}^{1,n}$ to the tangent space of H_T^+ is positive definite and gives a Riemannian metric on H_T^+. The space obtained from H_T^+ equipped with the metric above is called the *hyperboloid model* of the n-dimensional hyperbolic space, and we denote it by $\mathbf{H}^n$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are points in H_T^+ and d denotes the hyperbolic distance between $\boldsymbol{x}$ and $\boldsymbol{y}$, then the following relation holds (see [Na, p. 45], [Ra, (3.2.2)] or [Th, Proposition 2.4.5(a)]):

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\cosh d. \tag{2.1}$$

A ray in L^+ started from the origin $\boldsymbol{o}$ corresponds to a point in the ideal boundary of $\mathbf{H}^n$. The set of such rays forms the sphere at infinity, and we denote it by S_∞^{n-1}. Then each ray in L^+ becomes a point at infinity of $\mathbf{H}^n$. The (standard) *hyperboloid of one sheet* H_S is defined to be $H_S := \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \}$.

Let us denote by $\mathcal{P}$ the radial projection from $\mathbf{E}^{1,n} - \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid x_0 = 0 \}$ to an affine hyperplane $\mathbf{P}_1^n := \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid x_0 = 1 \}$ along the ray from the origin $\boldsymbol{o}$. The projection $\mathcal{P}$ is a homeomorphism on $\mathbf{H}^n$ to the n-dimensional open unit ball $\mathbf{B}^n$ in $\mathbf{P}_1^n$ centered at the origin $\boldsymbol{i} := (1, 0, 0, \ldots, 0)$ of $\mathbf{P}_1^n$, which gives the *projective model* of $\mathbf{H}^n$. The affine hyperplane $\mathbf{P}_1^n$ contains not only $\mathbf{B}^n$ and its set theoretic boundary $\partial\mathbf{B}^n$ in $\mathbf{P}_1^n$, which is canonically identified with S_∞^{n-1}, but also the outside of the compactified projective model $\overline{\mathbf{B}^n} := \mathbf{B}^n \sqcup \partial\mathbf{B}^n \approx \mathbf{H}^n \sqcup S_\infty^{n-1}$. In this identification, the points near the intersection $S \cap \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid x_0 = 0 \}$ are mapped to an end of $\mathbf{P}_1^n$. So we can naturally extend $\mathcal{P}$ to the mapping from $\mathbf{E}^{1,n} - \{\boldsymbol{o}\}$ to the n-dimensional real projective space $\mathbf{P}^n := \mathbf{P}_1^n \sqcup \mathbf{P}_\infty^n$, where $\mathbf{P}_\infty^n$ is the set of lines in the affine hyperplane $\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid x_0 = 0 \}$ through $\boldsymbol{o}$. But we use the notation $\mathcal{P}$ for the mapping obtained as above to save letters since there would be no confusion. We denote by $\operatorname{Ext}\overline{\mathbf{B}^n}$ the exterior of $\overline{\mathbf{B}^n}$ in $\mathbf{P}^n$.

We call an affine hyperplane in $\mathbf{E}^{1,n}$ through the origin a *linear* hyperplane. A vector subspace of $\mathbf{E}^{1,n}$ is said to be *time-like* if it has a time-like vector, *space-like* if every nonzero vector in it is space-like, or *light-like* otherwise. Suppose P is a time-like linear hyperplane, and let R be a half-space in $\mathbf{E}^{1,n}$ bounded by P. Then we can associate a unique vector $\boldsymbol{w} \in H_S$ so that $\langle \boldsymbol{w}, \boldsymbol{q} \rangle \leq 0$ for any $\boldsymbol{q} \in R$. This establishes a well-known duality between points on H_S and half-spaces in $\mathbf{E}^{1,n}$ bounded by time-like linear hyperplanes. Now we give an generalization of this duality. For an arbitrary vector $\boldsymbol{u}$ in $\mathbf{E}^{1,n}$, we define a half-space $R_{\boldsymbol{u}}$ and a hyperplane $P_{\boldsymbol{u}}$ in $\mathbf{E}^{1,n}$ as follows:

$$R_{\boldsymbol{u}} := \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{u} \rangle \leq \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle - 1}{2} \right\},$$

$$P_{\boldsymbol{u}} := \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{u} \rangle = \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle - 1}{2} \right\} = \partial R_{\boldsymbol{u}}.$$

We denote by $\Gamma_{\boldsymbol{u}}$ (resp. $\Pi_{\boldsymbol{u}}$) the intersection of $R_{\boldsymbol{u}}$ (resp. $P_{\boldsymbol{u}}$) and H_T^+. We call $\boldsymbol{u}$ a *normal vector* to $P_{\boldsymbol{u}}$ (or $\Pi_{\boldsymbol{u}}$).

By the definition, a hyperplane $P_{\boldsymbol{x}}$ is linear if and only if $\boldsymbol{x} \in H_S$. Then $\Pi_{\boldsymbol{x}}$ is a geodesic hyperplane in $\mathbf{H}^n$. Let $\boldsymbol{y}$ be a point in $\mathbf{H}^n$, and we denote by d the signed distance between $\Pi_{\boldsymbol{x}}$ and $\boldsymbol{y}$, that is, the hyperbolic distance (in the usual sense) of $\Pi_{\boldsymbol{x}}$ and $\boldsymbol{y}$ with signature positive (resp. negative) if $\boldsymbol{y} \in \Gamma_{\boldsymbol{x}}$ (resp. $\boldsymbol{y} \notin \Gamma_{\boldsymbol{x}}$), that is, if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 0$ (resp. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle > 0$). Then there is a following well-known relationship between $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ and d (see, for example, [Ra, Theorem 3.2.12]):

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\sinh d. \tag{2.2}$$

For two different geodesic hyperplanes in $\mathbf{H}^n$, the following theorem is a well-known one:

Theorem 2.1 (see [Ra, Theorem 3.2.6, 3.2.7 and 3.2.9]) *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two points in H_S with $\boldsymbol{x} \neq \pm\boldsymbol{y}$, and we denote by N the vector subspace of $\mathbf{E}^{1,n}$ spanned by $\boldsymbol{x}$ and $\boldsymbol{y}$.*

(1) $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| < 1 \iff$ *N is space-like*
$\iff$ *$\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ intersect in H_T^+.*

(2) $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| > 1 \iff$ *N is time-like*
$\iff$ *$\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ are disjoint, and $N \cap H_T^+$ is a unique common orthogonal geodesic line to $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$.*

(3) $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| = 1 \iff$ *N is light-like*
$\iff$ *$P_{\boldsymbol{x}} \cap P_{\boldsymbol{y}}$ is light-like. So $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ meet at infinity.* □

For two geodesic hyperplanes $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ in $\mathbf{H}^n$ (so $\boldsymbol{x}$, $\boldsymbol{y} \in H_S$), we call $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ are *ultraparallel* if the condition of Theorem 2.1(2) holds, and *parallel* if the condition of Theorem 2.1(3) holds. Next we suppose $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ intersect, that is, the condition of Theorem 2.1(1) holds. Then we have the following relation (see [Th, Proposition 2.4.5(c)] and [SW, Lemma 2.7]):

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\cos\theta, \tag{2.3}$$

where θ is the dihedral angle between $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ which is measured in $\Gamma_{\boldsymbol{x}} \cap \Gamma_{\boldsymbol{y}}$. We note that this relation holds even if $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$ are parallel. In this case we regard θ as 0.

For an arbitrary point $\boldsymbol{u}$ in H_S, $P_{\boldsymbol{u}} \cap \mathbf{P}^n$ becomes a hyperplane in $\mathbf{P}^n$, moreover $P_{\boldsymbol{u}}$ intersects $\mathbf{B}^n$. Since $\mathcal{P}(\boldsymbol{u})$ is a point in $\operatorname{Ext} \overline{\mathbf{B}^n}$, the cone consisting of lines through $\mathcal{P}(\boldsymbol{u})$ and a point in $P_{\boldsymbol{u}} \cap \partial \mathbf{B}^n$ is tangent to $\partial \mathbf{B}^n$. We call $P_{\boldsymbol{u}} \cap \mathbf{P}^n$ the *polar hyperplane* of $\mathcal{P}(\boldsymbol{u})$ in $\mathbf{P}^n$, and $\mathcal{P}(\boldsymbol{u})$ the *pole* of $P_{\boldsymbol{u}} \cap \mathbf{P}^n$ (see, for example, [Ke, p. 544]). For an arbitrary point $\boldsymbol{v}$ in $\operatorname{Ext} \overline{\mathbf{B}^n}$, we denote by $\Omega(\boldsymbol{v})$ its polar hyperplane and by $\Psi(\boldsymbol{v})$ the hyperplane in $\mathbf{B}^n$ with pole $\boldsymbol{v}$, i.e., $\Psi(\boldsymbol{v}) := \Omega(\boldsymbol{v}) \cap \mathbf{B}^n$.

2.2 What is $\Pi_{\boldsymbol{u}}$?

In this subsection we classify $\Pi_{\boldsymbol{u}}$ with respect to the position of $\boldsymbol{u}$. We first note that, if $\boldsymbol{u}$ is the origin of $\mathbf{E}^{1,n}$, then $P_{\boldsymbol{u}}$ is an empty set, so is $\Pi_{\boldsymbol{u}}$.

Case 1. Suppose $\boldsymbol{u}$ is a time-like vector, i.e., $\boldsymbol{u} \in \left\{\boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0\right\}$. Then, since $-\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$, we can rewrite the definition of $P_{\boldsymbol{u}}$ as follows:

$$P_{\boldsymbol{u}} = \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \left\langle \boldsymbol{x}, \frac{\boldsymbol{u}}{\sqrt{-\langle \boldsymbol{u}, \boldsymbol{u} \rangle}} \right\rangle = \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle - 1}{2\sqrt{-\langle \boldsymbol{u}, \boldsymbol{u} \rangle}} \right\}.$$

Now we can easily check that the right side of the definition is less than -1. So, for $\Pi_{\boldsymbol{u}}$ being non-empty, the height of $\boldsymbol{u}$ must be positive. Then, by equation (2.1), $\Pi_{\boldsymbol{u}}$ is the set of points in the hyperbolic space each of which is $|\log(-\langle \boldsymbol{u}, \boldsymbol{u} \rangle)|/2$ away from $n(\boldsymbol{u}) = \boldsymbol{u}/\sqrt{-\langle \boldsymbol{u}, \boldsymbol{u} \rangle}$, which means that $\Pi_{\boldsymbol{u}}$ is the sphere of radius $|\log(-\langle \boldsymbol{u}, \boldsymbol{u} \rangle)|/2$ with center $n(\boldsymbol{u})$. We here note that $\Pi_{\boldsymbol{u}} = \{\boldsymbol{u}\}$ if and only if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = -1$.

Case 2. Suppose $\boldsymbol{u}$ is a space-like vector, i.e., $\boldsymbol{u} \in S = \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0 \right\}$. In this case we can rewrite the definition of $P_{\boldsymbol{u}}$ as follows:

$$P_{\boldsymbol{u}} = \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \left\langle \boldsymbol{x}, \frac{\boldsymbol{u}}{\sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle}} \right\rangle = \frac{\langle \boldsymbol{u}, \boldsymbol{u} \rangle - 1}{2\sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle}} \right\}.$$

By equation (2.2), $\Pi_{\boldsymbol{u}}$ is the set of points in the hyperbolic space each of which is $|\log\langle \boldsymbol{u}, \boldsymbol{u} \rangle|/2$ away from the geodesic hyperplane $\Pi_{n(\boldsymbol{u})}$. We call such a hypersurface $\Pi_{\boldsymbol{u}}$ an *equidistant hypersurface*, and $\Pi_{n(\boldsymbol{u})}$ the *axial hyperplane* of $\Pi_{\boldsymbol{u}}$ (cf. [Fe, p. 39]). We here note that $\Pi_{\boldsymbol{u}}$ is geodesic if and only if $\boldsymbol{u} \in H_S = \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \right\}$.

Case 3. Suppose $\boldsymbol{u}$ is a light-like vector, i.e., $\boldsymbol{u} \in L = \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \right\}$. In this case we can rewrite the definition of $P_{\boldsymbol{u}}$ as follows:

$$P_{\boldsymbol{u}} = \left\{ \boldsymbol{x} \in \mathbf{E}^{1,n} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{u} \rangle = -\frac{1}{2} \right\}.$$

Since the right side of the definition is negative, for $\Pi_{\boldsymbol{u}}$ being non-empty, the height of $\boldsymbol{u}$ must be positive. Then the set $\Pi_{\boldsymbol{u}}$ is called a *horosphere* whose *center* is the ray through $\boldsymbol{u}$.

Summarizing previous discussions, we obtain the following proposition:

Proposition 2.2 *The correspondences between points in* $\mathbf{E}^{1,n}$ *and geometric behaviors of* $\Pi_{\boldsymbol{u}}$ *are as follows:*

$\boldsymbol{u}$	$\langle \boldsymbol{u}, \boldsymbol{u} \rangle$	u_0	$\Pi_{\boldsymbol{u}} = P_{\boldsymbol{u}} \cap H_T^+$
T^+ (H_T^+)	< 0 $(= -1)$	> 0	sphere (point)
L^+	$= 0$	> 0	horosphere
S (H_S)	> 0 $(= 1)$	—	equidistant hypersurface (geodesic hyperplane)
otherwise, i.e., $\boldsymbol{u} \in \{\boldsymbol{o}\} \sqcup T^- \sqcup L^-$			$\emptyset$

2.3 Widths

We next define the *width* of a point in $T^+ \sqcup L^+ \sqcup S$, and observe its relationship to the Lorentzian norm.

Definition 2.3 Let $\boldsymbol{u}$ be a point in $T^+ \sqcup L^+ \sqcup S$. Then the *width*, say $\delta_{\boldsymbol{u}}$, is defined as follows:

(1) If $\boldsymbol{u} \in T^+$, then $\delta_{\boldsymbol{u}}$ is the signed radius of $\Pi_{\boldsymbol{u}}$, where the sign is defined to be positive (resp. negative) if $|\langle \boldsymbol{u}, \boldsymbol{u} \rangle| \leq 1$ (resp. $|\langle \boldsymbol{u}, \boldsymbol{u} \rangle| \geq 1$).

(2) If $\boldsymbol{u} \in S$, then $\delta_{\boldsymbol{u}}$ is the signed distance between $\Pi_{\boldsymbol{u}}$ and $\Pi_{n(\boldsymbol{u})}$, where the sign is defined to be positive (resp. negative) if $|\langle \boldsymbol{u}, \boldsymbol{u} \rangle| \leq 1$ (resp. $|\langle \boldsymbol{u}, \boldsymbol{u} \rangle| \geq 1$).

(3) If $\boldsymbol{u} \in L^+$, then $\delta_{\boldsymbol{u}} := (-\log(\boldsymbol{u}, \boldsymbol{u}))/2$, where $(\cdot, \cdot)$ means the Euclidean inner product, that is, $(\boldsymbol{u}, \boldsymbol{u}) := u_0^2 + u_1^2 + \cdots + u_n^2$ if $\boldsymbol{u} = (u_0, u_1, \ldots, u_n)$.

The discussion in previous subsection implies the following proposition:

Proposition 2.4 *Suppose* $\boldsymbol{u} \in T^+ \sqcup L^+ \sqcup S$. *Then the following relation holds:*

$$\delta_{\boldsymbol{u}} = \begin{cases} -\dfrac{1}{2} \log |\langle \boldsymbol{u}, \boldsymbol{u} \rangle| & \textit{if} \quad \boldsymbol{u} \in T^+ \sqcup S, \\[2ex] -\dfrac{1}{2} \log (\boldsymbol{u}, \boldsymbol{u}) & \textit{if} \quad \boldsymbol{u} \in L^+. \end{cases} \qquad \square$$

3 Definition of a tilt

"Tilts" are defined on "faces" of "weighted" n-simplices in the projective model $\mathbf{B}^n$, and a "weighted" n-simplex is a "generalized" n-simplex with weights at each vertex. So in this section we first define generalized n-simplices in $\mathbf{B}^n$, secondly define weighted n-simplices, and finally define tilts.

3.1 Generalized n-simplices

The projective model $\mathbf{B}^n$ has the advantage that it enable us to describe polyhedra in $\mathbf{H}^n$ in terms of Euclidean terminology. For example, we can regard an ideal polyhedron in $\mathbf{H}^n$ as an Euclidean polyhedron in $\mathbf{P}^n_1$ whose vertices lie in $\partial\mathbf{B}^n$. Using this advantage, in this subsection we define generalized n-simplices in $\mathbf{B}^n$.

Let $V = \{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a set of independent points in $\mathbf{P}^n$, and let $V_{\text{in}} := \{\boldsymbol{v} \in V \mid \boldsymbol{v} \in \overline{\mathbf{B}^n}\}$ and $V_{\text{ex}} := \{\boldsymbol{v} \in V \mid \boldsymbol{v} \in \operatorname{Ext}\overline{\mathbf{B}^n}\} = V - V_{\text{in}}$. Without loss of generality, we may assume $V_{\text{ex}} = \{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ and $V_{\text{in}} = \{\boldsymbol{v}_{k+1}, \boldsymbol{v}_{k+2}, \ldots, \boldsymbol{v}_n\}$ for some $k \in \{-1, 0, 1, \ldots, n\}$, by changing indices if necessary. This notation means that $V_{\text{ex}} = \emptyset$ and $V_{\text{in}} = V$ when $k = -1$, and that $V_{\text{ex}} = V$ and $V_{\text{in}} = \emptyset$ when $k = n$. Now we suppose V satisfies the following two conditions:

Condition 1. If V_{ex} has more than one point, then for arbitrary different points $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ in V_{ex} hyperplanes $\Psi(\boldsymbol{v}_i)$ and $\Psi(\boldsymbol{v}_j)$ with poles $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ respectively do not intersect in $\mathbf{B}^n$.

Condition 2. The set V_{in} is wholly contained in one connected component of $\overline{\mathbf{B}^n} - \bigcup_{i=0}^{k} \Omega(\boldsymbol{v}_i)$.

We note that, when $k = -1$, Condition 2 means that $V \subset \overline{\mathbf{B}^n}$.

For each point $\boldsymbol{v}_i$ in V_{ex}, there is a unique point $\boldsymbol{v}'_i$ in H_S such that $\mathcal{P}(\boldsymbol{v}'_i) = \boldsymbol{v}_i$ and $V_{\text{in}} \subset R_{\boldsymbol{v}'_i}$. Let $|\boldsymbol{v}'_0\boldsymbol{v}'_1 \cdots \boldsymbol{v}'_k\boldsymbol{v}_{k+1}\boldsymbol{v}_{k+2} \cdots \boldsymbol{v}_n|$ be the affine simplex in $\mathbf{E}^{1,n}$ with vertex set $\{\boldsymbol{v}'_0, \boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_k, \boldsymbol{v}_{k+1}, \boldsymbol{v}_{k+2}, \ldots, \boldsymbol{v}_n\}$. Since the points in V are independent in $\mathbf{P}^n$, vectors $\{\boldsymbol{v}'_0, \boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_k, \boldsymbol{v}_{k+1}, \boldsymbol{v}_{k+2}, \ldots, \boldsymbol{v}_n\}$ are linearly independent in $\mathbf{E}^{1,n}$, namely the hyperplane through $n+1$-points $\boldsymbol{v}'_0$, $\boldsymbol{v}'_1, \ldots,$ $\boldsymbol{v}'_k$, $\boldsymbol{v}_{k+1}$, $\boldsymbol{v}_{k+2}, \ldots,$ $\boldsymbol{v}_n$ does not contain the origin $\boldsymbol{o}$. Thus we can define $\mathcal{P}(|\boldsymbol{v}'_0\boldsymbol{v}'_1 \cdots \boldsymbol{v}'_k\boldsymbol{v}_{k+1}\boldsymbol{v}_{k+2} \cdots \boldsymbol{v}_n|)$, an n-simplex in $\mathbf{P}^n$ with vertex set V, and denote it by $|\boldsymbol{v}_0\boldsymbol{v}_1 \cdots \boldsymbol{v}_n|$. We note that, if $V_{\text{ex}} = \emptyset$, $|\boldsymbol{v}_0\boldsymbol{v}_1 \cdots \boldsymbol{v}_n|$ is just the n-dimensional affine simplex in $\mathbf{P}^n_1 \approx \mathbf{R}^n$ with vertex set V.

Definition 3.1 Under the assumptions stated above, the *generalized n-simplex*

Δ_V *in* $\mathbf{B}^n$ *with vertex set* V is defined as follows:

$$\Delta_V := \begin{cases} \mathbf{B}^n \cap |v_0 v_1 \cdots v_n| & \text{if } \ V \subset \overline{\mathbf{B}^n}, \\ \mathbf{B}^n \cap |v_0 v_1 \cdots v_n| \cap \bigcap_{i=0}^{k} R_{v_i'} & \text{if } \ V \cap \operatorname{Ext} \overline{\mathbf{B}^n} \neq \emptyset \quad \text{(see Figure 1).} \end{cases}$$

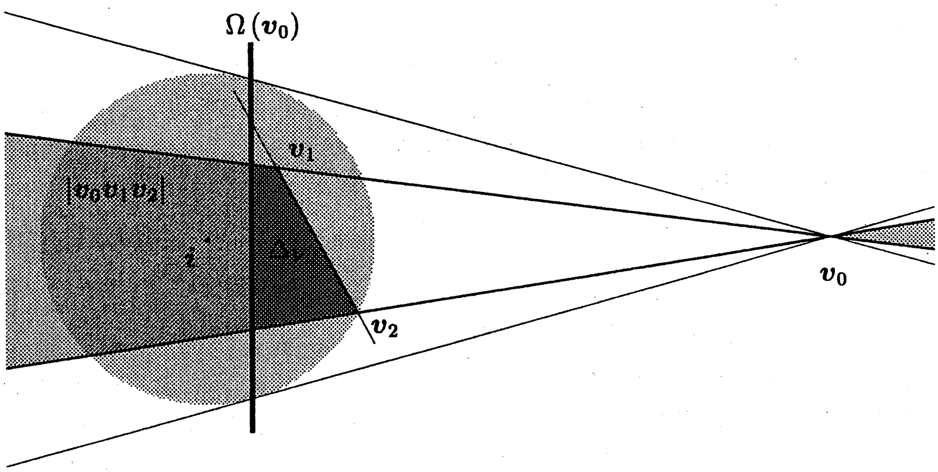

Figure 1: An example of a generalized 2-simplex in $\mathbf{B}^2$

Each face of Δ_V is either contained in a face of $|v_0 v_1 \cdots v_n|$ or in $\Psi(v_i)$ for some $v_i \in V_{\text{ex}}$. We call the former an *internal face* of Δ_V, and the later an *external face* of Δ_V (cf. [Ko1, Ko2]). For each vertex v_i of Δ_V, we denote by $\mathcal{F}_i$ the hyperplane in $\mathbf{P}^n$ through n points $\{v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$. If an internal face of Δ_V coincides with $\mathcal{F}_i \cap \Delta_V$ for some $v_i \in V$, then we call the face the *opposite face* of v_i, and denote it by Φ_i. By the definitions of the notation, we have an injective correspondence from the internal faces of Δ_V to the vertex set. We here note that this correspondence may not be surjective (see Figure 2). We may use the symbol of opposite faces to denote internal faces without referring to vertices. Let Φ_i and Φ_j be internal faces, and $\mathcal{F}_i$ and $\mathcal{F}_j$ their corresponding geodesic hyperplanes in the previous sense. Then we say that Φ_i and Φ_j (with $i \neq j$) are *parallel* (resp. *ultraparallel*, *intersecting*) if $\mathcal{P}^{-1}(\mathcal{F}_i) \cap H_T^+$ and $\mathcal{P}^{-1}(\mathcal{F}_j) \cap H_T^+$ are parallel (resp. ultraparallel, intersecting) (cf. Theorem 2.1). The *dihedral angle between* Φ_i *and* Φ_j is defined to be the dihedral angle between $\mathcal{P}^{-1}(\mathcal{F}_i) \cap H_T^+$ and $\mathcal{P}^{-1}(\mathcal{F}_j) \cap H_T^+$ measured in $\mathcal{P}^{-1}(\Delta_V) \cap H_T^+$. By Condition 1, we can see that each connected component of external faces is totally geodesic. We also note that a vertex of Δ_V as a polyhedron in hyperbolic space is not a "vertex" of the generalized n-simplex Δ_V if it is made from the intersection of an external face and an edge of $|v_0 v_1 \cdots v_n|$ (see Definition 3.1).

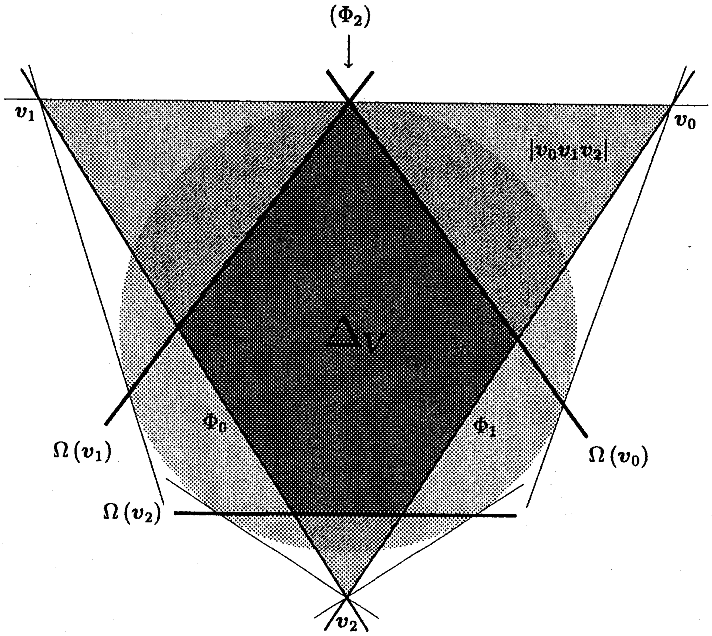

Figure 2: A generalized 2-simplex with one degenerate internal face

3.2 Weighted n-simplices

We recall that Δ_V is a generalized n-simplex with vertex set V. At each vertex, we give a real number called *weight*. Let W be the set of weights of all vertices. Then we call a triplet (Δ_V, V, W) a *weighted n-simplex in* $\mathbf{B}^n$. Now Definition 2.3 imply the following proposition:

Proposition 3.2 (lift proposition) *For a weighted n-simplex (Δ_V, V, W) in the projective model $\mathbf{B}^n$, there exists a unique affine n-simplex $\widehat{\Delta_V}$ in $\mathbf{E}^{1,n}-\{o\}$ with vertex set $\widehat{V}$ satisfying the following four conditions:*

(1) $\widehat{V} \subset T^+ \sqcup L^+ \sqcup S$;

(2) $\mathcal{P}\left(\widehat{V}\right) = V$;

(3) For any $\boldsymbol{u} \in \widehat{V} \cap S$, *we have* $\Delta_V \subset R_{n(\boldsymbol{u})} \cap \mathbf{B}^n$;

(4) For any $\boldsymbol{u} \in \widehat{V}$, *the width* $\delta_{\boldsymbol{u}}$ *is equal to the weight of* $\mathcal{P}(\boldsymbol{u})$. □

We call $\widehat{\Delta_V}$ the *lift* of the weighted n-simplex (Δ_V, V, W) in $\mathbf{B}^n$, $\widehat{V}$ the *lift* of the vertex set V, and $\boldsymbol{u}$ the lift of the vertex $\mathcal{P}(\boldsymbol{u}) \in V$. We here note that condition (2) means $\widehat{V}$ is a set of linearly independent vectors in $\mathbf{E}^{1,n}$. We also note that $\mathcal{P}\left(\widehat{\Delta_V}\right)$ does not always coincide with Δ_V, though $\mathcal{P}\left(\widehat{V}\right) = V$.

3.3 Definition of tilts and the tilt proposition

R. C. Penner gave in [Pe, Proposition 2.6(b)] a criterion of convexity of the lifts of adjacing two (2-dimensional) ideal triangles along a face. J. R. Weeks independently gave in [We1, Proposition 3.1] a criterion of convexity when simplices are 2 and 3-dimensional ideal simplices. This criterion is expressed by using "tilts," and allow him to make the hyperbolic structures computation program "SnapPea" (cf. [We2]). He also provided an efficient formula, called the *tilt formula*, to obtain tilts from intrinsic geometry of the simplex when its dimension is two (see [We1, Theorem 3.2]) and three (see [We1, Theorem 5.1]). M. Sakuma and J. R. Weeks generalized the tilt formula to general dimensions in [SW]. The idea of R. C. Penner is translated by M. Näätänen in [Nä, Lemma 3.3] into the case where simplices are triangles, and by the author in [Us2, Proposition 3.5(2)] into the case where simplices are truncated triangles (i.e., orthogonal hexagons). In this subsection, using Weeks' method, we obtain a criterion of convexity when two weighted n-simplices in $\mathbf{B}^n$ are adjacent along internal faces. Now we start with the definition of the tilt of a weighted n-simplex in $\mathbf{B}^n$ relative to an internal face.

Fix a weighted n-simplex (Δ_V, V, W) in $\mathbf{B}^n$, and take an internal face Φ_i of Δ_V. Then there is a unique point $\boldsymbol{m}_i$ in H_S such that $\Phi_i \subset P\boldsymbol{m}_i \cap \mathbf{B}^n$ and $\Delta_V \subset R\boldsymbol{m}_i \cap \mathbf{B}^n$. We define the *normal* vector $\boldsymbol{p}$ to the lift $\widehat{\Delta_V}$ of (Δ_V, V, W) by the condition that $\langle \boldsymbol{p}, \boldsymbol{x} \rangle = -1$ for all $\boldsymbol{x} \in \widehat{\Delta_V}$.

Definition 3.3 Under the assumptions stated above, the *tilt* t_i *of* (Δ_V, V, W) *relative to* Φ_i is defined as follows:

$$t_i := \langle \boldsymbol{m}_i, \boldsymbol{p} \rangle \ .$$

Let (Δ_{V_0}, V_0, W_0) and (Δ_{V_1}, V_1, W_1) be two weighted n-simplices in $\mathbf{B}^n$, and let Φ_0 (resp. Φ_1) be an internal face of (Δ_{V_0}, V_0, W_0) (resp. (Δ_{V_1}, V_1, W_1)). Then we say that (Δ_{V_0}, V_0, W_0) and (Δ_{V_1}, V_1, W_1) are *adjacent along* Φ_0 and Φ_1 if $\widehat{\Delta_{V_0}} \cap \widehat{\Delta_{V_1}} = \widehat{\Phi_0} = \widehat{\Phi_1}$, where $\widehat{\Phi_0}$ (resp. $\widehat{\Phi_1}$) is the lift of Φ_0 (resp. Φ_1) in $\widehat{\Delta_{V_0}}$ (resp. $\widehat{\Delta_{V_1}}$). Now we call Φ_0 and Φ_1 *joint faces*. For convenience we additionally assume that $V_0 = \{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$, $V_1 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n, \boldsymbol{v}_{n+1}\}$, and that the joint faces are opposite faces of $\boldsymbol{v}_0$ and $\boldsymbol{v}_{n+1}$. We denote by t_0 (resp. t_1) the tilt of (Δ_{V_0}, V_0, W_0) (resp. (Δ_{V_1}, V_1, W_1)) relative to Φ_0 (resp. Φ_1). Then the following proposition correspondent with [We1, Proposition 3.1] holds.

Proposition 3.4 (tilt proposition) *Under the assumptions stated above, the dihedral angle formed by* $\widehat{\Delta_{V_0}}$ *and* $\widehat{\Delta_{V_1}}$ *is convex (flat, concave respectively) in* $\mathbf{E}^{1,n}$ *if and only if* $t_0 + t_1 < 0$ $(= 0, > 0$ *respectively).* □

4 Tilt formulas

As we saw in the previous section, tilts are defined on internal faces of generalized n-simplices. But when $n = 2$, internal faces may be degenerate, that is, some of opposite faces may not exist in $\mathbf{B}^2$ (see Figure 2). Then we cannot defined the tilt on the degenerated internal face. But once the dimension is greater than two, the following proposition guarantees the existance of all internal faces.

Proposition 4.1 *Suppose n is greater than or equal to three. Then, for any weighted n-simplex (Δ_V, V, W) in $\mathbf{B}^n$, the opposite face Φ_i of an arbitrary vertex $\boldsymbol{v}_i \in V$ exists in $\mathbf{B}^n$.*

Proof of Proposition 4.1. All we have to show is that the opposite face Φ_n intersects $\mathbf{B}^n$ when $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1} \in \operatorname{Ext}\overline{\mathbf{B}^n}$ and each line $l(\boldsymbol{v}_i\boldsymbol{v}_j)$ in $\mathbf{P}^n$ through $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$, where $0 \leq i < j \leq n-1$, touches $\partial\mathbf{B}^n$. Let $\boldsymbol{w}_1$ (resp. $\boldsymbol{w}_2$) be the tangent point of $\partial\mathbf{B}^n$ and $l(\boldsymbol{v}_0\boldsymbol{v}_1)$ (resp. $l(\boldsymbol{v}_0\boldsymbol{v}_2)$). Then $\boldsymbol{w}_1$ does not coincide with $\boldsymbol{w}_2$ when $n \geq 3$. Since n-dimensional ball $\overline{\mathbf{B}^n}$ is convex, the line $l(\boldsymbol{w}_1\boldsymbol{w}_2)$ intersects $\mathbf{B}^n$. Thus $l(\boldsymbol{w}_1\boldsymbol{w}_2) \cap \mathbf{B}^n$ is a (non-empty) segment contained in the opposite face Φ_n. This completes the proof. □

4.1 Generalized distances

Previous tilt formulas, for exmaple [SW, Theorem 2.1], suggest that we have to measure hyperbolic distances between geometric objects defined by weighted vertices and their opposite faces. But as the vertex $\boldsymbol{v}_0$ and its opposite face in Figure 1, they may intersect. So, to denote our tilt formula, we have to define a sort of unification of distances and angles, which we call generalized distances defined below.

Definition 4.2 Let $\boldsymbol{x}$ be a point in H_S, and $\boldsymbol{y}$ an arbitrary point in $T^+ \sqcup (R_{\boldsymbol{x}} \cap L^+) \sqcup (R_{\boldsymbol{x}} \cap S)$. Then the *generalized distance d between $\boldsymbol{x}$ and $\boldsymbol{y}$* is defined as follows:

Case 1. If $\boldsymbol{y} \in R_{\boldsymbol{x}} \cap L^+$, then d is defined to be the signed distance between $\Pi_{\boldsymbol{x}}$ and $\Pi_{\boldsymbol{y}}$.

Case 2. If $\boldsymbol{y} \in T^+$ or $\boldsymbol{y} \in S$ with $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq -\sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}$ (that is, $\Pi_{\boldsymbol{x}}$ and $\Pi_{n(\boldsymbol{y})}$ are parallel or ultraparallel), then $d := d_n - \delta_{\boldsymbol{y}}$, where d_n is the signed distance between $\Pi_{\boldsymbol{x}}$ and $\Pi_{n(\boldsymbol{y})}$, and $\delta_{\boldsymbol{y}}$ is the width of $\boldsymbol{y}$.

Case 3. If $\boldsymbol{y} \in S$ with $(0 \geq) \langle \boldsymbol{x}, \boldsymbol{y} \rangle > -\sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}$, that is, if $\Pi_{\boldsymbol{x}}$ and $\Pi_{n(\boldsymbol{y})}$ intersect, then $d := \sqrt{-1}\,\theta - \delta_{\boldsymbol{y}}$, where θ is the dihedral angle between $\Pi_{\boldsymbol{x}}$ and $\Pi_{n(\boldsymbol{y})}$ measured in $\Gamma_{\boldsymbol{x}} \cap \Gamma_{n(\boldsymbol{y})}$.

By the definition of the generalized distance together with Proposition 2.4, we can obtain the following proposition:

Proposition 4.3 *Let $\boldsymbol{x}$ be a point in H_S. For an arbitrary point $\boldsymbol{y} \in T^+ \sqcup (R_{\boldsymbol{x}} \cap L^+) \sqcup (R_{\boldsymbol{x}} \cap S)$, the following equality holds:*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\frac{e^d + \nu\, e^{-d}}{2},$$

where $\nu := \langle \boldsymbol{y}, \boldsymbol{y} \rangle$, and d is the generalized distance between $\boldsymbol{x}$ and $\boldsymbol{y}$. □

4.2 The case where the dimension is greater than two

In this subsection we suppose the dimension n is greater than or equal to three. Fix a weighted n-simplex (Δ_V, V, W) in $\mathbf{B}^n$. Then Proposition 4.1 guarantees that all internal faces of Δ_V exist in $\mathbf{B}^n$, namely we can always define the tilt t_i for each internal face Φ_i. We denote by $\widehat{V} = \{\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ the lift of V, and we define $\nu_i := \langle \boldsymbol{u}_i, \boldsymbol{u}_i \rangle$. Let d_i be the generalized distance between $\boldsymbol{m}_i$ and $\boldsymbol{u}_i$, where we recall that $\boldsymbol{m}_i$ is the point in H_S such that $\Phi_i \subset P_{\boldsymbol{m}_i} \cap \mathbf{B}^n$ and $\Delta_V \subset R_{\boldsymbol{m}_i} \cap \mathbf{B}^n$. Now we define Q_i as follows:

$$Q_i := \frac{2}{e^{d_i} + \nu_i\, e^{-d_i}}.$$

We denote by θ_{ij} the dihedral angle between Φ_i and Φ_j, that is, the dihedral angle between $\Pi_{\boldsymbol{m}_i}$ and $\Pi_{\boldsymbol{m}_j}$ measured in $\Gamma_{\boldsymbol{m}_i} \cap \Gamma_{\boldsymbol{m}_j}$. We note that $\theta_{ij} = 0$ if Φ_i and Φ_j are parallel. Then we have the following theorem:

Theorem 4.4 (tilt formula for $n \geq 3$) *Under the notation defined above, the tilt of a weighted n-simplex relative to each of its (codimension one) internal faces may be computed as follows:*

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} 1 & -\cos\theta_{01} & -\cos\theta_{02} & \cdots & -\cos\theta_{0n} \\ -\cos\theta_{10} & 1 & -\cos\theta_{12} & \cdots & -\cos\theta_{1n} \\ -\cos\theta_{20} & -\cos\theta_{21} & 1 & \cdots & -\cos\theta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\cos\theta_{n0} & -\cos\theta_{n1} & -\cos\theta_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{pmatrix}.$$

We may say the $(n+1) \times (n+1)$ matrix on the right side of the formula denoted above the *Gram matrix of the generalized n-simplex* Δ_V (cf. [Vi, p. 39]). The proof of this theorem is a word-by-word interpretation of that of Theorem 2.1 in [SW].

4.3 The case where the dimension is two

As we saw in Figure 2, some internal faces of a weighted 2-simplex (Δ_V, V, W) in $\mathbf{B}^2$ may be degenerate. So Theorem 4.4 does not always hold when the dimension n is two. But under the assumption that all internal faces exist, an analogue of Theorem 4.4 holds. We here note that $\Pi_{\boldsymbol{m}_i}$ and $\Pi_{\boldsymbol{m}_j}$ may be ultraparallel for some $\boldsymbol{m}_i, \boldsymbol{m}_j \in H_S$ with $i \neq j$ (see Figure 1 again). So we should replace each element $-\cos\theta_{i\,j}$ of the Gram matrix in the previous theorem by $-\cosh\delta_{i\,j}$, where $\delta_{i\,j}$ is the generalized distance between $\boldsymbol{m}_i$ and $\boldsymbol{m}_j$.

From now on, we consider the case where some internal faces are degenerate. For example we assume that only the opposite face of the vertex $\boldsymbol{v}_2 \in V$ is degenerate (see Figure 2 again). In this case, we put $\boldsymbol{m}_2 := \sqrt{\nu_1}\,\boldsymbol{u}_0 + \sqrt{\nu_0}\,\boldsymbol{u}_1$. Then $\boldsymbol{m}_2$ is a non-zero vector in L. Now we can show that two sets $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $\{-Q_0\,\boldsymbol{m}_0, -Q_1\,\boldsymbol{m}_1, -Q_2\,\boldsymbol{m}_2\}$ form two bases of $\mathbf{E}^{1,2}$ and are dual to each other, where $Q_2 := -\langle \boldsymbol{m}_2, \boldsymbol{u}_2 \rangle^{-1} = -\left(\langle \boldsymbol{u}_0, \boldsymbol{u}_2 \rangle \sqrt{\nu_1} + \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle \sqrt{\nu_0}\right)^{-1} (\neq 0)$. Now using equations $\langle \boldsymbol{m}_0, \boldsymbol{m}_2 \rangle = -{Q_0}^{-1}\sqrt{\nu_1}$ and $\langle \boldsymbol{m}_1, \boldsymbol{m}_2 \rangle = -{Q_1}^{-1}\sqrt{\nu_0}$, we can easily obtain the following corollary:

Corollary 4.5 (tilt formula for $n = 2$ with one degenerate internal face) *Under the assumptions stated above, the following relation holds:*

$$\begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = \begin{pmatrix} 1 & -\cosh\delta_{0\,1} & -{Q_0}^{-1}\sqrt{\nu_1} \\ -\cosh\delta_{1\,0} & 1 & -{Q_1}^{-1}\sqrt{\nu_0} \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix}. \qquad \square$$

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