数理解析研究所講究録
1170 巻 2000 年 82-94

交通流の離散モデルにおける数理

龍谷大理工 西成 活裕 (Katsuhiro Nishinari)

1 Introduction

Traffic phenomena attracts much attention for physicists in recent years. It shows complex phase transition from free to congested state, and many theoretical models have been proposed so far to explain the transition[1]–[7]. Among them we will focus on deterministic cellular automaton (CA) models in this paper. CA models are quite simple and has flexibility, and suitable for computer simulations of discrete phenomena.

The rule–184 CA[8] has been widely used as a prototype of deterministic model of traffic flow, and several variations of it has been proposed. First, Fukui and Ishibashi proposed a high speed model[6] of the rule–184 CA, i.e., cars move more than one site per a time if there are enough space ahead. Second, Fuks and Boccara proposed a "monitored traffic model"[9], which is considered as a kind of quick start(QS) model. In the model, driver can see further than just one site forward and he predict that the obstacle car in front of him will move forward when two sites ahead is empty. Thus cars will start *quickly* compared with the rule–184 CA. Third, M.Takayasu and H.Takayasu proposed a slow start(SlS) model[10], in which cars stops at a time cannot move at the next time and wait one time step to move forward. This represents an asymmetry of stop and start behavior of cars. We call these variants as the rule–184 CA family in this paper.

Let us see here real observed data in the Hanshin expressway taken by the Japan Highway Public Corporation[12]. Fig.1 shows the flow–density diagram, which is called the fundamental diagram. We find a discontinuity at the occupancy $\sim 25\%$, and there seems to exist multiple states around the critical occupancy. Many other diagrams show this type of graphs and the flow-density curve shows a shape of "inverse λ"[12].
There is another observed result that above a certain critical density, a perturbation on an uniform flow will give rise to the formation of a jam, and "stop-and-go" wave propagates backward[13]. This means that cars accelerate and decelerate alternately in the downstream jam. We consider these two facts as criteria for judging which CA rules

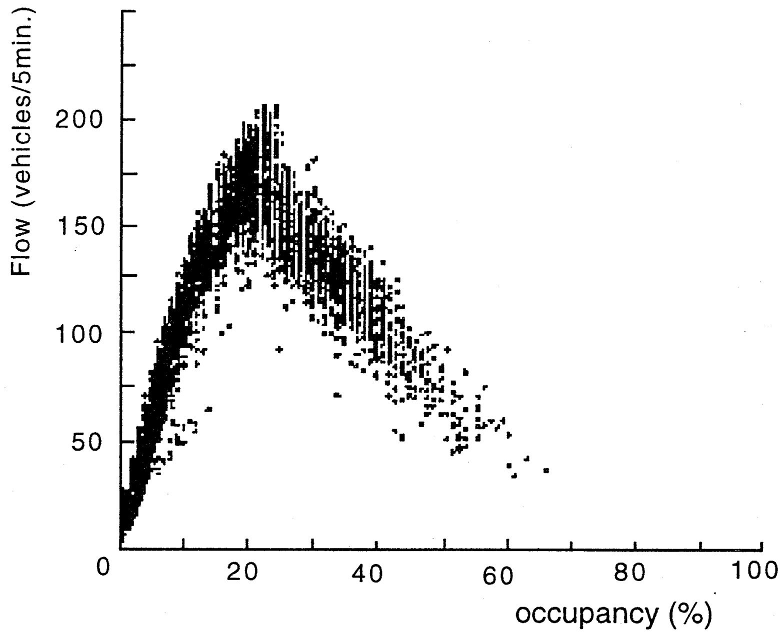

Figure1: Fundamental diagram of the Hanshin expressway taken by the Japan Highway Public Corporation.

are suitable for traffic model. In the rule–184 CA family, only in the SlS model there are the multiple state near the critical density and the "stop-and-go" wave propagates backward.

Recently, a multi-value generalization of the rule–184 model has been proposed by using the ultra discrete method[14]. The model is

$$U_j^{t+1} = U_j^t + \min(U_{j-1}^t, L - U_j^t) - \min(U_j^t, L - U_{j+1}^t), \tag{1}$$

where U_j^t represents the number of cars at site j at time t, and L is an integer. Each site are assumed to hold L cars at most. We can prove that if $L > 0$ and $0 \leq U_j^t \leq L$ for any j at a certain t, then $0 \leq U_j^{t+1} \leq L$ holds for any j. (1) is obtained from the Burgers' equation, and then it is called Burgers CA(BCA)[15]. BCA contains the rule–184 CA as a special case $L = 1$. We have introduced the positive integer L in (1), and physical meanings of it can be considered as following three ways: first, the road is L-lane freeway in a coarse sense, and effect of lane changes of cars is not considered explicitly. Second, we consider single-lane freeway and U_j^t/L represents probability of existing of a car at site j and time t. Third, the length of one site is long enough to contain the number of cars at most L in a single-lane freeway. In the Appendix, we have shown a direct proof that BCA with $L = 2$ can be considered as a two lane model of the rule–184 CA.

BCA does not show the multiple state, but we have shown that its high speed extension shows the state around the critical density[16]. The model is considered as a one of multi-value generalization of the FI model. This will be treated again in Sec.2.3. Another models in the rule–184 CA family, such as the SlS model and QS model, have been only treated as "0" and "1" CA so far. Thus in this paper, we will generalize the whole rule–184 CA family to multi-value CAs', and investigate its properties in detail. Multi-value generalization of CAs' has following advantages: we can consider general algebraic properties of CAs' which are free of particular properties that comes from two-valueness. Moreover, by introducing the parameter L we may apply those models with flexibility as other transport phenomena, such as granular flow and internet packet transportation.

2 Multi-value generalization

In this section, we will present five CA models, which are multi-value extensions of the rule–184 family.

2.1 Multi-value QS model

First, we will generalize the QS model to a multi-value one. In the model, cars know not only the number of vacant spaces in their next site, but also know those in the two sites ahead. Then cars in the site j know the number of cars that can move from the site $j+1$ to $j+2$ in the next time, i.e., $\min(U_{j+1}^t, L-U_{j+2}^t)$, while they do not know it in the BCA. Thus, vacant spaces at the site $j+1$ is considered to be $L - U_{j+1}^t + \min(U_{j+1}^t, L - U_{j+2}^t)$ in this model. Therefore the CA rule in this model is given by

$$\begin{aligned} U_j^{t+1} &= U_j^t + \min(U_{j-1}^t, L - U_j^t + \min(U_j^t, L - U_{j+1}^t)) \\ &\quad - \min(U_j^t, L - U_{j+1}^t + \min(U_{j+1}^t, L - U_{j+2}^t)) \\ &= U_j^t + \min(U_{j-1}^t, 2L - U_j^t - U_{j+1}^t) - \min(U_j^t, 2L - U_{j+1}^t - U_{j+2}^t). \qquad (2) \end{aligned}$$

This is a multi-value extension of the QS model. Cars will move forward by expecting that their preceding cars also move forward. From (2), cars will move to the next site by considering the sum of the vacant spaces in next two sites. In the case of $L = 1$, this model corresponds to the rule–3212885888 CA of a neighborhood size "5" by Wolfram's terminology[8].

2.2 Multi-value SlS model

In the SlS model, standing cars cannot move soon at the next time, and they can move at two time steps later if there are vacant spaces in their next site. In the multi-value case, we should distinguish standing cars and moving cars in each sites, and only standing cars need to wait one time step. The number of cars at the site $j-1$ blocked by cars in front of them at time $t-1$ is represented by $U_{j-1}^{t-1} - \min(U_{j-1}^{t-1}, L - U_j^{t-1})$. These cannot move at time t, then the maximum number of cars that move to the site j at t is given by $U_{j-1}^t - \{U_{j-1}^{t-1} - \min(U_{j-1}^{t-1}, L - U_j^{t-1})\}$. Therefore, considering the number of cars entering into and escaping from site j, the multi-value generalized SlS CA is given by

$$\begin{aligned} U_j^{t+1} = U_j^t \quad &+ \quad \min\left[U_{j-1}^t - \{U_{j-1}^{t-1} - \min(U_{j-1}^{t-1}, L - U_j^{t-1})\}, L - U_j^t\right] \\ &- \quad \min\left[U_j^t - \{U_j^{t-1} - \min(U_j^{t-1}, L - U_{j+1}^{t-1})\}, L - U_{j+1}^t)\right] \end{aligned} \tag{3}$$

We note that this model includes T^2 model in the case $L = 1$, which cannot be represented by the usual Wolfram's number because time neighborhood size is "3".

2.3 EBCA2 model

In our previous paper[16], we propose the EBCA model which is a high speed extension of BCA to a velocity "2", i.e., cars can move two site at a time if the successive two sites are not fully occupied. In this paper, we refer the model as EBCA2, because we have assumed that cars moving two sites have priority to those moving one site[16].

EBCA2 is given by

$$U_j^{t+1} = U_j^t + \min(b_{j-1}^t + a_{j-2}^t, L - U_j^t + a_{j-1}^t) - \min(b_j^t + a_{j-1}^t, L - U_{j+1}^t + a_j^t), \tag{4}$$

where $a_j^t \equiv \min(U_j^t, L - U_{j+1}^t, L - U_{j+2}^t)$ and $b_j^t \equiv \min(U_j^t, L - U_{j+1}^t)$. In [16], we study the case only $L = 1, 2$. In this paper, we will show some properties of general L in the next section. It is noted that (4) includes the FI model with velocity "2" in the case $L = 1$, which rule number is 3436170432.

2.4 EBCA1 model

There is another possibility when we extend BCA to the velocity "2". In contrast to EBCA2, we consider that cars moving one site have priority to those which can move two sites. This new model is called EBCA1 in this paper.

Let us consider the evolutional rule of this model. One time step consists of two successive procedures. First, cars move according to BCA, i.e., the number of cars at site j that can move according to BCA rule is given by $b_j^t \equiv \min(U_j^t, L - U_{j+1}^t)$. Next, only those cars that move one site, which is represented by b_j^t, can move more one site according to BCA if the site in front of them are not fully occupied. That is, the number of cars that move two sites in one time step is given by $\min(b_j^t, L - U_{j+2}^t - b_{j+1}^t + b_{j+2}^t)$, where the last term in min represents vacant spaces at site $j+2$ after the first procedure. Therefore, considering the number of cars entering into and escaping from site j, the evolutional rule of EBCA1 is given by

$$\begin{aligned} U_j^{t+1} &= U_j^t + b_{j-1}^t - b_j^t \\ &\quad + \min(b_{j-2}^t, L - U_j^t - b_{j-1}^t + b_j^t) - \min(b_{j-1}^t, L - U_{j+1}^t - b_j^t + b_{j+1}^t) \\ &= U_j^t + \min(b_{j-1}^t + b_{j-2}^t, L - U_j^t + b_j^t) - \min(b_j^t + b_{j-1}^t, L - U_{j+1}^t + b_{j+1}^t). \quad (5) \end{aligned}$$

It is noted that (5) of $L = 1$ differs from the FI model. EBCA1 becomes the rule–3372206272 CA in the case $L = 1$.

2.5 Multi-value SlS model with a higher velocity

Let us consider a generalization of the SlS rule to the velocity "2". To this purpose, we will consider a model of a combination of EBCA1 and the SlS model. First, we generalize the slow-start rule. In the case of the velocity "2", standing cars will move at most one site by the BCA rule instead of stopping at the next time step. They can move with the velocity "2" only after two time steps later. That is, if once cars stop, then they will *gradually* increase their speed to the maximum velocity "2". It is noted that in this model cars can move at least one site at every time step whether they have stopped or not in the preceding time. The number of standing cars at the site $j-2$ at time $t-1$ is given by $U_{j-2}^{t-1} - b_{j-2}^{t-1}$, where b_j^t is the same definition as before. These cannot move two sites due to the slow-start rule, then the maximum number of cars that can move from the site $j-2$ to j is given by $\min(U_{j-2}^t - (U_{j-2}^{t-1} - b_{j-2}^{t-1}), L - U_{j-1}^t)$. Therefore, following the discussion in Sec.3.2 and Sec.3.4, the evolutional rule is given by

$$\begin{aligned} U_j^{t+1} &= U_j^t + b_{j-1}^t - b_j^t \\ &\quad + \min(\min(U_{j-2}^t - (U_{j-2}^{t-1} - b_{j-2}^{t-1}), L - U_{j-1}^t), L - U_j^t - b_{j-1}^t + b_j^t) \\ &\quad - \min(\min(U_{j-1}^t - (U_{j-1}^{t-1} - b_{j-1}^{t-1}), L - U_j^t), L - U_{j+1}^t - b_j^t + b_{j+1}^t) \\ &= U_j^t + \min(U_{j-2}^t - U_{j-2}^{t-1} + b_{j-1}^t + b_{j-2}^{t-1}, L - U_{j-1}^t + b_{j-1}^t, L - U_j^t + b_j^t) \\ &\quad - \min(U_{j-1}^t - U_{j-1}^{t-1} + b_j^t + b_{j-1}^{t-1}, L - U_j^t + b_j^t, L - U_{j+1}^t + b_{j+1}^t). \end{aligned} \tag{6}$$

This model becomes a generalization of the T^2 model to the maximum speed "2" in the case $L = 1$.

3 Fundamental diagram and multiple state

The fundamental diagram of new CA models discussed in the previous section is studied in detail in this section. All models in Sec.2 are in conserved form such as

$$\Delta_t U_j^t + \Delta_j q_j^t = 0, \tag{7}$$

where Δ_t and Δ_j are forward difference operator with respect to indicated variables, and q_j^t represents the traffic flow. In the followings, we will consider a periodic road, or a circuit. The average density ρ^t and average flow Q^t over entire system is defined by

$$\rho^t \equiv \frac{1}{KL}\sum_{j=1}^{K} U_j^t, \tag{8}$$

$$Q^t \equiv \frac{1}{KL}\sum_{j=1}^{K} q_j^t. \tag{9}$$

where K is the number of sites in a period. Since in our models, the average density is a conserved quantity, then we will write it as ρ.

Fig.2.A–E are the density–flow diagram of each models with $L = 2$. We plot the flow Q^t for $t = 2K$, which is sufficiently long enough to relax an initial configration. We superpose many values of Q^{2K} starting from random initial conditions in writing the diagram. We see that in fig.2.B–E there are multiple states around the critical density.

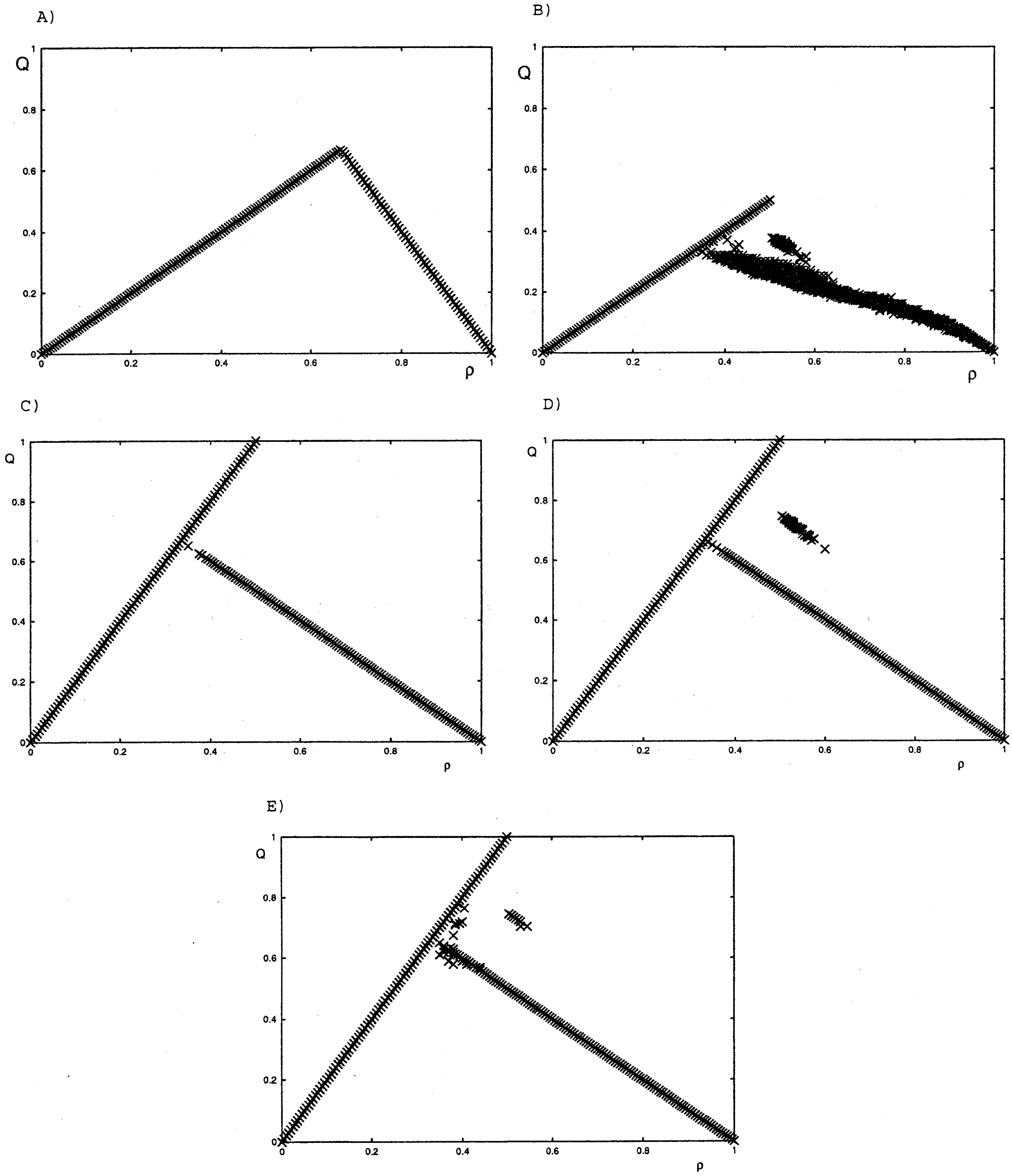

Figure 2: Fundamental diagram of multi-value CA models. A)QS model B)SlS model C)EBCA2 D)EBCA1 E)SlS–EBCA1 with $L = 2$ and $K = 100$.

From fig.2.B, the traffic flow in congested state will never relax to a steady state in the

SlS model. In other models, the relaxation to steady states will be realized in a relatively short time. From fig.2.B, D and E, we see that a new small branch exists near critical density. Stabilities of these branches will be studied in the next section.

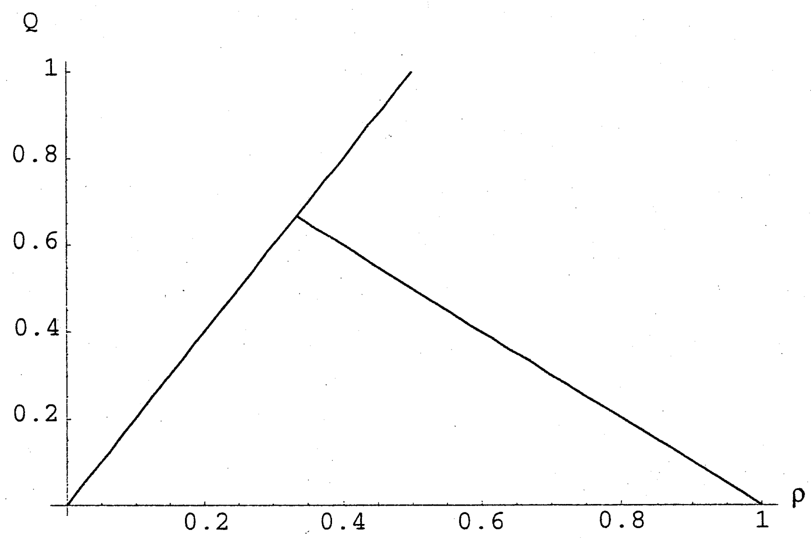

Figure 3: Fundamental diagram of EBCA1 with $L = 1$.

Fig.3 is the fundamental diagram of EBCA1 with $L = 1$. There exist a multiple state even in the case $L = 1$, i.e., in the rule–3372206272 CA. In deterministic two-value CA models, only the SlS model proposed by M.Takayasu and H.Takayasu is known to show metastable state. EBCA1 becomes the next example of such kind of models in two-value CA.

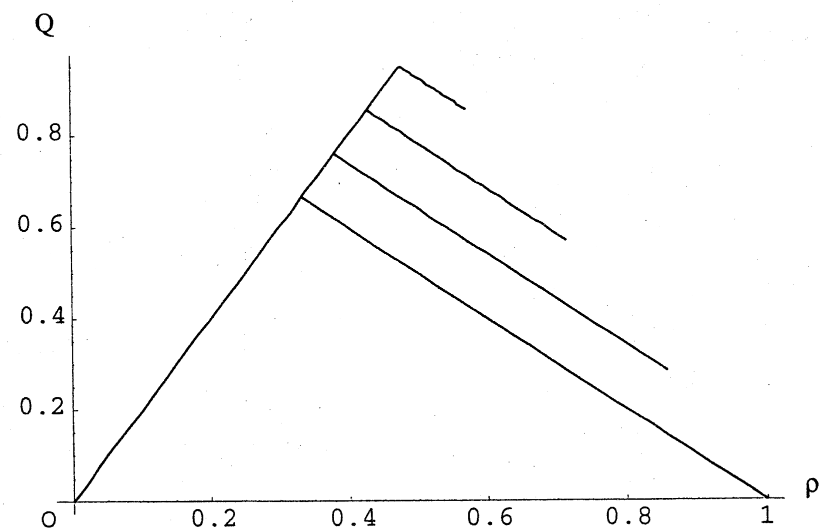

Figure 4: Fundamental diagram of EBCA2 with $L = 7$.

At the end of this section, we will comment on the case $L > 3$. The fundamental diagram of EBCA2 with $L = 7$ is given in Fig.4. We see that there are many branches around the critical density. The small branches, or "hat shapes" come from the rule–184 CA; the two numbers that their sum is L, i.e., 0 and L, 1 and $L-1$, etc., evolve like the rule–184 CA of the two numbers under EBCA2, and they form the hat shape in the fundamental diagram[15]. For example, the truth value table of EBCA2 with $L = 7$ in the case $U_j^t \in \{1, 6\}$ is expressed symbolically by

$$\frac{U_{j-1}^t U_j^t U_{j+1}^t}{U_j^{t+1}} = \frac{111}{1}, \frac{116}{1}, \frac{161}{1}, \frac{166}{6}, \frac{611}{6}, \frac{616}{6}, \frac{661}{1}, \frac{666}{6},$$

then we see that $U_j^t \in \{1, 6\}$ for all t. This is nothing but the rule–184 CA for "1" and "6" and this corresponds to the branch $Q = -\rho + 8/7$ in Fig.4.

4 Stability of flow

Fig.5.A–D shows an instability of uniform flow of each models at the critical density.

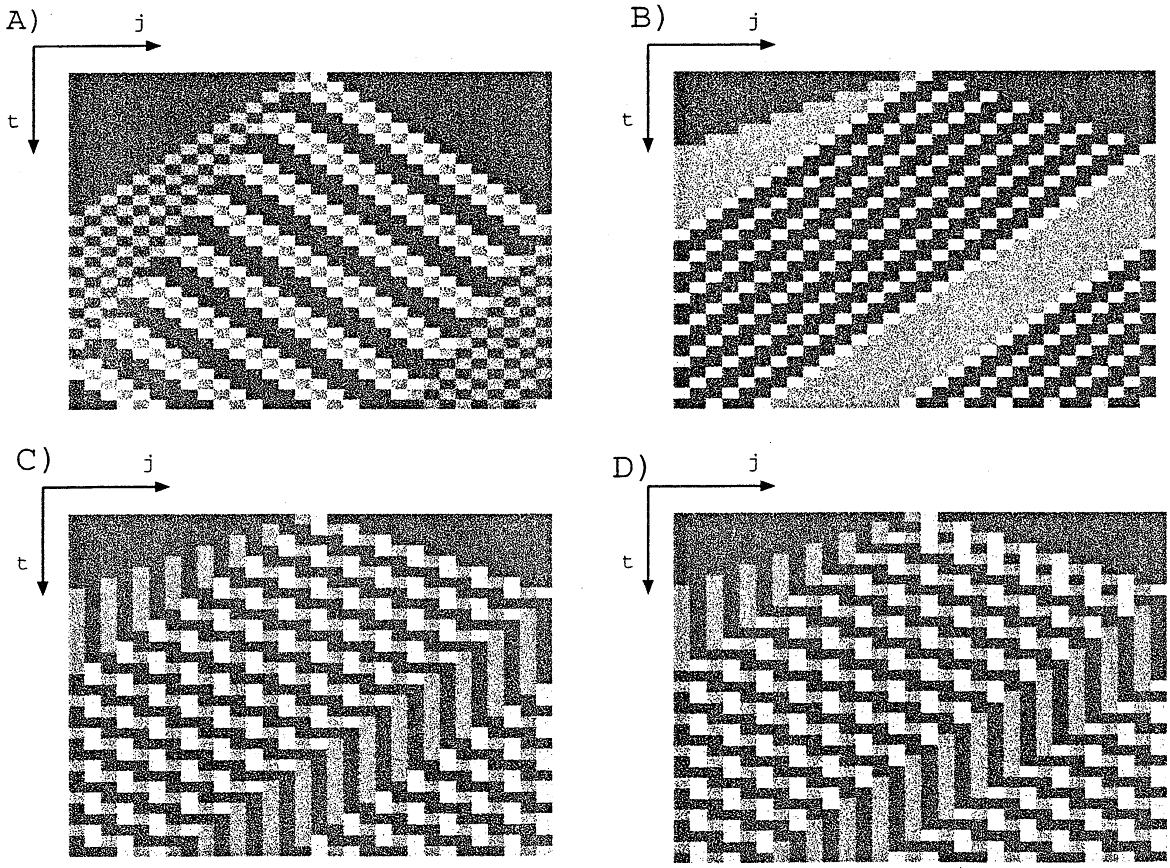

Figure 5: Instability of Uniform flow in A)SlS model B)EBCA2 C)EBCA1 D)SlS–EBCA1. 0,1,2 are represented by white, black and gray squares.

We can see the "stop-and-go" wave propagates backward from the perturbated place of uniform flow in Fig.5A, C and D, i.e., the SlS model, the EBCA1 model and the SlSEBCA1 model, respectively. The time evolution of the EBCA1 and SlSEBCA1 model show almost same pattern. This is explained as follows. The string "$\cdots 111020 \cdots$" and "$\cdots 121212 \cdots$" are both steady states of two models. These are represented at B and C in Fig.6, respectively.

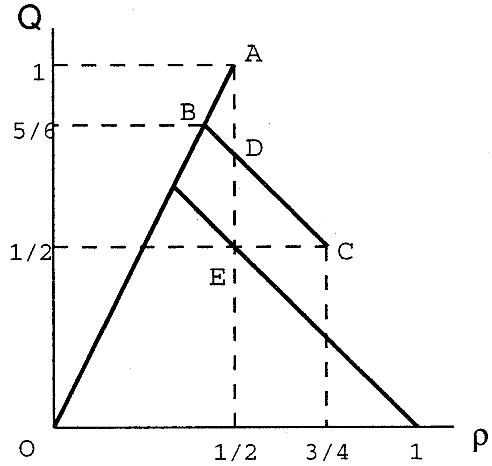

Figure 6: Schematic fundamental diagram of EBCA1 and SlSEBCA1

If we add a perturbation to the state at A, then the flow decreases and transits to the state D. This is the case in Fig.5. In the lower branch $Q = -\rho + 1$ of congested phase, the string "$\cdots 2222 \cdots$" appears, while in the upper branch B–C, the site "2" is not last more than two successive sites. We see in Fig.7 that if a perturbation to the uniform flow A is large enough, then the string "$\cdots 2222 \cdots$" appears and direct transition from A to D occurs.

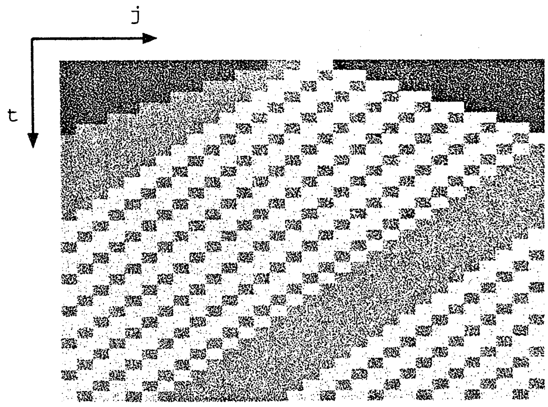

Figure 7: Instability of uniform flow of EBCA1.

5 Concluding Discussions

In this paper, multi-value extension of the rule-184 family is studied. We conclude that EBCA1 model is good for traffic model. The SlS model is also good for traffic model, but if we consider a higher speed model, then the difference between EBCA1 and high speed SlS model(SlS-EBCA1) becomes little. We can easily deal EBCA1 than the SlS model because EBCA1 uses only two successive time steps.

We investigate CAs' with $L = 2$, and only in the case of EBCA2 we have commented a result of general L. It is observed in other models with $L > 3$ that there appear small branches like those of EBCA2, but they have complicated nature than EBCA2. The general results of the multi-value family and its two-dimensional extension will appear elsewhere.

References

[1] H. Greenberg, Oper. Res., **7** (1959) pp.79.

[2] B. S. Kerner and P. Konhäuser, Phys. Rev. E, **48** (1993) p.2335.

[3] G. F. Newell, Oper. Res., **9** (1961) p.209.

[4] M. Bando, K. Hasebe, A. Nakayama, A. Shibata and Y. Sugiyama, Phys. Rev. E. **51** (1995) p.1035.

[5] K. Nagel and M. Schreckenberg, J. Phys. I France **2** (1992) p.2221.

[6] M. Fukui and Y. Ishibashi, J. Phys. Soc. Jpn., **65** (1996) p.1868.

[7] D. E. Wolf, M. Schreckenberg and A. Bachem (editors), *Workshop in Traffic and Granular Flow*, (World Scientific, Singapore, 1996).

[8] S. Wolfram, *Theory and Applications of Cellular Automata* (World Scientific, Singapore, 1986).

[9] H. Fukś and N. Boccara, Int. J. Mod. Phys. C, **9** (1998), p.1.

[10] M. Takayasu and H. Takayasu, Fractals **1** (1993) p.860.

[11] Traffic Engineering (in Japanese), T. Sasaki and Y. Iida, Eds. (Kokumin Kagakusha, 1992).

[12] K. Nishinari and M. Hayashi (editors), *Traffic Statistics in Tomei Express Way*, (The Mathematical Society of Traffic Flow, Japan, 1999).

[13] R. Herman, E.W. Montroll, R.B. Potts and R.W. Rothery, Opns. Res.,**7** (1959) p.86.

[14] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, Phys. Rev. Lett. **76** (1996) p.3247.

[15] K. Nishinari and D. Takahashi, J. Phys. A **31** (1998) p.5439.

[16] K. Nishinari and D. Takahashi, J. Phys. A **32** (1999) p.93.

Appendix

In this appendix, we will show a direct proof that a two-lane model of the rule–184 CA corresponds to BCA with $L = 2$. The two-lane model is shown in Fig.A1.

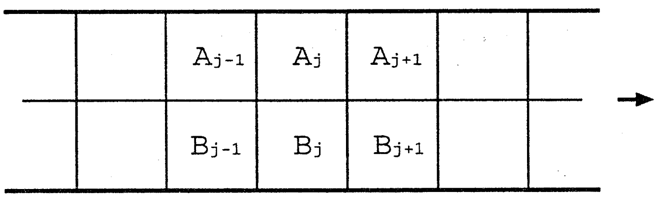

Figure A1: Two lane model of the rule–184 CA.

Cars move according to the rule–184 CA in each lane except the case of a lane change. If a car exists in the site j in the A-lane and the site in front of it is occupied, then the car feel like changing its lane. When the site j and $j+1$ in the B-lane is empty, then the car will change its lane and go to the site $j+1$ in the B-lane. This is the same changing rule for a car in the B-lane. Thus the lane change rule is represented by

$$\begin{aligned} (A_j^t, A_{j+1}^t, B_j^t, B_{j+1}^t) &= (1,1,0,0) \quad \text{then} \quad B_{j+1}^{t+1} = 1, \\ (A_j^t, A_{j+1}^t, B_j^t, B_{j+1}^t) &= (0,0,1,1) \quad \text{then} \quad A_{j+1}^{t+1} = 1. \end{aligned}$$

for all j and t.

Then time evolution of the A-lane is given by

$$\begin{aligned} A_j^{t+1} &= A_j^t + \min(A_{j-1}^t, 1 - A_j^t) - \min(A_j^t, 1 - A_{j+1}^t) \\ &\quad + \min(1 - A_{j-1}^t, 1 - A_j^t, B_{j-1}^t, B_j^t) - \min(A_j^t, A_{j+1}^t, 1 - B_j^t, 1 - B_{j+1}^t), \end{aligned} \tag{10}$$

where the first and second terms in right side hand of (10) represent the rule–184 CA, and the third and fourth terms represent lane change of cars that come from and go to the B-lane, respectively. We replace A to B, then we obtain time evolution for the B-lane as

$$\begin{aligned} B_j^{t+1} &= B_j^t + \min(B_{j-1}^t, 1 - B_j^t) - \min(B_j^t, 1 - B_{j+1}^t) \\ &\quad + \min(1 - B_{j-1}^t, 1 - B_j^t, A_{j-1}^t, A_j^t) - \min(B_j^t, B_{j+1}^t, 1 - A_j^t, 1 - A_{j+1}^t). \end{aligned} \tag{11}$$

If we put

$$U_j^t = A_j^t + B_j^t, \tag{12}$$

then summing (10) and (11), we can derive (1) with $L = 2$ by simple calculations with formulae $\min(A, B) + \min(A, 1 - B) = A$ and $\min(A, B) + \min(C, D) = \min(A + C, A + D, B + C, B + D)$, and the fact that $0 \le A_j^t \le 1$ and $0 \le B_j^t \le 1$ for all j and t.