

On the Rumin complex

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The moduli of complex structures seems to be related to the hyperbolicity. Here, we treat the moduli of isolated singularities. Let (V, o) be a normal isolated singularity in a complex euclidean space C^N . Consider the link M , defined by the intersection of (V, o) and the real hypersphere $S_\epsilon^{2N-1}(o)$, centered at the origin o with the radius of ϵ . Over this link M , a CR structure is naturally induced from V . Concerning the moduli problem of isolated singularities, Kuranishi initiated the deformation theory of CR structures. Of course, the problem of constructing a versal family of deformations of isolated singularities is done by several authors, almost 30 years ago (for example, Grauert, Douady, etc.), with more direct method (rather, in algebraic geometry). However, our approach (in CR geometry) has one geometric aspect. In order to introduce "several new methods in topology (Symplectic structure, Seiberg-Witten invariants, etc.)" to the moduli of isolated singularities, our method is definitely more accessible (we are directly treating links, contact structures and CR structures).

The purpose of this survey is to introduce the brief sketch of the Rumin's method in the deformation theory of CR structures and establish a versal family in the 5-dimensional case (the precise proof will be published elsewhere). Obviously, the moduli of CR structures should be related to "Seiberg-Witten equation in CR structures". In the future, I would like to discuss the relation of our setting and "Seiberg-Witten equation in CR structures". But, at the present time, our project in this direction is in process. Here, we only sketch our joint work ([A-G-L[1]], [A-G-L[2]]), briefly. Anyway, we start with the deformation theory of CR structures.

1 Standard Deformation Complex

Let $(M, {}^0T')$ be a CR structure. This means that M is a C^∞ differentiable manifold with real odd dimension and ${}^0T'$ is a sub-vector bundle of the complexified tangent bundle $C \otimes TM$ satisfying:

$${}^0T'' \cap {}^0T' = 0, \dim_C \frac{C \otimes TM}{{}^0T'' + {}^0T'} = 1 \quad (1)$$

$$[\Gamma(M, {}^0T''), \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T''), \quad (2)$$

where ${}^0T' = \overline{{}^0T''}$.

Now we assume that there is a global real vector field ξ satisfying; for every point p of M ,

$$\xi_p \notin {}^0T''_p + {}^0T'_p. \quad (3)$$

We set

$$T' := {}^0T' + C \otimes \xi, \quad (4)$$

where $C \otimes \xi$ means the line bundle generated by ξ . For brevity, we use the notation F for this line bundle. By using this T' , we set a C^∞ vector bundle decomposition

$$C \otimes TM = T' + {}^0T''. \quad (5)$$

Following [A1], we introduce a first order differential operator $\bar{\partial}_{T'}$ from $\Gamma(M, T')$ to $\Gamma(M, T' \otimes ({}^0T'')^*)$ ($\Gamma(M, T')$ (resp. $\Gamma(M, T' \otimes ({}^0T'')^*)$) means the space of C^∞ sections of T' (resp. $T' \otimes ({}^0T'')^*$), by: for $u \in \Gamma(M, T')$, $X \in {}^0T''_p$, p is a point of M ,

$$\bar{\partial}_{T'} u(X) = [X, u]_{T'}, \quad (6)$$

where $[X, u]_{T'}$ means the projection of $[\tilde{X}, u]$ to T' according to (5), and \tilde{X} means a C^∞ extension of X to M ($\bar{\partial}_{T'} u(X)$ doesn't depend on the extension). Like the case for scalar valued differential forms, we can introduce $\bar{\partial}_{T'}^{(p)}$ operators, $p = 1, 2, \dots$ (see [A1]) and we have a differential complex

$$\begin{aligned} 0 \rightarrow \Gamma(M, T') \xrightarrow{\bar{\partial}_{T'}} \Gamma(M, T' \otimes ({}^0T'')^*) \xrightarrow{\bar{\partial}_{T'}^{(1)}} \Gamma(M, T' \otimes \wedge^2 ({}^0T'')^*) \rightarrow \\ \rightarrow \Gamma(M, T' \otimes \wedge^p ({}^0T'')^*) \xrightarrow{\bar{\partial}_{T'}^{(p)}} \Gamma(M, T' \otimes \wedge^{p+1} ({}^0T'')^*) \rightarrow \dots \end{aligned} \quad (7)$$

with $\bar{\partial}_{T'}^{(p+1)} \bar{\partial}_{T'}^{(p)} = 0$.

This complex is called the standard deformation complex. We, briefly recall the deformation theory of CR structures. Let $(M, {}^0T'')$ be a CR structure.

Definition 1 Let E be a complex subbundle of the complexified tangent bundle $C \otimes TM$ satisfying:

$$E \cap \bar{E} = 0.$$

We call this pair (M, E) an almost CR structure.

Almost CR structures satisfying a certain condition can be parametrized by elements of $\Gamma(M, T' \otimes ({}^0T'')^*)$ as follows.

Definition 2 An almost CR structure (M, E) is at finite distance from $(M, {}^0T'')$ if and only if the composition map of the inclusion map of E into $C \otimes TM$, and the projection map of $C \otimes TM$ to ${}^0T''$ according to (5), is isomorphic.

Then we have

Theorem 1 (Proposition 1.1 in [A]). The almost CR structure, which is at finite distance from $(M, {}^0T'')$, corresponds to an element ϕ in $\Gamma(M, \text{Hom}({}^0T'', T')) = \Gamma(M, T' \otimes ({}^0T'')^*)$, bijectively. The correspondence is that; for ϕ in $\Gamma(M, T' \otimes ({}^0T'')^*)$,

$${}^\phi T'' = \{ X' : X' = X + \phi(X), X \in {}^0T'' \}.$$

And the following theorem explains when this almost CR structure $(M, {}^\phi T'')$ is really a CR structure.

Theorem 2 (Theorem 2.1 in [A]). An almost CR structure $(M, {}^\phi T'')$, which is at finite distance from $(M, {}^0T'')$ is a CR structure (this means that our $(M, {}^\phi T'')$ satisfies the integrability condition), if and only if our ϕ is a solution of the non linear partial differential equations

$$\begin{aligned} P(\phi) &= \bar{\partial}_{T'}^{(1)} \phi + R_2(\phi) + R_3(\phi) \\ &= 0. \end{aligned}$$

Namely the linear term of $P(\phi) = 0$ is $\bar{\partial}_{T'}^{(1)} \phi$. And the real difficult problem, in solving this non-linear partial differential equation, is that: the non-linear term $R_2(\phi)$ includes the first order derivatives of ϕ , and our $\bar{\partial}_{T'}$ complex is subelliptic (the Neumann operator gains only 1). So, in CR case, the method in complex manifolds is not available. In order to overcome this difficulty, we introduce E_j structure in [A1]. Henceforth, we assume strongly convexity.

2 E_j structure

We recall several results obtained in [A1]. Let $(M, {}^0T'')$ be a strongly pseudo convex CR structure (this is an abstract of my talk, so we omit the notion of strongly pseudo convex CR structures). We set

$$\Gamma_j = \{u : u \in \Gamma(M, {}^0T' \otimes \wedge^j({}^0T'')^*), (\bar{\partial}_{T'}^{(j)} u)_{F \otimes \wedge^{j+1}({}^0T'')^*} = 0\} \quad (8)$$

Here $(\bar{\partial}_{T'}^{(j)} u)_{F \otimes \wedge^{j+1}({}^0T'')^*}$ means the projection of $\bar{\partial}_{T'}^{(j)} u$ to $F \otimes \wedge^j({}^0T'')^*$ according to the C^∞ vector bundle decomposition

$$T' \otimes \wedge^{j+1}({}^0T'')^* = {}^0T' \otimes \wedge^{j+1}({}^0T'')^* + F \otimes \wedge^{j+1}({}^0T'')^*$$

induced by (4) and (5). In the definition of Γ_j , the first order derivative of u appears. But, actually as u takes its value in ${}^0T'$, $(\bar{\partial}_{T'}^{(j)} u)_{F \otimes \wedge^{j+1}({}^0T'')^*} = 0$ is an algebraic condition. We see this more precisely. We consider a bundle map:

$${}^0T' \otimes \wedge^j({}^0T'')^* \rightarrow F \otimes \wedge^{j+1}({}^0T'')^*$$

defined by; for u in ${}^0T' \otimes \wedge^j({}^0T'')^*$,

$$(\bar{\partial}_{T'}^{(j)} u)_{F \otimes \wedge^{j+1}({}^0T'')^*}.$$

Then, we have the following theorems.

Theorem 3 (Proposition 2.1 in [A1]). *There is a sub-vector bundle E_j of ${}^0T' \otimes \wedge^j({}^0T'')^*$, satisfying $\Gamma(M, E_j) = \Gamma_j$. And especially, our CR structure is strongly pseudo convex, $E_0 = 0$.*

Theorem 4 (Theorem 2.2 in [A1]).

$$\bar{\partial}_{T'}^{(j)} \Gamma_j \subset \Gamma_{j+1},$$

that is to say, $(\Gamma_j, \bar{\partial}_j)$, where

$$\bar{\partial}_j = \bar{\partial}_{T'}^{(j)} |_{\Gamma_j}$$

is a sub-differential complex of the standard deformation complex of $(\Gamma(M, T' \otimes \wedge^j({}^0T'')^*), \bar{\partial}_{T'}^{(j)})$.

So we have a sub-differential complex of the standard deformation complex;

$$0 \rightarrow 0 \rightarrow \Gamma(M, E_1) \xrightarrow{\bar{\partial}_1} \Gamma(M, E_2) \xrightarrow{\bar{\partial}_2} \Gamma(M, E_3). \quad (9)$$

And for this complex, we have

Theorem 5 (Theorem 2.4 in [A1]). This subcomplex $(\Gamma(M, E_j), \bar{\partial}_j)$ recovers the cohomology group of the standard deformation complex. More precisely, the inclusion map induces the isomorphism map

$$\frac{\text{Ker } \bar{\partial}_j}{\text{Im } \bar{\partial}_{j-1}} \simeq \frac{\text{Ker } \bar{\partial}_{T'}^{(j)}}{\text{Im } \bar{\partial}_{T'}^{(j-1)}} \text{ if } j \geq 2,$$

and if $j = 1$, the inclusion map induces the surjective map

$$\text{Ker } \bar{\partial}_1 \rightarrow \frac{\text{Ker } \bar{\partial}_{T'}^{(1)}}{\text{Im } \bar{\partial}_{T'}} \rightarrow 0.$$

Furthermore for this complex,

Theorem 6 (Theorem 4.1 in [A1]) If $\dim_{\mathbb{R}} M = 2n - 1 \geq 7$, then at E_2 , there is a sub-elliptic estimate. So this estimate insures the Kodaira-Hodge decomposition theorem over $\Gamma(M, E_2)$. That is to say, we put the L^2 norm on $\Gamma(M, E_i)$ and complete these spaces. We denote $\Gamma_2(M, E_i)$ for these completed Hilbert spaces. Let $\mathbf{H} = \{u; u \in \Gamma(M, E_2), \bar{\partial}_1^* u = 0, \bar{\partial}_2 u = 0\}$. Then, there is a Neumann type operator N from $\Gamma_2(M, E_2)$ to itself and the harmonic operator H from $\Gamma_2(M, E_2)$ to \mathbf{H} satisfying;

- (1) $NH = HN = 0$,
- (2) for $u \in \Gamma_2(M, E_2)$, $u = Hu + \bar{\partial}_2^* \bar{\partial}_2 Nu + \bar{\partial}_1 \bar{\partial}_1^* Nu$.
- (3) $\|Nu\| \leq c \|u\|''$, for $u \in \Gamma_2(M, E_2)$, where c is a positive constant, independent of u .

For the norm $\| \cdot \|''$, see [A1].

3 The new complex inspired by Rumin

Let

$$H_0 = \{v : v \in \Gamma(M, T'), (\bar{\partial}_{T'} v)_{F \otimes ({}^0 T''^*)} = 0\}. \quad (10)$$

Instead of (7), we introduce the following differential complex.

$$0 \longrightarrow H_0 \xrightarrow{\bar{\partial}_{T'}} \Gamma(M, E_1) \xrightarrow{\bar{\partial}_1} \Gamma(M, E_2). \quad (11)$$

By the definition of H_0 , $\bar{\partial}_{T'} u$ is in $\Gamma(M, {}^0 T' \otimes ({}^0 T''^*))$ (because $F \otimes ({}^0 T''^*)$ term vanishes). And $\bar{\partial}_{T'}^{(1)} \bar{\partial}_{T'} u = 0$ is obvious. So, (9) makes sense as a differential complex. We have to explain what H_0 is like. First, we note that H_0 does not come from the space of C^∞ sections of a vector bundle over M . H_0 is a graph of the following first order differential operator. For

$u \cdot \xi$ in $\Gamma(M, F)$, where u is a C^∞ function on M , we set an element ψ_u of $\Gamma(M, {}^0T')$ by:

$$(\bar{\partial}_{T'}(\psi_u + u \cdot \xi))_{F \otimes ({}^0T'')^*} = 0, \quad (12)$$

that is to say,

$$[X, \psi_u]_F + (Xu)\xi + u[X, \xi]_F = 0, \quad \text{for } X \in \Gamma(M, {}^0T''). \quad (13)$$

This is possible. Since our CR structure is strongly pseudo convex, ψ_u is uniquely determined. Hence we have a first order differential operator from $\Gamma(M, F)$ to H_0 defined by; for $u \cdot \xi$ in $\Gamma(M, F)$, $\psi_u + u \cdot \xi$ in H_0 , and so we have a differential complex.

$$\begin{array}{ccccccc} & & \Gamma(M, F) & & & & \\ & & \rho \downarrow & & & & \\ 0 & \longrightarrow & H_0 & \xrightarrow{\bar{\partial}_{T'}} & \Gamma(M, E_1) & \xrightarrow{\bar{\partial}_1} & \Gamma(M, E_2). \end{array}$$

We note that ψ_u includes a first derivative of u . We see this more explicitly. By using a moving frame $\{e_1^\lambda, e_2^\lambda\}$ of ${}^0T''|_{U_\lambda}$, satisfying:

$$-\sqrt{-1}[e_i^\lambda, \bar{e}_j^\lambda]_F = \delta_{ij}\xi, \quad (14)$$

where $[e_i^\lambda, \bar{e}_j^\lambda]_F$ means the F part of $[e_i^\lambda, \bar{e}_j^\lambda]$ according to (2.4), we write down ψ_u . Set

$$\psi_u = \psi_1^\lambda \bar{e}_1^\lambda + \psi_2^\lambda \bar{e}_2^\lambda. \quad (15)$$

Then,

$$[e_i^\lambda, \psi_1^\lambda \bar{e}_1^\lambda + \psi_2^\lambda \bar{e}_2^\lambda]_F + (e_i^\lambda u)\xi + u[e_i^\lambda, \xi]_F = 0, \quad (16)$$

that is to say,

$$\sqrt{-1}\psi_i^\lambda \xi + (e_i^\lambda u)\xi + u[e_i^\lambda, \xi]_F = 0, \quad i = 1, 2. \quad (17)$$

So

$$\psi_i^\lambda = \sqrt{-1}e_i^\lambda u + 0\text{-th order term of } u. \quad (18)$$

We set a second order linear differential operator D from $\Gamma(M, F)$ to $\Gamma(M, E_1)$, by: for $u \cdot \xi$, where u is a C^∞ function,

$$D(u \cdot \xi) = \bar{\partial}_{T'}(\rho(u \cdot \xi)). \quad (19)$$

We would like to explain about this second order differential operator and Rumin's one. For this, we set a C^∞ vector bundle decomposition over complex valued differential k forms (here we write this space by $\Lambda^k(C \otimes TM)^*$) as follows.

$$\begin{aligned} \Lambda^k(C \otimes TM)^* &= \sum_{r+s=k-1, r, s \geq 0} \theta \wedge \wedge^r ({}^0T')^* \wedge \wedge^s ({}^0T'')^* \\ &+ \sum_{r+s=k, r, s \geq 0} \theta \wedge \wedge^r ({}^0T')^* \wedge \wedge^s ({}^0T'')^*. \end{aligned} \quad (20)$$

Here θ is defined by:

$$\begin{cases} \theta(\xi) = 1, \\ \theta(X) = 0, \quad X \in {}^0T', \\ d\theta(\xi, X) = 0 \quad X \in {}^0T'. \end{cases}$$

(if necessary, we change ξ). By using this decomposition, we introduce; for $i \geq 1$,

$$\begin{aligned} F^{n-2, i} &= \{u : u \in \Gamma(M, \theta \wedge \wedge^{n-2} ({}^0T')^* \wedge \wedge^i ({}^0T'')^*), \\ &\quad (du)_{\wedge^{n-1} ({}^0T')^* \wedge \wedge^{i+1} ({}^0T'')^*} = 0\}. \end{aligned} \quad (21)$$

Here $(du)_{\wedge^{n-1} ({}^0T')^* \wedge \wedge^{i+1} ({}^0T'')^*}$ means the $\wedge^{n-1} ({}^0T')^* \wedge \wedge^{i+1} ({}^0T'')^*$ part of du according to the above decomposition. For $i = 0$, we set

$$\Gamma(M, \wedge^{n-1} (T')^*). \quad (22)$$

Then we have a complex version of the Rumin complex

$$\Gamma(M, \wedge^{n-1} (T')^*) \xrightarrow{D} \Gamma(M, F^{n-2, 1}) \xrightarrow{d''} \Gamma(M, F^{n-2, 2}).$$

We note that: if the canonical line bundle is trivial, this is nothing but our new complex.

Our main result is as follows.

Theorem 7 *For this complex, a subelliptic estimate on $\Gamma(M, F^{n-2, 1})$ holds (see [A-G-L[2]]).*

And so, this assures that:

Theorem 8 *The following family is parametrized by a finite dimensional analytic space.*

$$\begin{cases} P(\phi) = 0, \\ \bar{\partial}_{T'}^* \phi = 0, \text{ for } \phi \in \Gamma(M, E_1). \end{cases}$$

Here $\bar{\partial}_{T'}^*$ means the adjoint operator in the our setting(not the standard one)(see [A-G-L[2]]). And this family is parametrized by a complex analytic space locally at o , complex analytically and versal in the sence of Kuranishi.

We note that the linear term of these non-linear equations is

$$\begin{cases} \bar{\partial}_1 \phi = 0, \\ \bar{\partial}_{T'}^* \phi = 0, \text{ for } \phi \in \Gamma(M, E_1). \end{cases}$$

This is the dual of the Dirac operator in our setting.

参考文献

- [A] T.Akahori, *Intrinsic formula for Kuranishi's $\bar{\partial}_b^\phi$* , Publication of RIMS, Kyoto Univ.14. No.3, pp615-641(1978).
- [A1] T.Akahori, *The new estimate for the subbundles E_j and its application to the deformation of the boundaries of strongly pseudo convex domains*, Invent.math.63, pp311-334 (1981).
- [A2] T.Akahori, *The new Neumann operator associated with deformations of strongly pseudo convex domains and its application to deformation theory*, Invent.math., Vol.68, pp317-352(1982).
- [A3] T.Akahori, *A mixed Hodge structure on a CR manifold*, MSRI preprint 1996-026.
- [A-M1] T.Akahori, K.Miyajima, *An analogy of Tian-Todorov theorem for deformations of CR structures*, Composition mathemaicae, Vol.85, pp57-85(1993).
- [A-M2] T.Akahori, K.Miyajima, *A remark on a Tian-Todorov type theorem for deformations of CR structures*, Bull. of Canadian , Vol.37(1), pp8-12(1994).
- [A-G-L[1]] T.Akahori, P.Garfield, J.M.Lee, *The bigraded Rumin complex on CR manifolds*, preprint.

- [A-G-L[2] T.Akahori,P.Garfield,L.M.Lee, *A priori estimate for the crucial degree of the Rumin complex and its application to deformation theory*, preprint
- [B-Mi] R.Buchweitz,J.Millson, *CR-geometry and deformations of isolated singularities*, Memoirs of AMS, Number 597, ppl-96(1887).
- [C-L] J.Chen,J.M.Lee, *A local slice theorem for 3-dimensional CR structures*, American Journal of Mathematics, 117, pp1249-1298(1995).
- [R] M.Rumin, *Formes différentielles sur les variétés de contact*, J. Differential Geom. 39, pp281-330(1994).
- [T] N.Tanaka, "A Differential Geometric Study on Strongly Pseudo-Convex Manifolds," Kinokuniya Company Ltd., Tokyo, 1975.