New coefficient inequalities for starlike and convex functions

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Abstract. The object of the present paper is to derive new coefficient inequalities for univalent and starlike, and univalent and convex functions defined in the open unit disk $U$. Our results are the improvements of the previous theorems given by J. Clunie and F. R. Keogh ([1]) and by H. Silverman ([2]).

1 Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad (a_1 = 1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in A$ is said to be univalent and starlike in $U$ if it satisfies

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0$$

for all $z \in U$. Also a function $f(z) \in A$ is said to be univalent and convex in $U$ if it satisfies

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0$$

for all $z \in U$.

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Clunie and Keogh ([1]) (also Silverman ([2])) have proved the following result: If $f(z) \in A$ satisfies
\[
\sum_{n=2}^{\infty} n|a_n| \leq 1,
\]
then $f(z)$ is univalent and starlike in $U$. If $f(z) \in A$ satisfies
\[
\sum_{n=2}^{\infty} n^2|a_n| \leq 1,
\]
then $f(z)$ is univalent and convex in $U$.
In the present paper, we consider new coefficient inequalities for functions $f(z)$ to be univalent and starlike, and univalent and convex in $U$.

2 Coefficient inequalities
Our main result for the coefficient inequality of $f(z)$ to be univalent and starlike in $U$ is contained in

**Theorem 1.** Let $f(z)$ be in the class $A$ and
\[
\max_{n \geq 1} n|a_n| = p|a_p|.
\]
If $f(z)$ satisfies
\[
\sum_{n=1, n \neq p}^{\infty} (|n-p|+p)|a_n| \leq p|a_p|,
\]
then $f(z)$ is univalent and starlike in $U$.

**Proof.** Applying the maximum principle of analytic functions, the following inequality holds true on $|z|=1$
\[
|zf'(z) - pf(z)| - |pf(z)| = \left| \sum_{n=1}^{\infty} (n-p)a_n z^n \right| - p \left| \sum_{n=1}^{\infty} a_n z^n \right|
\]
\[
\leq \sum_{n=1}^{\infty} |n-p| |a_n| |z|^n - p \left( |a_p| |z|^p - \sum_{n=1, n \neq p}^{\infty} |a_n| |z|^n \right)
\]
\[
= \sum_{n=1, n \neq p}^{\infty} (|n-p|+p) |a_n| - p |a_p| \leq 0.
\]
Therefore, it follows that
\[
\left| \frac{zf'(z)}{f(z)} - p \right| < p
\]
for all $z \in U$. This shows that $f(z)$ is univalent and starlike in $U$. \qed
Remark 1. If
\[ \max_{n \geq 1} n |a_n| = |a_1| = 1, \]
then Theorem 1 becomes the result by Clunie and Keogh ([1]) (also by Silverman ([2]).

Corollary 1. If a function \( f(z) \in A \) satisfies
\[ \max_{n \geq 1} n |a_n| = 2|a_2| \]
and
\[ \sum_{n=3}^{\infty} n |a_n| \leq 2|a_2| - 3, \]
then \( f(z) \) is univalent and starlike in \( U \).

By means of the definitions between starlike functions and convex functions, it follows that \( f(z) \in A \) is univalent and convex in \( U \) if and only if \( zf'(z) \) is univalent starlike in \( U \). Therefore Theorem 1 gives us

Theorem 2. Let \( f(z) \) be in the class \( A \) and
\[ \max_{n \geq 1} n^2 |a_n| = p^2 |a_p|. \]
If \( f(z) \) satisfies
\[ \sum_{n=1, n \neq p}^{\infty} n(|n-p|+p) |a_n| \leq p^2 |a_p|, \]
then \( f(z) \) is univalent and convex in \( U \).

Remark 2. If
\[ \max_{n \geq 1} n^2 |a_n| = |a_1| = 1, \]
then Theorem 2 becomes the result by Silverman ([2]).

Corollary 2. If a function \( f(z) \in A \) satisfies
\[ \max_{n \geq 1} n^2 |a_n| = 4|a_2| \]
and
\[ \sum_{n=3}^{\infty} n |a_n| \leq 4|a_2| - 3, \]
then \( f(z) \) is univalent and convex in \( U \).
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