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離散的でない horoball height を持つクライン群の例
An example of Kleinian groups with indiscrete horoball heights

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1 Introduction

Let M be a hyperbolic 3-manifold with a cusp, i.e., M is the quotient of the hyperbolic 3-space $\mathbb{H}^3$ by a torsion-free Kleinian group Γ with a parabolic element. For simplicity, we suppose that M contains precisely one cusp, i.e., all parabolic fixed points of Γ are equivalent with respect to the action of Γ. We shall identify $\mathbb{H}^3$ with the upper half of the Euclidean 3-space $\mathbb{E}^3$ so that ∞ becomes a parabolic fixed point of Γ. Let C be the *maximal cusp* of M. The *horoball pattern*, $\mathcal{H}(M)$, of M is the set of horoballs in $\mathbb{H}^3$ which project onto C and the centers are distinct from ∞. Let $h : \mathbb{H}^3 \to \mathbb{R}_+$ be the height function defined by using the coordinate of $\mathbb{E}^3$. Then the discreteness of $h(\mathcal{H}(M)) \subset \mathbb{R}_+$ is an invariant of M.

Theorem 1.1. *Suppose that* Γ *is geometrically finite. Then* $h(\mathcal{H}(M))$ *is discrete in* $\mathbb{R}_+$.

It is natural to expect that there exists a manifold M such that $h(\mathcal{H}(M))$ is indiscrete in $\mathbb{R}_+$. The main result in this paper is the following theorem. For any quasi-Fuchsian group of the once-punctured torus, Γ, we can define the end invariant $\lambda(\Gamma) = (\lambda^-(\Gamma), \lambda^+(\Gamma)) \in \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \Delta$, where Δ is the diagonal of $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$. It is proved in [7] that λ is a bijective map from the closure of the quasi-Fuchsian space of the once-punctured torus to $\overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \Delta$ and that λ^{-1} is continuous.

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Theorem 1.2. *Let λ_∞ be the real number which has the expansion into the continued fraction*

$$\lambda_\infty = [2,3,4,\dots] = \frac{1}{2+\frac{1}{3+\frac{1}{4+\cdots}}}.$$

Put $\Gamma_\zeta = \lambda^{-1}(\lambda_\infty, \zeta)$ and $M_\zeta = \mathbb{H}^3/\Gamma_\zeta$ for any $\zeta \in \mathbb{H}^2$. Then $h(\mathcal{H}(M_\zeta))$ is indiscrete in $\mathbb{R}_+$.

2 Horoball pattern

Let M be a hyperbolic 3-manifold with a single cusp. Let $\Pi : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma = M$ be the universal covering. The *maximal cusp* of M is defined as follows: Let v be a parabolic fixed point of Γ and Γ_v the stabilizer of v in Γ. Then Γ_v consists of the parabolic elements in Γ which stabilizes v. There exists a horoball H centered at v such that the quotient H/Γ_v is embedded in M. The set $H/\Gamma_v \subset M$ is called a *cusp* of M. If we gradually expand H then H/Γ_v eventually has a self-intersection in M. The maximal cusp is the subset H/Γ_v of M with this maximal size. Let $\mathcal{H}(M)$ be the set of horoballs in $\mathbb{H}^3$ which project onto the maximal cusp and the centers are distinct from v.

We shall identify $\mathbb{H}^3$ with the upper half of $\mathbb{E}^3$, i.e., $\mathbb{H}^3 = \{(x,y,z) \in \mathbb{E}^3 \mid z > 0\}$, so that v is identified with ∞. (Note that $\partial\mathbb{H}^3$ is identified with $\mathbb{C} \cup \{\infty\}$.) For a point $(x,y,z) \in \mathbb{H}^3$, we define $h(x,y,z) = z$.

Definition 2.1. For a set $X \subset \mathbb{H}^3$, the *Euclidean height* $h(X)$ of X is defined by

$$h(X) = \sup\{h(x) \mid x \in X\}.$$

We remark that the discreteness of $h(\mathcal{H}(M)) \subset M$ is independent of the choice of a parabolic fixed point v and an identification of $\mathbb{H}^3$ with the upper half space.

In the following, we prove a stronger version of Theorem 1.1 (Theorem 2.3).

Definition 2.2. (1) The *rank* of a parabolic fixed point v of Γ is the rank of an abelian group Γ_v.

(2) Suppose that the rank of v is one. We say that v is *doubly cusped* if there exist two open round disks in $\Omega(\Gamma)$ which are disjoint and stabilized by Γ_v, where $\Omega(\Gamma)$ denotes the domain of discontinuity of Γ.

(3) A parabolic fixed point of Γ is said to be *bounded* if (i) it is of rank 2 or (ii) it is of rank 1 and doubly cusped.

Theorem 2.3. *Suppose that ∞ is a bounded parabolic fixed point of Γ. Then $h(\mathcal{H}(M))$ is discrete in $\mathbb{R}_+$.*

We remark that Theorem 1.1 follows immediately from Theorem 2.3 and Proposition 2.4 below. (See [6, Chapter VI, Proposition A.10] for example.)

Proposition 2.4. *Suppose that Γ is geometrically finite. Then any parabolic fixed point of Γ is bounded.*

Proof of Theorem 2.3. Since ∞ is bounded, there exists a compact subset K of $\mathbb{C}$ with the following property: For any $w \in \Lambda(\Gamma) - \{\infty\}$, there exists $\gamma \in \Gamma_\infty$ such that $\gamma w \in K$, where $\Lambda(\Gamma)$ denotes the limit set of Γ. Suppose that $h(\mathcal{H}(M))$ is indiscrete in $\mathbb{R}_+$. Note that $\gamma H \in \mathcal{H}(M)$ for any $\gamma \in \Gamma$ and $H \in \mathcal{H}(M)$ and that each element of Γ_∞ keeps the Euclidean heights of horoballs as it is a Euclidean parallel translation of the upper half space. Thus there exists a sequence of horoballs $\{H_n\} \subset \mathcal{H}(M)$ such that the sequence $\{h(H_n)\}$ converges to some point $h_\infty \in \mathbb{R}_+$, $h(H_n) \neq h_\infty$ for any $n \in \mathbb{N}$ and that the centers of H_n ($n \in \mathbb{N}$) are contained in K. By taking a subsequence, which we denote by the same symbol, we may assume that the horoballs H_n ($n \in \mathbb{N}$) are distinct from one another. Then, from the definition, they are mutually disjoint in the interior. Since $\{h(H_n)\}$ converges to $h_\infty \in \mathbb{R}_+$, there exist two positive numbers h_+ and h_- such that $h_- \leq h(H_n) \leq h_+$ for any $n \in \mathbb{N}$. Thus the Euclidean volume of each H_n ($n \in \mathbb{N}$) is bounded below by a positive number. On the other hand, each H_n is contained in the set $\{(x, y, z) \,|\, (x, y) \in B(K, h_+/2),\ z \leq h_+\}$ whose Euclidean volume is equal to $\mathrm{Area}(B(K, h_+/2))h_+ < \infty$, where $B(K, h_+/2)$ denotes the $(h_+/2)$-neighborhood of K. This is a contradiction. $\square$

3 Punctured torus groups

For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{C})$ with $c \neq 0$, the isometric hemisphere $Ih(\gamma)$ of γ is the Euclidean hemisphere with equator $\{z \in \mathbb{C} \,|\, |cz + d| = 1\}$. For a Kleinian group Γ, let $\mathcal{I}(\Gamma)$ be the set of isometric hemispheres defined by

$$\mathcal{I}(\Gamma) = \{Ih(\gamma) \,|\, \gamma \in \Gamma,\ \gamma(\infty) \neq \infty\}.$$

In this section, we study the Euclidean heights of isometric hemispheres which support faces of the Ford domain of a once-punctured torus group. This can be used to prove Theorem 1.2 by the following lemma.

Lemma 3.1. *For a hyperbolic 3-manifold with a single cusp $M = \mathbb{H}^3/\Gamma$, $h(\mathcal{H}(M))$ is discrete in $\mathbb{R}_+$ if and only if $h(\mathcal{I}(\Gamma))$ is discrete in $\mathbb{R}_+$.*

Proof. Let H be a horoball in $\mathbb{H}^3$ which projects onto the maximal cusp. We can see that $h(\gamma H) = 1/(|c|^2 h(\partial H))$ for any $\gamma \in \Gamma$ with $\gamma(\infty) \neq \infty$. Thus we have

$$h(\mathcal{H}(M)) = \{1/(|c|^2 h(\partial H)) \mid \gamma \in \Gamma,\ \gamma(\infty) \neq \infty\}.$$

On the other hand, from the definition, we have

$$h(\mathcal{I}(\Gamma)) = \{1/|c| \mid \gamma \in \Gamma,\ \gamma(\infty) \neq \infty\}.$$

Thus it is obvious that $h(\mathcal{H}(M))$ is discrete in $\mathbb{R}_+$ if and only if $h(\mathcal{I}(\Gamma))$ is discrete in $\mathbb{R}_+$. $\square$

Let T be the once-punctured torus and $\rho_0' : \pi_1(T) \to PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$ its Fuchsian representation. The *quasi-Fuchsian space* QF of the once-punctured torus is the set of quasi-conformal deformations of ρ_0 quotiented by the conjugation in $PSL(2,\mathbb{C})$ and equipped with the algebraic topology. We denote the closure of QF in the representation space of $\pi_1(T)$ by $\overline{QF}$. In this paper, we loosely identify an element of $\overline{QF}$ and its image in $PSL(2,\mathbb{C})$.

For any $\Gamma \in QF$, $\mathbb{H}^3/\Gamma$ is homeomorphic to $T \times (-1,1)$ and hence has two ends $\mathcal{E}^{\pm}$. We can associate an end invariant $\lambda(\Gamma) = (\lambda^-(\Gamma), \lambda^+(\Gamma))$ with Γ as follows:

(1) If the end $\mathcal{E}^{\epsilon}$ is geometrically finite, then $\lambda^{\epsilon}(\Gamma)$ is the marked conformal structure of the Riemann surface at infinity.

(2) If the end $\mathcal{E}^{\epsilon}$ is geometrically infinite, then $\lambda^{\epsilon}(\Gamma)$ is the ending lamination of the end.

Then each $\lambda^{\pm}(\Gamma)$ is defined as the point in the closure of the Teichmüller space of T, which is isomorphic to $\overline{\mathbb{H}^2}$.

Theorem 3.2 ([7]). $\lambda : \overline{QF} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \Delta$ *is bijective and* λ^{-1} *is continuous.*

To prove Theorem 1.2, it is convenient to study the representations of $\pi_1(T)$. (See [4] and [2, 3] for detail.) The once-punctured torus T has the symmetry τ depicted in Figure 1. Let $\mathcal{O}$ be the quotient of T by $\langle\tau\rangle$, which is the orbifold $S^2(\infty,2,2,2)$. Let $p : T \to T/\langle\tau\rangle = \mathcal{O}$ be the covering projection.

By the following proposition, we can study the elements of $\overline{QF}$ by using a representation of $\pi_1^{orb}(\mathcal{O})$. In the rest of this paper, we regard QF as a set of representations of $\pi_1^{orb}(\mathcal{O})$.

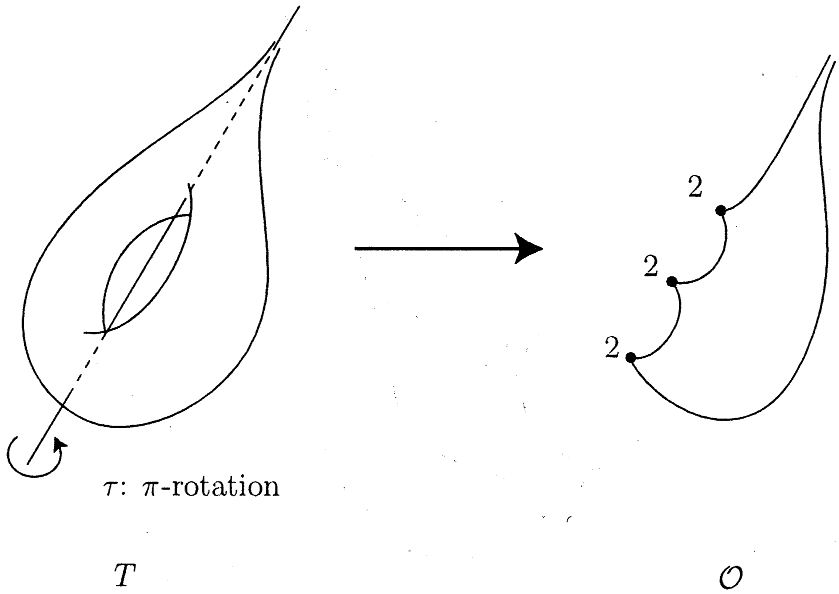

Figure 1: Covering $p : T \to \mathcal{O}$

Proposition 3.3. *For any $\rho \in \overline{QF}$, there exists a unique representation $\widetilde{\rho} : \pi_1^{orb}(\mathcal{O}) \to PSL(2, \mathbb{C})$ such that $\widetilde{\rho} \circ p_* = \rho$.*

We can see that the fundamental group of $\mathcal{O}$ has the following presentation:

$$\pi_1^{orb}(\mathcal{O}) = \langle P_0, Q_0, R_0 \mid P_0^2 = Q_0^2 = R_0^2 = 1 \rangle,$$

where each P_0, Q_0 and R_0 is represented by a loop which goes around a branch point. (See Figure 2.) Put $K = R_0 Q_0 P_0$. Then K is represented by a loop which goes around the puncture.

Definition 3.4 (Elliptic generators). (1) A triple (P, Q, R) of elements of $\pi_1^{orb}(\mathcal{O})$ is called an *elliptic generator triple* if the following conditions are satisfied:

(i) $\pi_1^{orb}(\mathcal{O}) = \langle P, Q, R \rangle$.

(ii) $P^2 = Q^2 = R^2 = 1$ and $RQP = K$.

(2) An element P of $\pi_1^{orb}(\mathcal{O})$ is said to be an *elliptic generator* if there exist $Q, R \in \pi_1^{orb}(\mathcal{O})$ such that (P, Q, R) is an elliptic generator triple.

Remark 3.5. For an elliptic generator triple (P, Q, R), put $A = KP$ and $B = K^{-1}R$. Then $p_*(\pi_1(T)) = \langle A, B \rangle$ and $ABA^{-1}B^{-1} = K^2$.

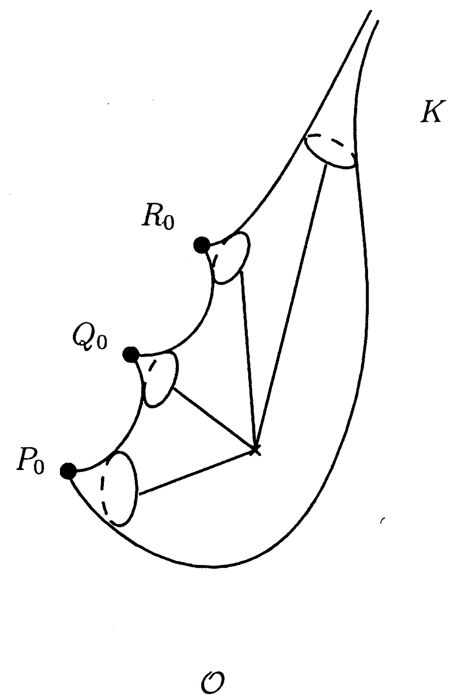

Figure 2: Generators of $\pi_1^{orb}(\mathcal{O})$

Let $\mathcal{D}^{(0)}$ be the isotopy classes of essential simple closed curves in T. Then $\mathcal{D}^{(0)}$ can be identified with $\mathbb{Q} \cup \{\infty\} \subset \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}^2$. Let σ_0 be the geodesic triangle in $\overline{\mathbb{H}^2}$ spanned by $\infty, 0, 1$, which we denote by $\langle \infty, 0, 1 \rangle$.

Definition 3.6 (Modular diagram). The *modular diagram* $\mathcal{D}$ is the simplicial complex defined by the triangulation $\{\gamma\sigma_0 \mid \gamma \in SL(2,\mathbb{Z})\}$ of $\mathbb{H}^2 \cup \mathcal{D}^{(0)}$.

By the definition, the element $KP \in p_*(\pi_1(T))$ is represented by an essential simple closed curve C in T for any elliptic generator P. We denote the isotopy class of C by $s(P)$, and call it the *slope* of P.

Lemma 3.7. (1) *For elliptic generators P and P', $s(P) = s(P')$ if and only if $P' = K^n P K^{-n}$ for some $n \in \mathbb{Z}$.*

(2) *For any elliptic generator triple (P, Q, R), the three points $s(P)$, $s(Q)$ and $s(R)$ span a triangle in $\mathcal{D}$.*

(3) *For any triangle σ in $\mathcal{D}$, there exists an elliptic generator triple (P, Q, R) such that $\sigma = \langle s(P), s(Q), s(R) \rangle$.*

Let $\rho : \pi_1(T) \to PSL(2,\mathbb{C})$ be a representation in $\overline{QF}$. Then ρ lifts to a representation $\widehat{\rho} : \pi_1(T) \to SL(2,\mathbb{C})$. We define the *Markoff map* $\phi : \mathcal{D}^{(0)} \to \mathbb{C}$ by $\phi(s(P)) = \operatorname{tr}\widehat{\rho}(KP)$.

Lemma 3.8. (1) *For any triangle* $\langle s_0, s_1, s_2 \rangle$ *in* $\mathcal{D}$,

$$\phi(s_0)^2 + \phi(s_1)^2 + \phi(s_2)^2 = \phi(s_0)\phi(s_1)\phi(s_2).$$

(2) *For any different triangles* $\langle s_0, s_1, s_2 \rangle$ *and* $\langle s_0, s_1, s_2' \rangle$ *in* $\mathcal{D}$,

$$\phi(s_2) + \phi(s_2') = \phi(s_0)\phi(s_1).$$

Remark 3.9. (1) By Lemma 3.8(2), a Markoff map is determined from the values at the vertices of a single triangle in $\mathcal{D}$.

(2) We can see that any Markoff map induces a unique representation of $\pi_1^{orb}(\mathcal{O})$ to $PSL(2,\mathbb{C})$.

In [5], Jorgensen studies the Ford domains of quasi-Fuchsian groups of the once-punctured torus. We can apply the argument to the boundary groups of quasi-Fuchsian space of once-punctured torus. (See [1] for an outline.) We can use several results obtained by this study. For the rest of this paper, we suppose that $\rho(K) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ for any $\rho \in \overline{QF}$.

Lemma 3.10. *Let* $\rho \in \overline{QF}$. *For any elliptic generator* P *with* $\rho(P)(\infty) \neq \infty$, $h(Ih(\rho(P)))$ *is equal to* $1/|\phi(s(P))|$, *where* ϕ *is a Markoff map which induces* ρ.

Definition 3.11. Let λ_∞ be the real number which has the expansion into the continued fraction $\lambda_\infty = [2, 3, \dots]$. For $\zeta \in \mathbb{H}^2$, let $\Gamma_\zeta = \lambda^{-1}(\lambda_\infty, \zeta)$ and ϕ_ζ be a Markoff map which induces $\rho_\zeta \in \overline{QF}$ with $\operatorname{Im}\rho_\zeta = \Gamma_\zeta$. (See Figure 3[1].)

Let s_n be the rational number which has the expansion into the continued fraction $s_n = [2, 3, \dots, n]$. Since any parabolic element in Γ_ζ is the image of an element which is conjugate in $\pi_1^{orb}(\mathcal{O})$ into the cyclic group $\langle K \rangle$, the following lemma holds.

Lemma 3.12. *No* $\phi_\zeta(s_n)$ $(n \in N)$ *is equal to* ± 2.

As a corollary to the characterization of the Ford domains of once-punctured torus groups, we have the following lemma (cf. [5, Lemma 5]).

[1]This figure is drawn by using OPTi [8].

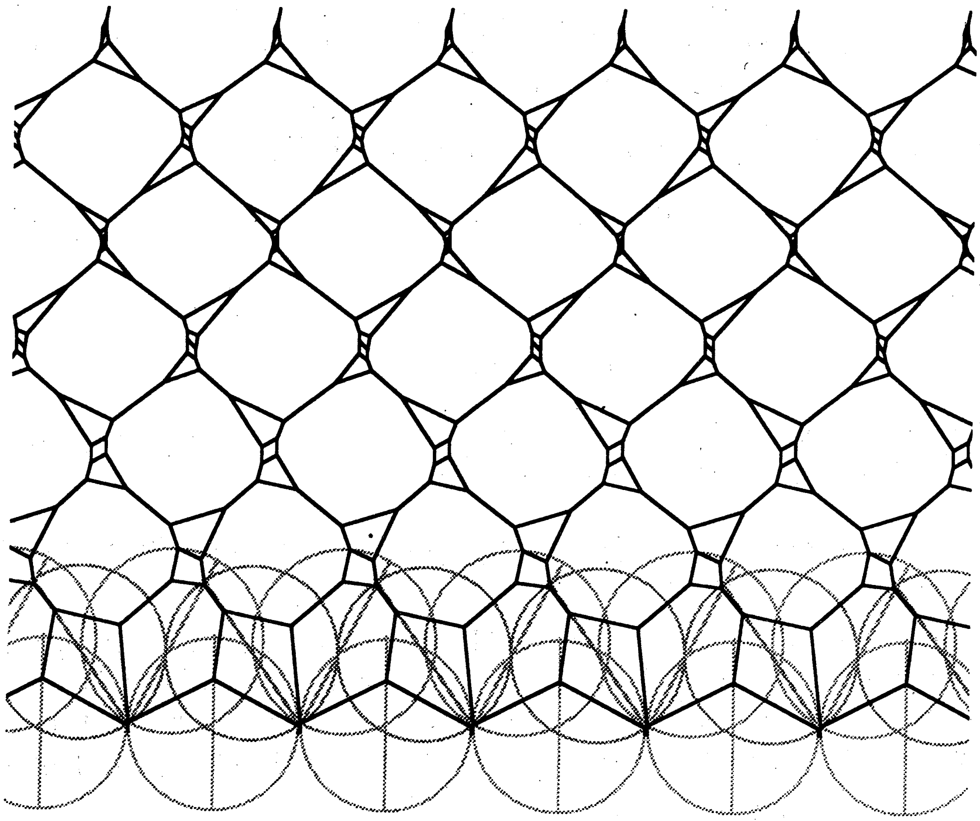

Figure 3: Ford domain of Γ_ζ for some $\zeta \in \mathbb{H}^2$

Lemma 3.13. *There exists a subsequence of* $\{s_n\}$*, which we denote by the same symbol, such that the sequence* $\{\phi_\zeta(s_n)\}$ *converges to one of* ± 2.

Proof of Theorem 1.2. Let s'_n and s''_n ($n \in \mathbb{N}$) be the points in $\mathcal{D}^{(0)}$ depicted in Figure 4. Let x_n (resp. y_n, z_n and w_n) be the value of ϕ_ζ at s_n (resp. s_{n-1}, s'_n and s''_n). Then, by Lemma 3.13, there exists a subsequence of $\{s_n\}$, which we denote by the same symbol, such that each $\{x_n\}$ and $\{y_n\}$ converges to one of ± 2. We may suppose that both $\{x_n\}$ and $\{y_n\}$ converge to 2, if necessary, by changing ϕ_ζ to another Markoff map which induces ρ_ζ. Then, by Lemma 3.8,

$$x_n^2 + y_n^2 + z_n^2 = x_n y_n z_n, \tag{3.1}$$

$$y_n + w_n = x_n z_n. \tag{3.2}$$

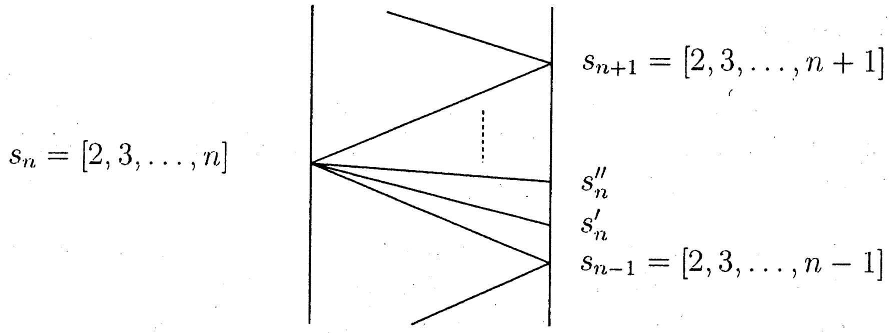

Figure 4: Slopes s_n, s'_n and s''_n

By (3.1), we have

$$z_n = \frac{x_n y_n \pm \sqrt{x_n^2 y_n^2 - 4(x_n^2 + y_n^2)}}{2}. \tag{3.3}$$

Thus, by taking a subsequence, $\{z_n\}$ converges to $2(1 + \epsilon\sqrt{-1})$ ($\epsilon \in \{\pm 1\}$). Then, by (3.2), $\{w_n\}$ converges to $2(1 + 2\epsilon\sqrt{-1})$.

Suppose that $h(\mathcal{I}(\Gamma_\varsigma))$ is discrete in $\mathbb{R}_+$. Then, by Lemma 3.10, each $\{|x_n| \mid n \in \mathbb{N}\}$, $\{|y_n| \mid n \in \mathbb{N}\}$, $\{|z_n| \mid n \in \mathbb{N}\}$ and $\{|w_n| \mid n \in \mathbb{N}\}$ is a finite set. Hence both $|x_n|$ and $|y_n|$ are equal to 2, $|z_n|$ is equal to $|2(1+\epsilon\sqrt{-1})| = 2\sqrt{2}$ and $|w_n|$ is equal to $|2(1 + 2\epsilon\sqrt{-1})| = 2\sqrt{5}$ for sufficiently large n. Put $x_n = 2e^{\theta_n\sqrt{-1}}$, $y_n = 2e^{\varphi_n\sqrt{-1}}$ and $z_n = 2\sqrt{2}e^{\psi_n\sqrt{-1}}$ for such n. Since both $\{x_n\}$ and $\{y_n\}$ converge to 2 and $\{z_n\}$ converges to $2(1 + \epsilon\sqrt{-1})$, both $\{\theta_n\}$ and $\{\varphi_n\}$ converge to 0 and $\{\psi_n\}$ converges to $\epsilon\pi/2$. By (3.2), we have $|x_n z_n - y_n| = |w_n|$ Thus

$$|4\sqrt{2}e^{(\theta_n+\psi_n)\sqrt{-1}} - 2e^{\varphi_n\sqrt{-1}}| = 2\sqrt{5},$$

and hence $\varphi_n - \theta_n - \psi_n = \epsilon\pi/4$. Then, by (3.3),

$$2\sqrt{2}e^{(\varphi_n - \theta_n - \epsilon\pi/4)\sqrt{-1}} = 2e^{(\theta_n+\varphi_n)\sqrt{-1}}\left(1 + \epsilon\sqrt{1 - (e^{-2\theta_n\sqrt{-1}} + e^{-2\varphi_n\sqrt{-1}})}\right),$$

and hence

$$e^{-2\varphi_n\sqrt{-1}} = 2\epsilon\sqrt{-1}e^{-4\theta_n\sqrt{-1}} + (1 - 2\epsilon\sqrt{-1})e^{-2\theta_n\sqrt{-1}}. \tag{3.4}$$

Note that the absolute value of the left hand side of (3.4) is equal to 1. Put $f(\theta) = |2\sqrt{-1}e^{-4\theta\sqrt{-1}} + (1 - 2\sqrt{-1})e^{-2\theta\sqrt{-1}}|$. Then $\frac{df}{d\theta}(0)$ is not equal to 0. Therefore θ_n is equal to 0 for sufficiently large n. This contradicts Lemma 3.12. □

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