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"Forbidden divisor" characterizations of epigroups with certain properties of group elements

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Introduction

A semigroup S is called an *epigroup* if, for any $a \in S$, there exists a positive integer n such that a^n is a *group element*, that is, belongs to a subgroup of S. Epigroups often occur in the literature under the name *group-bound semigroups*; the term *epigroup* which is shorter and more flexible was suggested by Shevrin who also promoted the idea of viewing epigroups as semigroups with an additional unary operation [5]. Indeed, it is well known (and easy to verify) that, for any element a of an epigroup S, there exists a unique maximal subgroup H of S which contains all but a finite number of powers of a. We denote the identity element of this group H by a^0. Then $aa^0 = a^0a \in H$ [2], and we denote the inverse of $aa^0 = a^0a$ in H by $\overline{a}$. Thus, we can introduce a new unary operation $a \mapsto \overline{a}$ on any epigroup.

For an epigroup S, we denote by $\mathrm{Gr}\, S$ the set of all its group elements, in other words, the union of all subgroups of S. In [5, §7] Shevrin has considered epigroups in which $\mathrm{Gr}\, S$ is a retract. (Recall that a subsemigroup T of a semigroup S is a *retract* if there is a homomorphism $S \to T$ which is identical on T.) He has shown [5, Lemma 8] that $\mathrm{Gr}\, S$ is a retract if and only if S satisfies the identity

$$\overline{\overline{ab}} = \overline{\overline{a}} \cdot \overline{\overline{b}} \tag{1}$$

and posed the problem of characterizing such epigroups S in terms of "forbidden divisors". By a *divisor* D of an epigroup S we mean any homomorphic image of a *subepigroup* of S (i.e. a subsemigroup of S closed under the operation $a \mapsto \overline{a}$). A "forbidden divisor" characterization for epigroups with a property Θ is a list of epigroups such that S satisfies Θ if and only if none of the epigroups listed divide S. Characterizations of such kind are known to be very handy, especially when the list of "forbidden divisors" consists of a few finite semigroups.

The aim of the present note is to solve Shevrin's problem by exhibiting a list of "forbidden divisors" containing one 5-element and three 4-element semigroups (see Section 2). We also obtain characterizations of similar flavour for epigroups distinguished by further natural properties of their group elements including the most important one "$\mathrm{Gr}\, S$ *is a subsemigroup in* S" (see Sections 1 and 3).

Every finite (more generally, every periodic) semigroup is an epigroup, whence the results of our note apply to corresponding classes of finite semigroups. It is worth mentioning that the operation $a \mapsto \overline{a}$ defined above is widely used in the theory of finite semigroups where the usual notation for what we have denoted by $\overline{a}$ is $a^{\omega-1}$. We have chosen to adopt the notation from [5] instead because it is more convenient when one has to apply the operation repeatedly as in (1) above. The rest of our notation is standard and follows [1].

1 Epigroups whose group elements form a subsemigroup

To formulate the main result of this section, we need a construction due to Reilly [4]. Let G be a group. Define a multiplication on the disjoint union $R(G)$ of G^0 with the cartesian square $G \times G$ by preserving the multiplication on G^0 and letting, for all $g \in G$, $[h,j],[k,\ell] \in G \times G$,

$$\begin{aligned} [h,j]g &= [h,jg], \\ g[h,j] &= [hg^{-1},j], \\ [h,j]0 &= 0[h,j] = 0, \\ [h,j][k,\ell] &= \begin{cases} [h,\ell] & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then $R(G)$ becomes a semigroup being obviously an epigroup. We note that $\operatorname{Gr} R(G)$ fails to form a subsemigroup provided that $|G| > 1$. Indeed, if e is the identity element of G and g another element, then both g and $[e,e]$ belong to $\operatorname{Gr} R(G)$, while their product $[e,e]g = [e,g]$ does not.

Denote by $\mathbf{C}_\infty$ (resp., $\mathbf{C}_p$) the infinite cyclic group (resp., the cyclic group of the prime order p) and by $\mathbf{V}$ the semigroup defined (in the class of semigroups with 0) by the presentation

$$\langle e, f \mid e^2 = e,\ f^2 = f,\ fe = 0 \rangle.$$

Obviously, $\mathbf{V}$ consists of 4 elements: e, f, ef and 0, and $\operatorname{Gr}\mathbf{V} = \{e, f, 0\}$ is not a subsemigroup.

Theorem 1.1 *Let S be an epigroup. The set $\operatorname{Gr} S$ forms a subsemigroup in S if and only if none of the semigroups $R(\mathbf{C}_\infty)$, $R(\mathbf{C}_p)$ (p runs over the set of all primes), and $\mathbf{V}$ divide S.*

Proof. Necessity. An element $a \in S$ is a group element if and only if $a = \overline{\overline{a}}$. Therefore $\operatorname{Gr} S$ forms a subsemigroup in S if and only if S satisfies the identity

$$\overline{\overline{\overline{\overline{a}} \cdot \overline{\overline{b}}}} = \overline{\overline{a}} \cdot \overline{\overline{b}}.$$

All identities of S are inherited by its divisors, and therefore, $\operatorname{Gr} D$ is a subsemigroup in each divisor D of S. We have observed that the group elements of none of the semigroups $R(\mathbf{C}_\infty)$, $R(\mathbf{C}_p)$, and $\mathbf{V}$ form a subsemigroup, hence none of the semigroups divide S.

Sufficiency. We start with an observation clarifying the role of the semigroup $\mathbf{V}$:

Proposition 1.2 *Let S be an epigroup. Then the semigroup $\mathbf{V}$ divides S if and only if there exist idempotents $e, f \in S$ such that $ef \notin \operatorname{Gr} S$.*

Proof. Necessity. If $ef \notin \operatorname{Gr} S$ for all idempotents $e, f \in S$, then S satisfies the identity

$$\overline{\overline{a^0 b^0}} = a^0 b^0,$$

and so does every divisor of S. Obviously the identity fails in $\mathbf{V}$, hence $\mathbf{V}$ does not divide S.

Sufficiency. Let e, f be two idempotents in S such that $ef \notin \operatorname{Gr} S$. Clearly, we may assume that S is generated by e and f (as an epigroup). Under this assumption, we are going to show that

$$S = \{e, f, ef\} \cup feS \cup efeS. \tag{2}$$

Indeed, each element in S can be represented as a unary semigroup term $t(e,f)$, the word "unary" meaning that, along with the multiplication, we may apply the unary operation $a \mapsto \overline{a}$. We shall induct on the construction of the term $t(e,f)$. If it involves no operation at all, then $t(e,f) = e$ or $t(e,f) = f$. Suppose that $t(e,f) = t_1(e,f) \cdot t_2(e,f)$, where $t_1(e,f)$ and $t_2(e,f)$ are "shorter" terms to which the induction assumption can be applied. If $t_1(e,f)$ belongs to $I = feS \cup efeS$, then obviously $t(e,f)$ does so. If $t_1(e,f) \in \{e,f,ef\}$, while $t_2(e,f)$ belongs to I, then again $t(e,f) \in I$ since $\{e,f,ef\}I \subseteq I$ as one readily checks. If $t_1(e,f), t_2(e,f) \in \{e,f,ef\}$, then $t(e,f) \in \{e,f,ef,fe,efe,fef,efef\} \subseteq \{e,f,ef\} \cup I$. Now suppose that $t(e,f) = \overline{s(e,f)}$ where again $s(e,f)$ is a "shorter" term. In view of the identity

$$\overline{a} = a\overline{a}^2 \tag{3}$$

holding in every epigroup (see [5, §1]), $t(e,f)$ belongs to $s(e,f)S$, whence $t(e,f)$ lies in I whenever $s(e,f)$ does. If $s(e,f) = e$ or $s(e,f) = f$, then $t(e,f) = e$ or, resp., $t(e,f) = f$. Finally, if $s(e,f) = ef$, then, using the identity (3) twice, we obtain

$$t(e,f) = \overline{ef} = ef \cdot \overline{ef} \cdot \overline{ef} = (ef)^2(\overline{ef})^3 \in efeS \subseteq I.$$

Now, from the decomposition (2) and the inclusion $\{e,f,ef\}I \subseteq I$ mentioned above, we conclude that $I = feS \cup efeS$ is in fact an ideal of S. Consider two cases: $ef \notin I$ and $ef \in I$.

Case 1: $ef \notin I$. Then we also have $e, f \notin I$ and $e \neq f$. Thus, the Rees quotient S/I consists of 4 elements: e, f, ef and 0, and the relations $e^2 = e$, $f^2 = f$ and $fe = 0$ hold. This means that S/I is isomorphic to the semigroup $\mathbf{V}$, whence $\mathbf{V}$ divides S, as required.

Case 2: $ef \in I$. This means that $ef = fes$ or $ef = efes$ for some $s \in S$. Since multiplying the former equality on the left by e yields the latter one, we may assume that $ef = efes$. Using (2) and the condition of the case under consideration, we see that there are two possibilities for the element s: either $s \in \{e,f\}$ or $s \in I$. If $s \in \{e,f\}$, then multiplying the equality $ef = efes$ by f on the right yields $ef = efef$, whence ef is an idempotent. This contradicts the hypothesis of our lemma (ef is not a group element). Hence we conclude that $s \in I = feS \cup efeS$ and $ef = efes \in (ef)^2S$. Obviously, $(ef)^2 \in efS$, and thus, we see that ef and $(ef)^2$ generate the same right ideal in S, in other words, these elements are in the Green relation $\mathscr{R}$.

We have proved that either the semigroup $\mathbf{V}$ divides S or $ef \mathrel{\mathscr{R}} (ef)^2$. Observe that the condition of our lemma is self-dual, and hence it must imply the dual conclusion as well. This dual conclusion says that either $\mathbf{V}$ divides S or $ef \mathrel{\mathscr{L}} (ef)^2$. However, if both $ef \mathrel{\mathscr{R}} (ef)^2$ and $ef \mathrel{\mathscr{L}} (ef)^2$, then $ef \mathrel{\mathscr{H}} (ef)^2$, and by Green's theorem [1, Theorem 2.16] the $\mathscr{H}$-class of the element ef is a group, a contradiction. ■

Recall that a semigroup S is called *E-solid* if for all idempotents $e,f,g \in S$ such that $e \mathrel{\mathscr{L}} g \mathrel{\mathscr{R}} f$, there exists an idempotent $h \in S$ such that $e \mathrel{\mathscr{R}} h \mathrel{\mathscr{L}} f$. For a group element a of S (not necessarily being an epigroup), we still denote by a^0 the identity element of the maximal subgroup containing a.

Lemma 1.3 *Every semigroup S in which the product of two arbitrary idempotents is a group element is E-solid.*

Proof. Take three idempotents $e,f,g \in S$ such that $e \mathrel{\mathscr{L}} g \mathrel{\mathscr{R}} f$. We then are in a position to apply a theorem by Miller and Clifford [1, Theorem 2.17] which shows that $e \mathrel{\mathscr{R}} ef \mathrel{\mathscr{L}} f$. Since ef is a group element, $ef \mathrel{\mathscr{H}} (ef)^0$ and $e \mathrel{\mathscr{R}} (ef)^0 \mathrel{\mathscr{L}} f$. ■

The rest of the proof closely follows the ideas of Pastijn and the author's paper [3], where *regular* semigroups whose group elements form a subsemigroup have been studied. In particular, the next two lemmas have been proved in [3] for E-solid regular semigroups.

Lemma 1.4 *Let S be a semigroup in which the product of two arbitrary idempotents is a group element, $a, b \in \operatorname{Gr} S$. If $ab \notin \operatorname{Gr} S$, then either $ab^0 \notin \operatorname{Gr} S$ or $a^0 b \notin \operatorname{Gr} S$.*

Proof. Clearly, $ab^0 \mathscr{L} a^0b^0 \mathscr{R} a^0b$. Suppose $ab^0, a^0b \in \operatorname{Gr} S$. Then since $a^0b^0 \in \operatorname{Gr} S$, we can substitute the corresponding idempotents for all these three elements getting $(ab^0)^0 \mathscr{L} (a^0b^0)^0 \mathscr{R} (a^0b)^0$. Since S is E-solid by Lemma 1.3, there exists an idempotent, say h, such that $(ab^0)^0 \mathscr{R} h \mathscr{L} (a^0b)^0$ and hence $ab^0 \mathscr{R} h \mathscr{L} a^0b$. On the other hand, $ab^0 \mathscr{R} ab \mathscr{L} a^0b$. Combining these two observations, we get $ab \mathscr{H} h$ and therefore $ab \in \operatorname{Gr} S$, a contradiction. ∎

On the set $E(S)$ of all idempotents of a given semigroup S we consider the usual partial order:

$$e \leq f \text{ if and only if } ef = fe = e.$$

Lemma 1.5 *Let S be a semigroup in which the product of two arbitrary idempotents is a group element, $a \in GrS$, $f \in E(S)$. If $af \notin \operatorname{Gr} S$, then there exists an idempotent d such that $d \leq a^0$ and $ad \notin \operatorname{Gr} S$.*

Proof. Let $d = (a^0f)^0a^0$. Then d is easily seen to be an idempotent, $d \leq a^0$, and $df = (a^0f)^0a^0f = a^0f$ since a^0f is a group element. This immediately implies that $ad \mathscr{L} d \mathscr{R} a^0f$. Were ad a group element, we would also have $(ad)^0 \mathscr{L} d \mathscr{R} (a^0f)^0$. Since S is E-solid by Lemma 1.3, there exists an idempotent, say h, such that $(ad)^0 \mathscr{R} h \mathscr{L} (a^0b)^0$ and hence $ad \mathscr{R} h \mathscr{L} (a^0f)^0$. On the other hand, $ad \mathscr{R} a(a^0f)^0 \mathscr{L} (a^0f)^0$. Combining these two observations, we get $a(a^0f)^0 \mathscr{H} h$, and therefore, $a(a^0f)^0 \in \operatorname{Gr} S$. Now Lemma 1.4 applies (with the element a^0f in the role of b) and we obtain that $a \cdot a^0f = af$ belongs to $\operatorname{Gr} S$, a contradiction. ∎

We need a construction from [3]. Let G be a group and L, R subgroups of G. Denote by G_L (${}_RG$) the collection of all left (resp., right) cosets of G with respect to L (resp., R) and by ${}_RG_L$ the collection of all *double cosets* with respect to the subgroups R and L, i.e. all subsets of the kind RgL where g runs over G. Now let A be any proper subset of ${}_RG_L$ containing RL. Define a multiplication on the disjoint union $T(G, L, R, A)$ of G^0 with the cartesian product $G_L \times {}_RG$ by preserving the multiplication on G^0 and letting, for all $g \in G$, $[hL, Rj], [kL, R\ell] \in G_L \times {}_RG$,

$$\begin{aligned} g[hL, Rj] &= [ghL, Rj], \\ [hL, Rj]g &= [hL, Rjg], \\ [hL, Rj]0 &= 0[hL, Rj] = 0, \\ [hL, Rj][kL, R\ell] &= \begin{cases} [hL, R\ell] & \text{if } RjkL \in A, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It is easy to check that $T(G, L, R, A)$ becomes an epigroup.

Our next lemma is a combination of Propositions 1.1 and 1.12 from [3].

Lemma 1.6 *Let G be a group, L, R subgroups of G, H a proper subgroup of G containing RL, and $A = \{RhL \mid h \in H\}$. Then one of the semigroups $R(\mathbf{C}_\infty)$, $R(\mathbf{C}_p)$ (p is a prime) divides $T(G, R, L, A)$.* ∎

We are ready to complete the proof of Theorem 1.1. Suppose that S is an epigroup such that the semigroup $\mathbf{V}$ does not divide S and $\mathrm{Gr}\, S$ is not a subsemigroup in S. We only need to prove that one of the semigroups $R(\mathbf{C}_\infty)$, $R(\mathbf{C}_p)$ (p is a prime) divides S. Proposition 1.2 and Lemmas 1.4 and 1.5 show that, using symmetry, we may assume that there exist $a \in \mathrm{Gr}\, S$ and $d \in E(S)$ such that $d \leq a^0$ and $ad \notin \mathrm{Gr}\, S$.

Denote by G the subgroup of S generated by a and let $H = \{g \in G \mid dgd \mathrel{\mathscr{H}} d\}$. First of all, we shall verify that H can be alternatively described as the set of all elements $g \in G$ such that gd is a group element. Indeed, each element of the form gd belongs to the $\mathscr{J}$-class D of d since $d = a^0 d = g^{-1} \cdot gd$ where g^{-1} is the group inverse of g in G. Let $I = SdS \setminus D$ and consider the principal factor SdS/I. By Munn's theorem [1, Theorem 2.55] it is a completely 0-simple semigroup, and an non-zero element of a completely 0-simple semigroup is a group element if and only if its square is not equal to 0. Lifting the situation back to S, we conclude that gd is a group element if and only if $(gd)^2 \in D$, and since $dgd = g^{-1} \cdot (gd)^2$, this is equivalent to $dgd \mathrel{\mathscr{H}} d$. In particular, the element a does not belong to H because ad is not a group element.

Our next step is to show that H is a subgroup of G. Take two arbitrary elements $g, h \in H$. Using the fact that $d \leq a^0$, we get $gd \mathrel{\mathscr{L}} hd \mathrel{\mathscr{R}} hdh^{-1}$. As shown in the previous paragraph, both gd and hd are group elements, so we can substitute the corresponding idempotents for them: $(gd)^0 \mathrel{\mathscr{L}} (hd)^0 \mathrel{\mathscr{R}} hdh^{-1}$. Obviously, hdh^{-1} is an idempotent too. In view of Proposition 1.2 and Lemma 1.3, the epigroup S is E-solid, whence there exists an idempotent $e \in S$ such that $(gd)^0 \mathrel{\mathscr{R}} e \mathrel{\mathscr{L}} hdh^{-1}$. Since $gd \mathrel{\mathscr{R}} (gd)^0$ and $hdh^{-1} \mathrel{\mathscr{L}} dh^{-1}$, this implies $gd \mathrel{\mathscr{R}} e \mathrel{\mathscr{L}} dh^{-1}$. Then, by Miller–Clifford's theorem [1, Theorem 2.17], we conclude that $dh^{-1} \mathrel{\mathscr{R}} dh^{-1}gd \mathrel{\mathscr{L}} gd$, whence $dh^{-1}gd \mathrel{\mathscr{H}} d$ and $h^{-1}g \in H$. Thus, H is indeed a subgroup.

Denote by U the subgroup of the $\mathscr{H}$-class containing d which is generated by all the elements dgd, $g \in H$, and by T the subsemigroup of S generated by G, U, and the ideal I. Then it is easy to see that $T = G \cup M \cup I$ where M is the set of all products of the form

$$t = hd(dg_1d)^{\varepsilon_1}(dg_2d)^{\varepsilon_2} \cdots (dg_nd)^{\varepsilon_n} dk$$

where $h, k, g_1, \ldots, g_n \in G$, $dg_1d, \ldots dg_nd \in U$, $\varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}$, and $(dg_id)^{-1}$ is the group inverse of dg_id in the group U. Clearly, $hd \mathrel{\mathscr{R}} t \mathrel{\mathscr{L}} dk$, whence M constitutes a single $\mathscr{D}$-class and $M \cup I/I$ is a completely 0-simple semigroup. In particular, $M \cup I/I$ is an epigroup, and since I (as an ideal of the epigroup S) is also an epigroup and ideal extensions preserve the property "to be an epigroup", $M \cup I$ is an epigroup as well. Since, in turn, $M \cup I$ is an ideal in T, and $T/M \cup I \cong G^0$ is an epigroup, the whole semigroup T is an epigroup too.

Consider the sets $L = \{g \in G \mid gd \mathrel{\mathscr{R}} d\}$ and $R = \{g \in G \mid dg \mathrel{\mathscr{L}} d\}$ which can be easily verified to be subgroups of H. Clearly, the left cosets with respect to L are in a one-to-one correspondence with the $\mathscr{R}$-classes of M, while the right cosets with respect to R index its $\mathscr{L}$-classes. Consider now the relation γ on T which coincides with the identity relation on G, with the Green relation $\mathscr{H}$ on M, and with the universal relation on I. It can be checked straightforwardly that γ is a congruence on T. Furthermore, the elements of M/γ are in a one-to-one correspondence with the pairs of the form $[hL, Rk]$ where hL runs over G_L and Rk runs over ${}_RG$. If we denote by A the set of all double cosets $RhL \in {}_RG_L$ such that $h \in H$, then we see that the product $[hL, Rj] \cdot [kL, R\ell]$ belongs to M/γ (and then equals to $[hL, R\ell]$) if and only if $RjkL$ is in A. Thus, T/γ is isomorphic to $T(G, L, R, A)$. We are in a position to apply Lemma 1.6 which immediately yields the desired conclusion. ■

2 Epigroups whose group elements form a retract

Again, we start with introducing certain finite semigroups that will play role of "forbidden divisors" in characterization results below. By $\mathbf{B}_2$ we denote the 5-element Brandt semigroup. It may be viewed as the semigroup formed by the 2×2-matrix units

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$$

together with the zero 2×2-matrix; alternatively, it may be introduced by the following presentation:

$$\mathbf{B}_2 = \langle a, b \mid aba = a,\ bab = b,\ a^2 = b^2 = 0\rangle.$$

We note that if G is an arbitrary group with more than 1 element, then the Reilly semigroup $R(G)$ considered in Section 1 contains a subsemigroup isomorphic to $\mathbf{B}_2$. Indeed, if h, k are two distinct elements of G, then the elements $[h,h], [h,k], [k,h], [k,k], 0 \in R(G)$ form such a subsemigroup.

By $\mathbf{A}_2$ we denote the 5-element idempotent-generated 0-simple semigroup. It can be given by the presentation

$$\mathbf{A}_2 = \langle a, b \mid aba = a^2 = a,\ bab = b,\ b^2 = 0\rangle,$$

or, alternatively, described as the semigroup formed by the following 2×2-matrices:

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}.$$

We note that the first 4 matrices form a subsemigroup isomorphic to the semigroup $\mathbf{V}$.

We shall also need the semigroup

$$\mathbf{L}_{3,1} = \langle a, e \mid e^2 = e = ea,\ a^2 = a^e\rangle = \{a, e, ae, a^2\}$$

and its dual $\mathbf{R}_{3,1}$.

Our next theorem answers Shevrin's question mentioned in the introduction:

Theorem 2.1 *Let S be an epigroup. The set* $\operatorname{Gr} S$ *is a retract of S if and only if none of the semigroups $\mathbf{B}_2$, $\mathbf{L}_{3,1}$, $\mathbf{R}_{3,1}$, and $\mathbf{V}$ divide S.*

Proof. Necessity. As discussed in the introduction, the property that $\operatorname{Gr} S$ is a retract of S holds if and only if S satisfies the identity (1). Since identities are inherited by divisors, it remains to verify that none of the semigroups $\mathbf{B}_2$, $\mathbf{L}_{3,1}$, $\mathbf{R}_{3,1}$, and $\mathbf{V}$ satisfy (1). Indeed, one readily sees that (1) fails in $\mathbf{B}_2$ if one sets $x = a$, $y = b$; in $\mathbf{V}$ if $x = e$, $y = f$; in $\mathbf{L}_{3,1}$ if $x = a$, $y = e$.

Sufficiency. Recall that a semigroup S is called *archimedean* if for all $a, b \in S$ there exists a positive integer n such that $a^n \in SbS$. It is well known that S is archimedean epigroup if and only if $\operatorname{Gr} S$ is a completely simple ideal in S [5, Proposition 3]. Our proof relies on the following "forbidden divisor" characterization of epigroups decomposable into a semilattice of archimedean semigroups:

Proposition 2.2 ([5, Theorem 3]) *An epigroup S decomposes into a semilattice of archimedean semigroups if and only if neither $\mathbf{B}_2$ nor $\mathbf{A}_2$ divides S.* ■

The second crucial ingredient of the proof is the following

Lemma 2.3 *Let an epigroup S be a semilattice Γ of the archimedean semigroups S_γ, $\gamma \in \Gamma$. If the semigroup $\mathbf{L}_{3,1}$ does not divide S, then for all $\alpha, \beta \in \Gamma$ such that $\alpha \geq \beta$ and for all $a \in S_\alpha$, $q \in \mathrm{Gr}\, S_\beta$,*

$$aq = \overline{\overline{a}}q.$$

Proof. Arguing by contradiction, assume that for some $\alpha, \beta \in \Gamma$ with $\alpha \geq \beta$ there exist $a \in S_\alpha$, $q \in \mathrm{Gr}\, S_\beta$ such that $aq \neq \overline{\overline{a}}q$. We note that a is not a group element (otherwise $a = \overline{\overline{a}}$). Since S is an epigroup, some power of a is a group element. Let $n > 1$ be the least integer such that $a^n \in Gr S_\alpha$.

Clearly, we may also assume that S is generated by a and q (as an epigroup). Then the completely simple semigroup $M = \mathrm{Gr}\, S_\beta$ is an ideal not only in S_β but also in the whole $S = S_\alpha \cup S_\beta$.

Consider the relation ρ on S that coincides with the Green relation $\mathscr{R}$ on M and with the equality relation on $S \setminus M$. It is known and easy to verify that ρ is a congruence. In order to show that the images of the elements aq and $\overline{\overline{a}}q$ are distinct in the quotient semigroup S/ρ, we suppose that $aq \mathrel{\mathscr{R}} \overline{\overline{a}}q$. Then, in particular, $\overline{\overline{a}}q = aqw$ for some $w \in M$. Multiplying this equality by the idempotent a^0 on the left, we get $\overline{\overline{a}}q = \overline{\overline{a}}qw$ whence $\overline{\overline{a}}qw = aqw$. Completely simple semigroups are known to satisfy the identity $x = x(yx)^0$. Applying it to aq and w in the role of x (resp., y), we obtain

$$aq = aq(waq)^0 = aqw \cdot aq \cdot \overline{waq} = \overline{\overline{a}}qw \cdot aq \cdot \overline{waq} = a^0 \cdot aq(waq)^0 = a^0 \cdot aq = \overline{\overline{a}}q,$$

in a contradiction to the choice of the elements a and q. Thus, we may consider S/ρ instead of S; in other words, we may assume that the ideal M consists of left zeros.

Now consider two cases: $\alpha > \beta$ and $\alpha = \beta$.

Case 1: $\alpha > \beta$. Then S_α is the disjoint union of $H = \mathrm{Gr}\, S_\alpha$ (which is then the cyclic group generated by $\overline{a}$) and the set $\{a, a^2, \ldots, a^{n-1}\}$, while $S_\beta = M = \{q, aq, a^2q, \ldots, a^{n-1}q\} \cup Hq$. Suppose that $aq \in \{a^2q, \ldots, a^{n-1}q\} \cup Hq$. If $aq = hq$ for some $h \in H$, we multiply this equality through on the right by a^0. Since a^0 is the identity element of the group H, we then get

$$\overline{\overline{a}}q = a^0 \cdot aq = a^0 \cdot hq = hq = aq,$$

a contradiction. If $aq = a^m q$ for $1 < m < n$, then also $aq = a^{2m-1}q = \ldots$, and iterating, we eventually arrive at the already excluded subcase $aq \in Hq$. Thus, we have proved that $aq \notin \{a^2q, \ldots, a^{n-1}q\} \cup Hq$. Clearly, this implies that $q \neq aq$, $q \notin \{a^2q, \ldots, a^{n-1}q\} \cup Hq$, and $a \notin \{a^2, \ldots, a^{n-1}\} \cup H$. Let τ be the partition of S with the following 4 classes: $\{a\}$, $\{q\}$, $\{aq\}$, and $S \setminus \{a, q, aq\}$. It is routine to check that τ is a congruence on S and $S/\tau \cong \mathbf{L}_{3,1}$.

Case 2: $\alpha = \beta$. Then $a^n \in M$ whence a^n is a left zero and $S = \{a, a^2, \ldots, a^{n-1}\} \cup M$, while $M = \{a^n, q, aq, a^2q, \ldots, a^{n-1}q\}$. Similarly to the previous case, one can easily verify that $q \neq aq$, $q, aq \neq \{a^n, a^2q, \ldots, a^{n-1}q\}$, $a \notin \{a^2, \ldots, a^{n-1}\}$. This allows to define the congruence τ by the same rule as above and to conclude that $S/\tau \cong \mathbf{L}_{3,1}$. ∎

Now we can complete the proof of Theorem 2.1. Suppose that none of the semigroups $\mathbf{B}_2$, $\mathbf{L}_{3,1}$, $\mathbf{R}_{3,1}$, and $\mathbf{V}$ divide our epigroup S. Since $\mathbf{V}$ is isomorphic to a subsemigroup in the semigroup $\mathbf{A}_2$, neither $\mathbf{B}_2$ nor $\mathbf{A}_2$ divides S whence S is a semilattice Γ of the archimedean semigroups S_γ, $\gamma \in \Gamma$, by Proposition 2.2. Let η be the corresponding congruence on S (the least semilattice congruence). Clearly, $s \,\eta\, \overline{\overline{s}}$ for all $s \in S$. Using this we observe that $\overline{\overline{ab}} \,\eta\, ab \,\eta\, \overline{\overline{a}} \cdot \overline{\overline{b}}$ for all $a, b \in S$. Thus, given two arbitrary elements $a, b \in S$, the elements $\overline{\overline{ab}}$ and $\overline{\overline{a}} \cdot \overline{\overline{b}}$ lie in one archimedean subsemigroup, say S_δ.

Since $\mathbf{B}_2$ is isomorphic to a subsemigroup in each of the semigroups $R(\mathbf{C}_\infty)$, $R(\mathbf{C}_p)$ (p is a prime), we may apply Theorem 1.1 to S concluding that $\operatorname{Gr} S$ is a subsemigroup. Hence, $\overline{\overline{ab}}, \overline{\overline{a}} \cdot \overline{\overline{b}} \in \operatorname{Gr} S_\delta$. If $a \in S_\alpha$, $b \in S_\beta$, then $\alpha \geq \delta$ and $\beta \geq \delta$. Since the semigroup $\mathbf{L}_{3,1}$ does not divide S, Lemma 2.3 applies (with $(ab)^0$ in the role of q) yielding

$$b(ab)^0 = \overline{\overline{b}}(ab)^0.$$

Multiplying this equality by a on the right, we get

$$ab(ab)^0 = a\overline{\overline{b}}(ab)^0.$$

Since $\overline{\overline{b}}(ab)^0$ is again a group element (recall that $\operatorname{Gr} S$ is a subsemigroup), we may apply Lemma 2.3 once more, this time with $\overline{\overline{b}}(ab)^0$ in the role of q. This gives

$$\overline{\overline{ab}} = ab(ab)^0 = a\overline{\overline{b}}(ab)^0 = \overline{\overline{a}} \cdot \overline{\overline{b}}(ab)^0. \tag{4}$$

By symmetry,

$$\overline{\overline{ab}} = (ab)^0 \overline{\overline{a}} \cdot \overline{\overline{b}}.$$

Hence $\overline{\overline{a}} \cdot \overline{\overline{b}}$ belongs to the $\mathscr{H}$-class of $\overline{\overline{ab}}$. However, since the idempotent $(ab)^0$ is the identity element of this $\mathscr{H}$-class, we have

$$\overline{\overline{a}} \cdot \overline{\overline{b}}(ab)^0 = \overline{\overline{a}} \cdot \overline{\overline{b}}.$$

Comparing this equality with (4), we obtain

$$\overline{\overline{ab}} = \overline{\overline{a}} \cdot \overline{\overline{b}},$$

as required. ■

3 Epigroups with other restrictions to group elements

In this section we have collected "forbidden divisor" characteriztions of epigroups whose group elements form a quasiideal, or a left ideal, or an ideal, or a retract ideal. Because of lack of space, the proofs of the first two of these results are not supplied here, but it should be mentioned that they are much easier than the proofs of Theorems 1.1 and 2.1.

The characterizations involve the following finite semigroups:

$$\mathbf{C} = \langle a, e \mid e^2 = e,\ ae = ea = a,\ a^2 = 0 \rangle = \{e, a, 0\};$$
$$\mathbf{Y} = \langle a, e, f \mid e^2 = e,\ f^2 = f,\ ef = fe = 0,\ ea = af = a \rangle = \{e, f, a, 0\};$$
$$\mathbf{P} = \langle a, e \mid e^2 = e,\ ea = a,\ ae = 0 \rangle = \{e, a, 0\},$$

and the semigroup $\mathbf{Q}$ which is dual to $\mathbf{P}$. We note that the semigroup $\mathbf{P}$ may be conveniently thought as the semigroup consisting of the following 2×2-matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This means, in patricular, that $\mathbf{P}$ embeds in both $\mathbf{B}_2$ and $\mathbf{V}$.

Recall that a set $I \subseteq S$ is called a *quasiideal* if $IS \cap SI \subseteq I$. A plain semigroup version of the following result has been found by Tishchenko and the author [6, Proposition 2]:

Theorem 3.1 *Let S be an epigroup. The set* $\mathrm{Gr}\, S$ *is a quasiideal of S if and only if none of the semigroups* $\mathbf{C}$, $\mathbf{Y}$, *and* $\mathbf{V}$ *divide S.* ∎

Theorem 3.2 *Let S be an epigroup. The set* $\mathrm{Gr}\, S$ *is a left ideal of S if and only if neither* $\mathbf{C}$ *nor* $\mathbf{P}$ *divides S.* ∎

Theorem 3.2 and its dual immediately imply

Corollary 3.3 *Let S be an epigroup. The set* $\mathrm{Gr}\, S$ *is an ideal of S if and only if neither* $\mathbf{C}$ *nor* $\mathbf{P}$ *nor* $\mathbf{Q}$ *divides S.* ∎

We may combine Corollary 3.3 and Theorem 2.1 in order to obtain a characterization of epigroups whose group elements form a retract ideal, a property that have been also considered in [5, §1]. Since the semigroup $\mathbf{P}$ is isomorphic to a subsemigroup in both $\mathbf{B}_2$ and $\mathbf{V}$, this characterization reduces to the following

Corollary 3.4 *Let S be an epigroup. The set* $\mathrm{Gr}\, S$ *is a retract ideal of S if and only if none of the semigroups* $\mathbf{C}$, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{L}_{3,1}$ *and* $\mathbf{R}_{3,1}$ *divide S.* ∎

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