数理解析研究所講究録
1163 巻 2000 年 9-17

DRAWING BERS EMBEDDINGS OF THE TEICHMÜLLER SPACE OF ONCE PUNCTURED TORI

大阪市立大学 理学部 小森 洋平 (Yohei Komori)
京都大学 理学部 須川 敏幸 (Toshiyuki Sugawa)
奈良女子大学 理学部 和田 昌昭 (Masaaki Wada)
奈良女子大学 理学部 山下 靖 (Yasushi Yamashita)

1. Introduction

Let Γ be a Fuchsian group acting on the unit disk $\mathbb{D}$ uniformizing a once punctured torus S and $B_2(\mathbb{D},\Gamma)$ the complex Banach space of holomorphic quadratic differentials for Γ on $\mathbb{D}$ with bounded norm. It is well known that the complex dimension of $B_2(\mathbb{D},\Gamma)$ is one and we can embed the Teichmüller space $T(\Gamma)$ of Γ in $B_2(\mathbb{D},\Gamma)$ by the Bers projection Φ as a bounded contractible open subset.

In 1972, Bers wrote ([Bers 1972] page 278)

> Unfortunately, there is no known method to decide whether a given $\phi\in B_2(L,G)$ belongs to $T(G)$. This is so even if $d=\dim B_2(L,G)<\infty$. Even the case $d=1$ is untractable.

(G is a Fuchsian group acting on the upper half plane and L is the lower half plane.)

In this paper we show the pictures of $\Phi(T(\Gamma))$ in $B_2(\mathbb{D},\Gamma)$ for several Γ and explain our algorithm to produce such pictures. See Figure 1 for example. We claim that the component located at the center of the picture is equal to $\Phi(T(\Gamma))$.

To describe the idea of the algorithm, let us recall some basic facts in Teichmüller theory. ([Shiga 1987])

For every ϕ in $B_2(\mathbb{D},\Gamma)$, there exists a locally univalent meromorphic function f_ϕ on $\mathbb{D}$ with $\{f_\phi,z\}=\phi(z)$ where $\{f,\cdot\}$ is the Schwarzian derivative of f. After certain normalization (see section 2), f_ϕ is uniquely determined by ϕ and induces a group homomorphism $\theta_\phi:\Gamma\to \mathrm{PSL}(2,\mathbb{C})$ defined by

$$f_\phi\circ\gamma=\theta_\phi(\gamma)\circ f_\phi,\quad \gamma\in\Gamma.$$

We call θ_ϕ the *holonomy representation* of Γ associated with $\phi\in B_2(\mathbb{D},\Gamma)$. We consider the set $K(\Gamma)$ of ϕ in $B_2(\mathbb{D},\Gamma)$ such that $\theta_\phi(\Gamma)$ is a Kleinian group. Then

Theorem 1.1 ([Shiga 1987]). *$\Phi(T(\Gamma))$ is equal to the component of $\operatorname{Int}K(\Gamma)$ containing the origin.*

Therefore we will draw the pictures of $K(\Gamma)$ in $B_2(\mathbb{D},\Gamma)$ for given Γ and that the algorithm involves the following two steps: for each element ϕ in $B_2(\mathbb{D},\Gamma)\cong\mathbb{C}$, we

Step 1: compute the holonomy representation θ_ϕ and
Step 2: decide whether the image $\theta_\phi(\Gamma)$ in $\mathrm{PSL}(2,\mathbb{C})$ is discrete.

Each step will be discussed in section 2 and 3.

2. Holonomy representation

In this section we will describe an algorithm which takes as input an element ϕ of $B_2(\mathbb{D},\Gamma)$ and returns a holonomy representation θ_ϕ.

2.1. **Monodromy homomorphism.** Let $\phi \in B_2(\mathbb{D},\Gamma)$. We associate with ϕ the meromorphic function $f_\phi = \eta_1/\eta_0$, where η_1 and η_2 are linearly independent solutions of the differential equation

$$2\eta'' + \phi\eta = 0, \tag{1}$$

normalized by the initial conditions

$$\begin{aligned} \eta_0(0) = 1 \quad \eta_0'(0) = 0 \\ \eta_1(0) = 0 \quad \eta_1'(0) = 1. \end{aligned}$$

Then $\{f_\phi, z\} = \phi(z)$ on $\mathbb{D}$ as expected.

To illustrate how we get the holonomy representation, let us consider the solutions of (1). In view of Γ-invariance of $\phi(z)dz^2$, we see that if η is a solution of (1), then so is $\gamma^*\eta := (\eta\circ\gamma)(\gamma')^{-1/2}$ for every $\gamma \in \Gamma$. In particular, since (η_0,η_1) is a basis of solutions of (1), we can write

$$\gamma^*\eta_0 = D\eta_0 + C\eta_1, \quad \gamma^*\eta_1 = B\eta_0 + A\eta_1,$$

for some complex numbers A, B, C and D. By setting

$$\theta_\phi(\gamma) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we have

$$f_\phi \circ \gamma = \frac{\gamma^*\eta_1}{\gamma^*\eta_0} = \frac{B\eta_0 + A\eta_1}{D\eta_0 + C\eta_1} = \frac{Af+B}{Cf+D} = \theta_\phi(\gamma)\circ f_\phi$$

for each γ and this is the desired homomorphism associated with ϕ. So our task is to compute such complex numbers A, B, C and D for each generator of group Γ. But to make our calculation easier, we will work with a 4-times punctured sphere.

2.2. **Commensurability relations.** Let Γ be a Fuchsian group uniformizing a once punctured torus T and (α,β) a *standard generator pair* of Γ, i.e. α and β freely generate Γ, both are hyperbolic, the commutator $[\alpha,\beta]$ is parabolic and the intersection number $\alpha\cdot\beta = 1$. Then T admits an intermediate covering space which is the plane $\mathbb{C}$ punctured at a lattice $L_\tau = \{m + n\tau; m, n \in \mathbb{Z}\}$ so that α and β correspond to the generators

$$z \to z+1, \quad z \to z+\tau$$

for L_τ. We may assume that $\Im\tau > 0$.

Now consider the 4-times punctured sphere S and the $(2,2,2,\infty)$-orbifold $\mathcal{O}$ (i.e., the orbifold with underlying space a punctured sphere and with three cone points of cone angle π) which have $\mathbb{C} - L_\tau$ as the common covering space. More precisely, let G_S and $G_\mathcal{O}$ be the groups of transformations on $\mathbb{C} - L_\tau$ generated by π-rotations about points in L_τ and $\frac{1}{2}L_\tau := \{(m+n\tau)/2; m, n \in \mathbb{Z}\}$ respectively. Then $S = (\mathbb{C}-L_\tau)/G_S$ and $\mathcal{O} = (\mathbb{C}-L_\tau)/G_\mathcal{O}$ (and $T = (\mathbb{C}-L_\tau)/L_\tau$). Note that we have finite coverings $T \to \mathcal{O}$ and $S \to \mathcal{O}$.

Let Γ_S and $\Gamma_\mathcal{O}$ be the covering group of the universal cover $\mathbb{D} \to (\mathbb{C} - L)_\tau \to S$ and $\mathbb{D} \to (\mathbb{C} - L_\tau) \to \mathcal{O}$ respectively. Since $L_\tau \lhd G_\mathcal{O}$ and $G_S \lhd G_\mathcal{O}$, we have $\Gamma \lhd \Gamma_\mathcal{O}$ and

$\Gamma_S \lhd \Gamma_{\mathcal{O}}$. In particular, $B_2(\mathbb{D}, \Gamma_{\mathcal{O}}) \subset B_2(\mathbb{D}, \Gamma)$ and $B_2(\mathbb{D}, \Gamma_{\mathcal{O}}) \subset B_2(\mathbb{D}, \Gamma_S)$. Since these are all 1-dimensional vector spaces, all three are equal and we conclude that $B_2(\mathbb{D}, \Gamma_S) = B_2(\mathbb{D}, \Gamma)$.

Recall that we can use a single global coordinate z on S:

$$S = \widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}.$$

Without loss of generality we may assume that the lattice points $0, 1, 1+\tau, \tau$ on $L_\tau \subset \mathbb{C}$ correspond to the punctures $0, 1, \infty, \lambda$ of S. Let $p : \mathbb{D} \to S \cong \widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$ be the projection and $B_2(S)$ the Banach space of bounded holomorphic quadratic differentials on S. By definition, the spaces $B_2(\mathbb{D}, \Gamma_S)$ and $B_2(S)$ are isomorphic via the pull-back $p_2^* : B_2(S) \to B_2(\mathbb{D}, \Gamma_S)$ defined by $p_2^*\psi = \psi \circ p \cdot (p')^2$. In particular, dimension of $B_2(S)$ is equal to one. Since the rational function

$$\psi_0(z) = \frac{1}{z(z-1)(z-\lambda)} \tag{2}$$

belongs to $B_2(S)$, ψ_0 forms a basis of the vector space $B_2(S)$. Therefore each element $\phi \in B_2(\mathbb{D}, \Gamma) = B_2(\mathbb{D}, \Gamma_S)$ can be written as $\phi = t\phi_0$ where t is a complex number and $\phi_0 = p_2^*(\psi_0)$.

2.3. **The monodromy of a 4-times punctured sphere.** Now for each $\phi = t\phi_0$, consider the developing map $f_\phi : \mathbb{D} \to \widehat{\mathbb{C}}$. Our idea is to compute f_ϕ on S instead of $\mathbb{D}$.

So we change the independent variable of $f_\phi(x)$ by function $x = p^{-1}(z)$ locally near $0 \in \mathbb{D}$ and refer to the independent variable z on $\widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$ near $p(0)$. Set $P = p^{-1}$ and $g(z) := f\phi(P(z))$. Then we have

$$\{g, z\} = \{f_\phi, P(z)\}(P'(z))^2 + \{P, z\} = t\psi_0(z) + \{P, z\}. \tag{3}$$

and to find g (or f_ϕ as a function of z) we must consider the corresponding linear second order equation

$$2y'' + \{g, z\}y = 0 \tag{4}$$

and express g as the ratio of two independent solutions of this equation. For $\{P, z\}$ we use the next lemma:

Lemma 2.1 ([Hempel 1988], [Kra 1989]). *$\{P, z\}$ is of the form*

$$\{P, z\} = \frac{1}{2z^2} + \frac{(1-\lambda)^2}{2(z-1)^2(z-\lambda)^2} + \frac{c(\lambda)}{z(z-1)(z-\lambda)}. \tag{5}$$

on S where $c(\lambda)$ is a constant determined by λ and called accessory parameter.

By the above lemma and (3), $\{g, z\}$ is globally defined on $\widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$. Combining (2),(3) and (5), the equation (4) to solve is

$$2y'' + \left(\frac{1}{2z^2} + \frac{(1-\lambda)^2}{2(z-1)^2(z-\lambda)^2} + \frac{t + c(\lambda)}{z(z-1)(z-\lambda)}\right) y = 0. \tag{6}$$

Now we describe the computation of the monodromy. Let γ_S be an element of Γ_S We start with an pair (y_0, y_1) of independent solutions of (6) from a certain point z_0 on S normalized by the initial conditions

$$\begin{aligned} y_0(z_0) &= 1 \quad y_0'(z_0) = 0 \\ y_1(z_0) &= 0 \quad y_1'(z_0) = 1. \end{aligned}$$

Then we continue them analytically along a closed path of S corresponding to γ_S. Returning to the starting point, we will arrive with a new pair of solutions (Y_0, Y_1). However, these new solutions must be linear combinations of the original solutions. Thus we have

$$Y_0 = Dy_0 + Cy_1, \quad Y_1 = By_0 + Ay_1,$$

for some complex numbers A, B, C and D. We define

$$\theta_\psi(\gamma_S) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for each $\gamma_S \in \Gamma_S$.

Let us compare g $(= f_\phi(p^{-1})$ and $h := y_1/y_0$. Though both g and h satisfy the same equation (6), the initial conditions at z_0 are not the same. This difference leads to a conjugation from one of $\theta_\phi(\Gamma_S)$ or $\theta_\psi(\Gamma_S)$ to the other. Therefore we have shown that

Lemma 2.2. *The monodromies θ_ϕ and θ_ψ are essentially the same. (Up to conjugacy)*

So we can do our calculations on S using (6).

2.4. **Punctured torus groups.** Now it seems like it's time to compute $\theta_\psi(\alpha)$ and $\theta_\psi(\alpha)$ by solving (6) along the corresponding paths in $S = \widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$. Unfortunately, though α and β are in Γ, they are not in Γ_S for which we have θ_ψ. In other words, α and β do not correspond to the closed paths in S. So we need a little more calculation to end this section.

First, observe that $\operatorname{tr}[\theta_\phi(\alpha), \theta_\phi(\beta)] = -2$. Set $x = \operatorname{tr}\theta_\phi(\alpha)$, $y = \operatorname{tr}\theta_\phi(\beta)$ and $z = \operatorname{tr}\theta_\phi(\alpha \cdot \beta)$. From the trace identity $2 + \operatorname{tr}[X, Y] = (\operatorname{tr} X)^2 + (\operatorname{tr} Y)^2 + (\operatorname{tr} XY)^2 - \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} XY$, we obtain:

$$x^2 + y^2 + z^2 = xyz. \tag{7}$$

Conversely, given any triple (x, y, z) satisfying (7), we can reconstruct the image of the group Γ up to conjugacy. We call such triple of complex numbers *Markov triple.*

Thus it suffices to compute x and y. Again by trace identity $\operatorname{tr} A \operatorname{tr} B = \operatorname{tr} AB + \operatorname{tr} AB^{-1}$

$$x = \sqrt{-\operatorname{tr}\theta_\psi(\alpha^2) + 2}, \quad y = \sqrt{-\operatorname{tr}\theta_\psi(\beta^2) + 2}.$$

Now we can calculate $\theta_\psi(\alpha^2)$ and $\theta_\psi(\beta^2)$ using equation (6) because α^2 and β^2 are in Γ_s The closed loop in S separating $\{0, 1\}$ and $\{\infty, \lambda\}$ corresponds to α^2 and the one separating $\{0, \lambda\}$ and $\{1, \infty\}$ corresponds to β^2 with suitable orientations.

3. Jorgensen's theory to decide discreteness

The input of the algorithm of this section is a Markov triple and the output is the answer "discrete" or "indiscrete".

The general idea is to try to construct the Ford fundamental region of the given Markov triple though it may not have a discrete group image in $\mathrm{PSL}(2,\mathbb{C})$. In this case the term"Ford fundamental region" does not make sense and our process of constructing it will fail. Then we will search for the evidence of its indiscreteness.

We have two remarks. First, this algorithm is based on the Jorgensen's theory on once punctured tori [Jorgensen]. The exposition of this theory with proofs and a generalization is in preparation in [Akiyoshi et al.]. Next, this algorithm may not halt in a finite time for some inputs. The cases where $\mathbb{H}^3/\theta_{(x,y,z)}(\Gamma)$ is a punctured torus bundle or geometrically infinite are the examples. In practice, we will stop our calculation at a certain time and answer "undecided".

3.1. Conditions for discreteness.

Before describing our procedure, let us recall the well known conditions for discreteness for the group action. The first one is Poincaré's fundamental polyhedron theorem and the next one is due to Shimezu and Leutbecher.

Theorem 3.1. *Let Ψ be a proper isometric side pairing for an convex polyhedron P in $\mathbb{H}^3$ such that the hyperbolic manifold M obtained by gluing together the sides of P by Ψ is complete. Then the group Γ generated by Ψ is discrete.*

Lemma 3.2. *Suppose that a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{C})$ contains $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have $|c| \geq 1$ if $c \neq 0$.*

We will use the theorem 3.1 to show that a group is discrete and lemma 3.2 to show that a group is indiscrete.

3.2. Isometric hemispheres.

Let (x, y, z) be a Markov triple. We can reconstruct θ up to conjugacy using Jorgensen's normalization [Jorgensen]:

$$\theta(\alpha) = \frac{1}{x}\begin{pmatrix} xy - z & y/x \\ xy & z \end{pmatrix}, \quad \theta(\beta) = \frac{1}{x}\begin{pmatrix} xz - y & -z/x \\ -xz & y \end{pmatrix}. \tag{8}$$

Note that

$$\theta(\alpha\beta) = \begin{pmatrix} x & -1/x \\ x & 0 \end{pmatrix}, \quad \theta(K) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \quad \text{where } K = [\alpha, \beta]. \tag{9}$$

The isometric hemispheres of α, $\alpha\beta$ and β are centered at $-z/xy$, 0 and y/zx with radii $1/y$, $1/x$ and $1/z$ respectively. The isometric hemispheres of α^{-1}, $(\alpha\beta)^{-1}$ and $\beta^{-1}K^{-1}$ are the translated images of the above three hemispheres by $z \mapsto z + 1$. Since $\theta(\Gamma)$ contains the action $\theta(K)$ of translation $z \mapsto z + 2$, we have an bi-infinite sequence of translated images of the above three isometric hemispheres with symmetry of translation by one. We denote these isometric hemispheres $I(\gamma)$ of $\gamma \in \Gamma$ in this sequence by

$$(10)\quad \dots, I_{-4} = I(\alpha^{-1}K), I_{-3} = I((\alpha\beta)^{-1}K), I_{-2} = I(\beta^{-1}), I_{-1} = I(\alpha), I_0 = I(\alpha\beta),$$
$$I_1 = I(\beta), I_2 = I(\alpha^{-1}), I_3 = I((\alpha\beta)^{-1}), I_4 = I(\beta^{-1}K^{-1}), I_5 = I(\alpha K^{-1}), \dots$$

Note that $I_n + 1 = I_{n+3}$ for any $n \in \mathbb{Z}$. Set $I_{(x,y,z)} := \{I_n\}$. We will try to find the set of Markov triples $\Sigma = \{(x, y, z), (x', y', z'), \dots\}$ such that the isometric hemispheres $I_{(x,y,z)}, I_{(x',y',z')}, \dots$ form the boundary of the Ford fundamental region. We begin by putting $\Sigma = \{(x, y, z)\}$ and check some conditions for this Σ whose output is one of:

Case 1: we have succeeded in constructing the Ford fundamental region so it is discrete.
Case 2: we have found that it is indiscrete,
Case 3: we need more isometric hemispheres to get the conclusion.

Observe that, if (x, y, z) is a Markov triple, then three kinds of *adjacent triples* $(yz - x, z, y)$, $(z, zx - y, x)$ and $(y, x, xy - z)$ are also Markov triples which give rise to different families of infinite isometric hemispheres. The output of case 3 tells us which adjacent triple is needed to (try to) construct the Ford fundamental region. After adding isometric hemispheres of the chosen adjacent triple to Σ, we check the above (but not yet mentioned) conditions again. We continue this process until we reach the cases 1 or 2.

Before starting the above main routine, we may have to replace the isometric hemispheres $I_{(x,y,z)}$ by its neighbors. So we first explain this process in 3.3 and then describe the main process which returns case 1, 2 or 3 in 3.4. In the following we denote (x, y, z) by (x_0, x_2, x_1) and the indices of x_i should be understood modulo three. Then the radius of I_n is $1/|x_n|$.

3.3. The initial process.

3.3.1. For a Markov triple (x_0, x_1, x_2), if $|x_i|$ is less than one for some $i \in \{0, 1, 2\}$, the group is indiscrete. This can be easily seen from Lemma 3.2 and (8). Therefore if this happens in this process or in the main routine below, we will stop our calculation and say (case 2). Otherwise go to 3.3.2.

3.3.2. First, to be a part of the boundary of the Ford region, we ask $I_n \cap I_{n+1} \neq$ for every $n \in \mathbb{Z}$ This is equivalent to the condition:

$$(11)\qquad \exists \text{ triangle with edge lengths } |x_0|, |x_1| \text{ and } |x_2|.$$

If this is not satisfied, one of x_i, say x_0, must be too big. Thus we replace (x_0, x_1, x_2) by the adjacent triple not containing x_0 which is $(x_1x_2 - x_0, x_2, x_1)$. Thus $\Sigma = \{(x_1x_2 - x_0, x_2, x_1)\}$ and go back to 3.3.1. Otherwise go to 3.3.3.

3.3.3. Next, we also want that each I_n does not covered by $I_{n-1} \cup I_{n+1}$. For $i \in \{0, 1, 2\}$, if

$$(12)\qquad |x_i| > |x_{i+1} + x_{i+2}I| \text{ and } |x_i| > |x_{i+1} - x_{i+2}I|,$$

then this condition is not satisfied and we replace the triple by the adjacent triple which does not contain x_i. If $i = 0$, $\Sigma = \{(x_1x_2 - x_0, x_2, x_1)\}$ and go to 3.3.1.

If we find a triple which satisfies both conditions, we go to 3.4.

3.4. The main process.

3.4.1. For any γ with $I(\gamma) \in I_{(x,y,z)}$ and $(x,y,z) \in \Sigma$, let $V(\gamma)$ be the visible part of $I(\gamma)$. If our configuration given by Σ forms the Ford region, $\theta(\gamma)(V(\gamma))$ must be equal to $V(\gamma^{-1})$. Besides, the action of $\theta(\gamma)$ is

1. π rotation around the axis on $I(\gamma)$ connecting (center of $I(\gamma) \pm I/x_i$) followed by
2. the translation $z \mapsto z \pm 1$.

The index i of x_i and the sign of the translation above depends on γ. Since our configuration has a symmetry of translation by one, This means that $V(\gamma)$ must be symmetric by the action of the above π rotation.

We claim that

Proposition 3.3. *This is also the sufficient condition for the isometric hemispheres to form the Ford fundamental region.*

The idea of the proof is to use the theorem 3.1. Since each face is symmetric, the face pairing is well defined. The properness of the face pairing comes from the "chain rule" of the isometric hemispheres. The completeness of the face pairing is easy.

In this case the answer is "discrete" and the result is (case 1). Otherwise goto 3.4.2.

3.4.2. To make our description of our algorithm simpler, suppose that any line segment $|I_n, i_{n+1}|$ where $|I_n, i_{n+1}|$ is the segment connecting the centers of I_n and i_{n+1}, does not intersect with $|I_{n+2}, i_{n+3}|$ for any n. Hence, if we look at the ideal boundary $\mathbb{C}$ of $\mathbb{H}^3$ from ∞, the ideal boundary is separated into two regions by the infinite graph with vertices $\{I_n\}$ and edges $\{|I_n, i_{n+1}|\}$.

Suppose that $V(n)$ is not symmetric for some n. We only consider the cases where I_n intersects with I_{n-2} or I_{n+2}, say I_{n+2}. Otherwise we stop trying to construct the Ford region and goto 3.4.3.

Thus we add the adjacent Markov triple with sequence I_n, I_{n+2} and something which is $(x_{n+2}, x_n x_{n+2} - x_{n+1}, x_n)$. The output is "(case 3) and add $(x_{n+2}, x_n x_{n+2} - x_{n+1}, x_n)$ to Σ".

3.4.3. Now we search for a Markov triple (x_0, x_1, x_2) with $|x_i|$ is less than one for some $i \in \{0,1,2\}$ to show that the group is indiscrete. We denote this condition by $(*)$.

We start with a Markov triple $\sigma \in \Sigma$. If σ satisfies $(*)$, we finish saying (case 2). If not, consider Markov triples adjacent to σ in the sense mentioned above and check $(*)$. If these three Markov triples do not satisfy $(*)$, we next consider the Markov triples which is adjacent to the above three. We continue this process until we find the one which satisfies $(*)$. In this case we say (case 2).

4. Pictures

We present two pictures produced by our method in the following pages.

5. Electronic availability

Files containing the program and pictures can be obtained from

`http://vivaldi.ics.nara-wu.ac.jp/~yamasita/Slice/` .

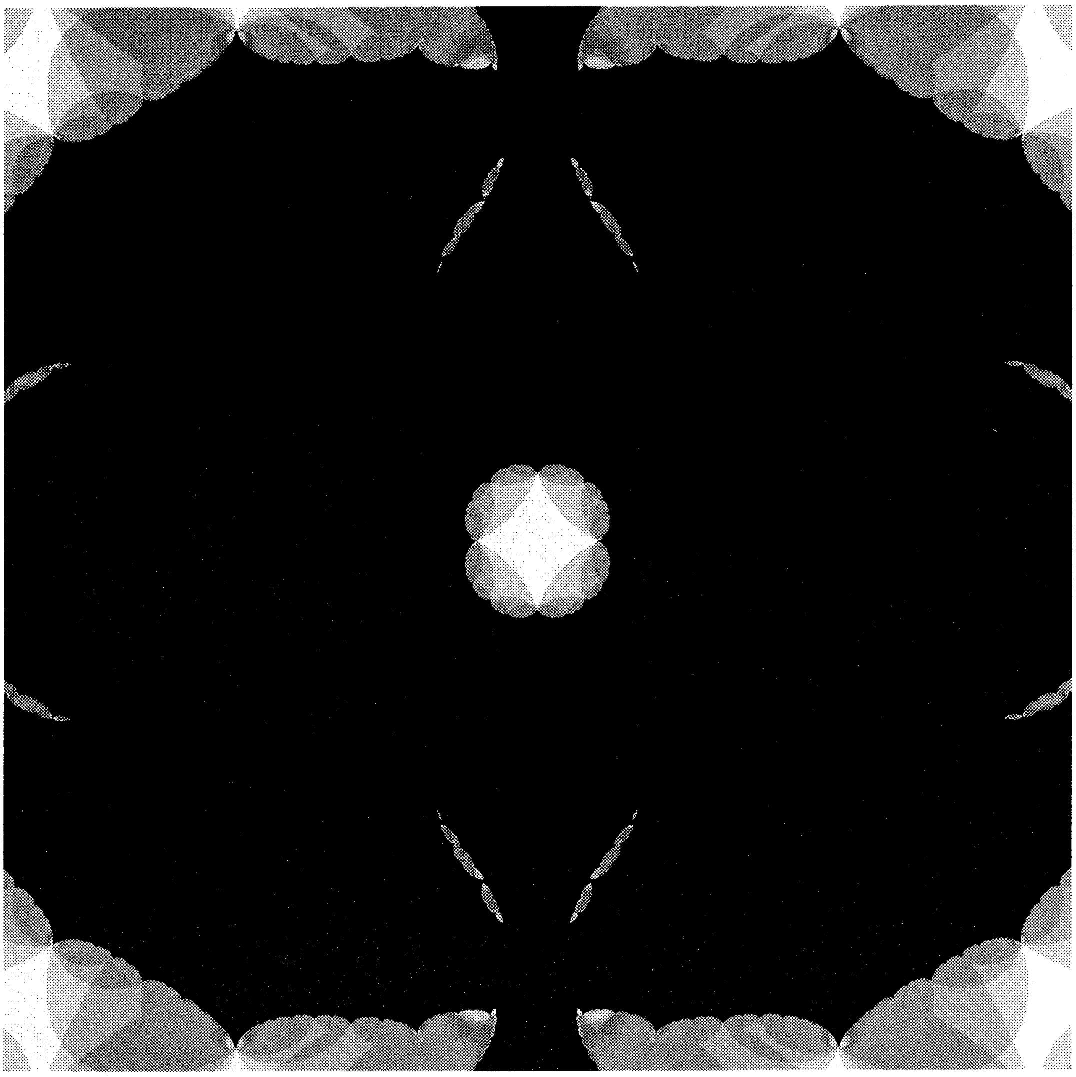

FIGURE 1. Γ = square torus

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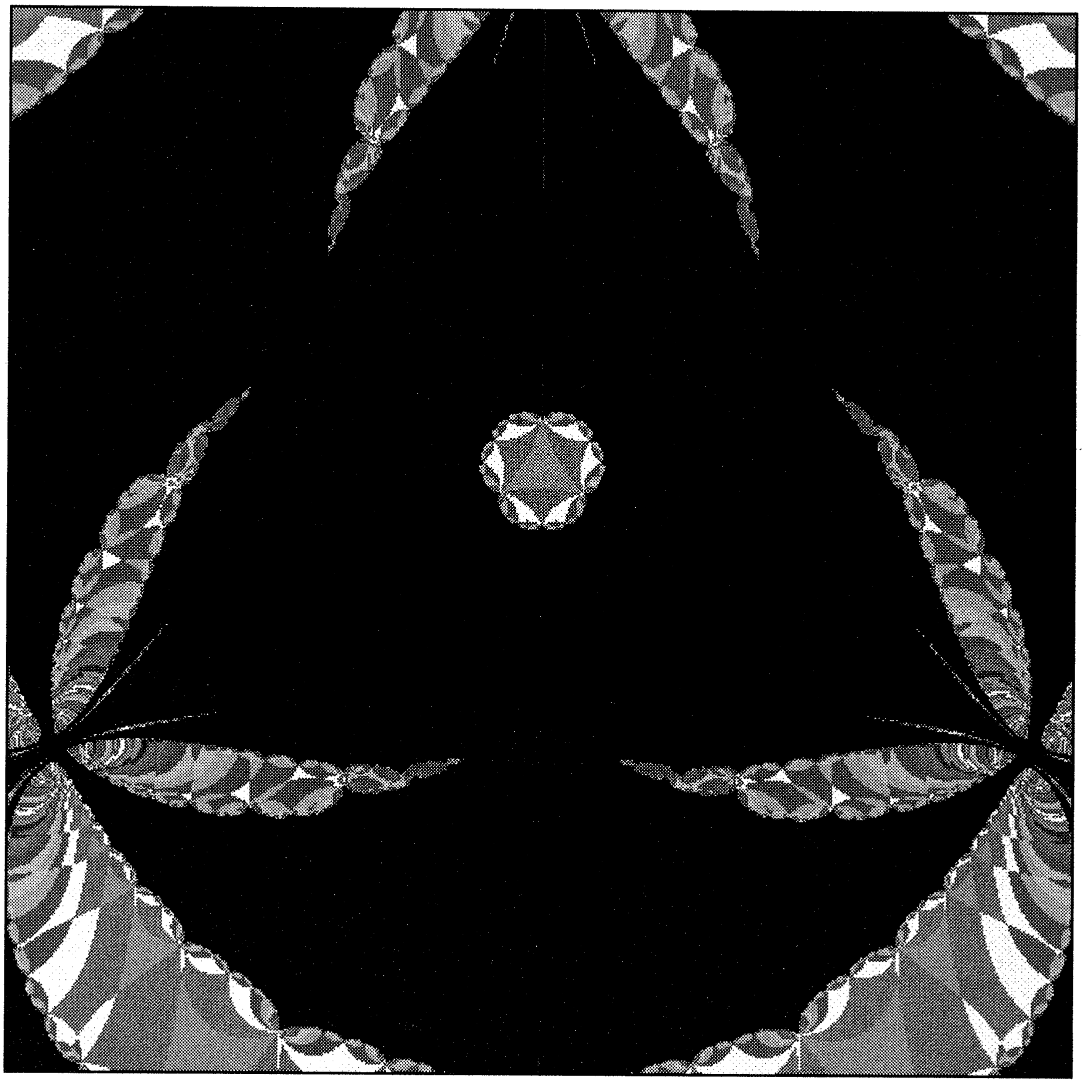

FIGURE 2. $\Gamma =$ hexagonal torus