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OPTIMISTIC CALCULATIONS ABOUT THE WITTEN–RESHETIKHIN–TURAEV INVARIANTS OF CLOSED THREE-MANIFOLDS OBTAINED FROM THE FIGURE-EIGHT KNOT BY INTEGRAL DEHN SURGERIES

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ABSTRACT. I calculate *optimistically* asymptotic behaviors of the WRT $SU(2)$ invariants for the three-manifolds obtained from the figure-eight knot by p-surgeries with $p=0,1,2,\ldots,10$, from which one can extract volumes and the Chern–Simons invariants of these closed manifolds. I conjecture that this also holds for general closed three-manifolds.

1. INTRODUCTION

In [2] R. Kashaev defined a link invariant by using quantum dilogarithm and confirmed that his invariants grow exponentially with the growth rates the hyperbolic volumes (times a constant) for three hyperbolic knots with small numbers of crossings. He also conjectured that this holds for every hyperbolic knot.

J. Murakami and I proved that Kashaev's link invariant is essentially (up to normalization) the same as the Jones polynomial colored with N-dimensional representation evaluated at the N-th root of unity. Moreover we generalized Kashaev's conjecture to the following conjecture.

Conjecture 1.1 (Volume Conjecture, [9]). *Let $J_N(K)$ be the N-colored Jones polynomial of a knot K evaluated at* $\exp\left(\dfrac{2\pi\sqrt{-1}}{N}\right)$. *Then*

$$\lim_{N\to\infty}\frac{\log|J_N(K)|}{N}=\frac{v_3}{2\pi}||K||,$$

where $||K||$ is the Gromov norm (or the simplicial volume) of the complement of K and v_3 is the hyperbolic volume of the regular ideal tetrahedron.

Recent developments toward the Volume Conjecture can be found in [3, 14, 13, 8, 10].

It is natural to ask whether a similar formula holds for closed three-manifolds replacing the colored Jones polynomial with the Witten–Reshetikhin–Turaev $SU(2)$ invariant associated with the N-th root of unity. But an argument using Heegaard splitting and Topological Quantum Field Theory tells us that the growth of the WRT invariant is a polynomial, showing that a similar limit in the Volume Conjecture vanishes. (After the first attempt of this work I learned this argument from D. Thurston and J. Roberts; it was also pointed out by S. Garoufalidis, V. Turaev and K. Walker independently.)

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The aim of this article is to be optimistic to calculate (fake) limits of the logarithms of the WRT invariants of closed three-manifolds obtained from the figure-eight knot by Dehn surgeries with integral coefficients. I will follow Kashaev's calculation in [2] formally and optimistically, and deduce an analytic function with integer parameter corresponding to the surgery coefficient. The function turns out to describe not only the (simplicial) volume but also the Chern–Simons invariant of the manifold. (J. Murakami told me to look at the Chern–Simons invariants after my earlier calculations. See [10] for a similar relation between the Chern–Simons invariants and the colored Jones polynomials for knots and links.)

I do not know what these optimistic calculations mean. But this is not a coincidence and there should be something behind it!

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2. The Witten–Reshetikhin–Turaev invariant

Let $J_n(K;t)$ be the colored Jones polynomial of a knot K associated with the n-dimensional representation of the Lie algebra $sl_2(\mathbb{C})$. We normalize $J_n(K;t)$ so that $J_2(K;t)$ is the Jones polynomial and $J_n(O;t) = \dfrac{t^{n/2}-t^{-n/2}}{t^{1/2}-t^{-1/2}}$ with O the unknot. Due to T. Le the colored Jones polynomial of the figure-eight knot 4_1 is

$$J_n(4_1;t) = \sum_{m=0}^{n-1}\prod_{l=1}^{m}\left(t^{(n+l)/2}-t^{-(n+l)/2}\right)\left(t^{(n-l)/2}-t^{-(n-l)/2}\right).$$

(K. Habiro obtained the same formula by a different technique.) Let M_p be the closed three-manifold obtained from S^3 by Dehn surgery along the figure-eight knot with coefficient $p \in \mathbb{Z}$. We denote by $\tau_N(M_p)$ the Witten–Reshetikhin–Turaev invariant of M_p associated with the Lie group $sl_2(\mathbb{C})$ and with level $N-2$. Then from [5, (1.9)],

$$\tau_N(M_p) = \sqrt{\frac{2}{N}}\sin\frac{\pi}{N}\exp\left(\frac{-3\pi\sqrt{-1}}{4}\right)q^{(3-p)/4}\sum_{n=1}^{N-1}[n]^2 q^{pn^2/4}J_n(4_1;q) \tag{2.1}$$

if $p>0$. Here $q=\exp\left(\dfrac{2\pi\sqrt{-1}}{N}\right)$ and $[n]=\dfrac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$. Note that our $J_n(K;q)$ is $[n]J_{K_0,n}$ with the notation in [5]. Since

$$\begin{aligned}
&[n]\prod_{l=1}^{m}\left(q^{(n+l)/2}-q^{-(n+l)/2}\right)\left(q^{(n-l)/2}-q^{-(n-l)/2}\right)\\
&=\frac{\prod_{l=-m}^{m}\left(q^{(n+l)/2}-q^{-(n+l)/2}\right)}{q^{1/2}-q^{-1/2}}\\
&=\frac{(1-q^{n-m})(1-q^{n-m+1})\cdots(1-q^{n+m})\times(-1)q^{-n(2m+1)/2}}{q^{1/2}-q^{-1/2}}
\end{aligned}$$

$$= \frac{-q^{-n(2m+1)/2}}{q^{1/2}-q^{-1/2}} \times \frac{(q)_{n+m}}{(q)_{n-m-1}}$$

we have

$$\tau_N(M_p) = P(N) \sum_{n=1}^{N-1} \sum_{m=0}^{n-1} \frac{(q)_n (q)_{n+m}}{(q)_{n-1}(q)_{n-m-1}} q^{n(pn/4-m)-n},$$

where $(q)_k = (1-q)(1-q^2)\cdots(1-q^k)$ and $P(N)$ is a function of N with polynomial growth. Note that this also holds for a non-positive p.

3. An optimistic limit

Suppose that we are given a function $S(N)$ of N by the following summation.

$$S(N) = P(N) \sum_{n_1,n_2,\ldots,n_k} q^{Q+L} \prod_{a=1}^{\alpha} (q)_{l_a}^{\varepsilon_a}, \tag{3.1}$$

where $P(N)$ is a function of N with polynomial growth, $\varepsilon_a = \pm 1$, l_a and L are linear functions of $n_1, n_2, \ldots, n_k$ (they do not depend on N but may have constant terms), $Q = \sum_{1\le i\le j\le k} r_{ij} n_i n_j$, and the summation runs over some range in $\{(n_1,n_2,\ldots,n_k) \mid 0 \le n_i \le N-1\,(i=1,2,\ldots,k)\}$. Note that $q = \exp\left(\frac{2\pi\sqrt{-1}}{N}\right)$ and we regard S as a function of N rather than q.

Then *an optimistic limit* of $\frac{2\pi\sqrt{-1}\log S(N)}{N}$, denoted by $\underset{N\to\infty}{\text{o-lim}} \frac{2\pi\sqrt{-1}\log S(N)}{N}$ is defined as follows.

First we replace $S(N)/P(N)$ with the following iterated integral $I(N)$ along some contours.

$$I(N) := \int\cdots\int \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,z_2,\ldots,z_k)\right] dz_1\, dz_2 \cdots dz_k. \tag{3.2}$$

Here $V(z_1,z_2,\ldots,z_k)$ is defined as follows. Put

$$\tilde{V}(z_1,z_2,\ldots,z_k) := -\sum_{a=1}^{\alpha} \varepsilon_a \left\{ \mathrm{Li}_2(x_a) - \frac{\pi^2}{6} \right\} + \sum_{1\le i\le j\le k} r_{ij} \log z_i \log z_j, \tag{3.3}$$

where $z_i = q^{n_i}$ and $x_a = q^{l_a'}$ with l_a' the degree one term in l_a, and $\mathrm{Li}_2(z)$ is Euler's dilogarithm defined by

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-u)}{u}\, du.$$

Next we consider the following system of partial differential equations:

$$\frac{\partial \tilde{V}(z_1,z_2,\ldots,z_k)}{\partial z_i} = 0 \quad (i=1,2,\ldots,k). \tag{3.4}$$

Since

$$\frac{\partial \tilde{V}(z_1,z_2,\ldots,z_k)}{\partial z_i} = \sum_{a=1}^{\alpha} \varepsilon_a l_{ai} \frac{\log(1-x_a)}{z_i} + \sum_{j=1}^{k} r_{ij} \frac{\log z_j}{z_i} + r_{ii} \frac{\log z_i}{z_i}$$

with $l_a' = \sum_{i=1}^{k} l_{ai} z_i$, (3.4) implies the following algebraic equations.

$$z_i^{r_{ii}} \prod_{j=1}^{k} z_j^{r_{ij}} \prod_{a=1}^{\alpha} (1-x_a)^{\varepsilon_a l_{ai}} = 1 \quad (i=1,2,\ldots,k). \tag{3.5}$$

Let $(\zeta_1,\zeta_2,\ldots,\zeta_k)$ be a solution to (3.5).

Definition 3.1 (optimistic limit). We put

$$V(\zeta_1,\zeta_2,\dots,\zeta_k) := \tilde{V}(\zeta_1,\zeta_2,\dots,\zeta_k) + 2\pi\sqrt{-1}\left(\sum_{j=1}^{k} c_j \log\zeta_j\right) \tag{3.6}$$

and call it *an optimistic limit* of $\dfrac{2\pi\sqrt{-1}\log S(N)}{N}$ as N goes to the infinity. It is denoted by $\operatorname*{o-lim}_{N\to\infty}\dfrac{2\pi\sqrt{-1}\log S(N)}{N}$. Here c_i is chosen so that $\dfrac{\partial\tilde{V}(\zeta_1,\zeta_2,\dots,\zeta_k)}{\partial z_i} + 2\pi\sqrt{-1}\sum_{j=1}^{k}\dfrac{c_j}{\zeta_j} = 0$ for every i.

Remark 3.2. The term $-\dfrac{\pi^2}{6}$ in (3.3) appears so that $V(1,1,\dots,1) = 0$ since $\operatorname{Li}_2(1) = \dfrac{\pi^2}{6}$ (see for example [6, (1.5)]). (I learned this from [11, §5].)

Remark 3.3. Note that V and $\tilde{V}$ satisfy the same algebraic equations (3.5) but different partial differential equations (3.4) and that the extra terms in (3.6) are necessary to choose an appropriate branch since Li_2 and log are multivalued functions. (I learned this from T. Takata.)

Remark 3.4. An optimistic limit is *not* well defined. There are many ambiguities both in choosing $I(N)$ (I did not say anything about the range of the summation in $S(N)$ and the contours in $I(N)$) and in choosing $(\zeta_1,\zeta_2,\dots,\zeta_k)$.

Remark 3.5. Following Kashaev [2], the behavior of $S(N)$ for large N may be approximated by $P(N)I(N)$ with suitably chosen contours. Moreover by using the saddle point method (see for example [7, §7.2]) we see that $I(N)$ (and $S(N)$) behaves like $\exp\left(\dfrac{N}{2\pi\sqrt{-1}}\operatorname*{o-lim}_{N\to\infty}\dfrac{2\pi\sqrt{-1}\log S(N)}{N}\right)$ for large N if we choose a solution $(\zeta_1,\zeta_2,\dots,\zeta_k)$ suitably. Even if this is not true I expect that there is a relation between an optimistic limit and the asymptotic behavior of $S(N)$.

4. Dehn surgery along the figure-eight knot

Put $k := 2$, $n_1 := n$, $n_2 := m$, $z := z_1$, $w := z_2$, $Q := pn^2/4 - mn$, $L := -n/2$, $\alpha := 4$, $l_1 := n$, $\varepsilon_1 := 1$, $l_2 := n-1$, $\varepsilon_2 := -1$, $l_3 := n+m$, $\varepsilon_3 := 1$, $l_4 := n-m-1$, and $\varepsilon_4 := -1$ in (3.1). Note that $x_1 = z$, $x_2 = z$, $x_3 = zw$, $x_4 = zw^{-1}$, $r_{11} = p/4$, $r_{12} = -1$, $r_{22} = 0$, $l_{11} = 1$, $l_{12} = 0$, $l_{21} = 1$, $l_{22} = 0$, $l_{31} = 1$, $l_{32} = 1$, $l_{41} = 1$, $l_{42} = -1$. Then

$$\tilde{V}(z,w) := -\operatorname{Li}_2(zw) + \operatorname{Li}_2\left(\frac{z}{w}\right) + \frac{p}{4}(\log z)^2 - \log z\log w$$

and

$$\begin{cases} \dfrac{\partial\tilde{V}}{\partial z} &= \dfrac{1}{z}\left\{\log z^{p/2} + \log\left(\dfrac{1-zw}{w-z}\right)\right\}, \\[2ex] \dfrac{\partial\tilde{V}}{\partial w} &= \dfrac{1}{w}\log\dfrac{(1-zw)(w-z)}{zw}. \end{cases}$$

Therefore (3.5) turns out to be

$$\begin{cases} z^{p/2}(1-zw) = w-z, \\[2ex] (1-zw)(w-z) = zw, \end{cases} \tag{4.1}$$

from which we have

$$
(4.2) \qquad \begin{cases} w = \dfrac{z + z^{p/2}}{z^{p/2} z + 1}, \\ z^2 - \left(\dfrac{z + z^{p/2}}{z^{p/2} z + 1} + 1 + \dfrac{z^{p/2} z + 1}{z + z^{p/2}} \right) z + 1 = 0. \end{cases}
$$

Remark 4.1. Since the second equation of (4.2) is symmetric with respect to z and z^{-1}, if ζ is a solution to it then so is ζ^{-1}. (This may be caused by the amphicheirality of the figure-eight knot.) Clearly $\overline{\zeta}$, the complex conjugate of ζ, also satisfies it. Therefore if (ζ, ω) is a solution to (4.2) then so are $(\overline{\zeta}, \overline{\omega})$, (ζ^{-1}, ω), and $(\overline{\zeta}^{-1}, \overline{\omega})$.

I will show calculations for $p = 0, 1, \ldots, 10$.

4.1. **6-surgery along the figure-eight knot.** I will describe the case where $p = 6$ in detail. Note that M_6 is hyperbolic. In this case there are the following six solutions to (4.2) due to MAPLE V:

$$
(\zeta_1, \omega_1),\ (\zeta_2, \omega_2),\ ({\zeta_1}^{-1}, \omega_1),\ (\overline{\zeta_2}, \overline{\omega_2}),\ ({\zeta_2}^{-1}, \omega_2),\ (\overline{\zeta_2}^{-1}, \overline{\omega_2}),
$$

where

$$
(\zeta_1, \omega_1) = (-0.8294835410 - 0.5585311587\sqrt{-1}, -2.205569430 \quad - 0.3703811357 \times 10^{-9}\sqrt{-1})
$$

and

$$
(\zeta_2, \omega_2) = (\ \ 0.3679390314 - 0.4972675889\sqrt{-1},\ \ 0.1027847152 - 0.6654569513\sqrt{-1}).
$$

Note that $|\zeta_1| = 1$ and so $\overline{\zeta_1} = {\zeta_1}^{-1}$ and $\overline{\zeta_1}^{-1} = \zeta_1$.

The partial derivatives $\dfrac{\partial \tilde{V}(\zeta, \omega)}{\partial z}$ and $\dfrac{\partial \tilde{V}(\zeta, \omega)}{\partial w}$ for (ζ_1, ω_1) and (ζ_2, ω_2) are as follows.

$$
\begin{cases} \dfrac{\partial \tilde{V}}{\partial z}(\zeta_1, \omega_1) & = 0.424142903 \times 10^{-10} - 6.283185309\sqrt{-1}, \\ \dfrac{\partial \tilde{V}}{\partial w}(\zeta_1, \omega_1) & = 0.2205569430 \times 10^{-9} - 0.2205569430 \times 10^{-9}\sqrt{-1}, \end{cases}
$$

and

$$
\begin{cases} \dfrac{\partial \tilde{V}}{\partial z}(\zeta_2, \omega_2) & = 0.3868858795 \times 10^{-9} + 0.5171193081 \times 10^{-9}\sqrt{-1}, \\ \dfrac{\partial \tilde{V}}{\partial w}(\zeta_2, \omega_2) & = 0.3623902007 \times 10^{-10} - 0.6757354228 \times 10^{-9}\sqrt{-1}. \end{cases}
$$

Therefore we have

$$
\begin{aligned}
V(\zeta_1, \omega_1) &= \tilde{V}(\zeta_1, \omega_1) + 2\pi\sqrt{-1}\log\zeta_1, \\
V(\zeta_2, \omega_2) &= \tilde{V}(\zeta_2, \omega_2), \\
V({\zeta_1}^{-1}, \omega_1) &= V(\zeta_1, \omega_1), \\
V(\overline{\zeta_2}, \overline{\omega_2}) &= \overline{V(\zeta_2, \omega_2)}, \\
V({\zeta_2}^{-1}, \omega_2) &= V(\zeta_2, \omega_2), \\
V(\overline{\zeta_2}^{-1}, \overline{\omega_2}) &= \overline{V(\zeta_2, \omega_2)},
\end{aligned}
$$

with

$$
V(\zeta_1, \omega_1) = 13.76750570 + 0.1 \times 10^{-8}\sqrt{-1}
$$

and

$$V(\zeta_2, \omega_2) = 1.340917487 + 1.284485301\sqrt{-1}$$

by MAPLE V.

So there are three optimistic limits (up to 10 digits); $V_1 := 13.76750570$, $V_2 := 1.340917487 + 1.284485301\sqrt{-1}$, and $\overline{V_2}$. By using SnapPea [12], we calculate $\mathrm{Vol}(M_6) = 1.2844853$ and $\mathrm{cs}(M_6) = 0.0679316734799$ and so we can write

$$V_2 = \mathrm{CS}(M_6) + \mathrm{Vol}(M_6)\sqrt{-1}.$$

since $0.0679316734799 \times 2\pi^2 = 1.34091748750\ldots$. Here $\mathrm{Vol}(M)$ and $\mathrm{cs}(M)$ are the volume and the Chern–Simons invariant [1] of a closed hyperbolic three-manifold M respectively, and $\mathrm{CS}(M) := 2\pi^2 \,\mathrm{cs}(M)$.

4.2. $5, 7, 8, 9, 10$**-surgeries along the figure-eight knot.** Similar calculations using MAPLE V for $p = 5, 7, 8, 9, 10$ give the following. Note that M_p is also hyperbolic in this case.

Observation 4.2. *For $p = 5, 6, 7, 8, 9, 10$ there is an optimistic limit $V(\zeta_p, \omega_p)$ of $\dfrac{2\pi\sqrt{-1}\log \tau_N(M_p)}{N}$ such that*

$$V(\zeta_p, \omega_p) = \mathrm{CS}(M_p) + \mathrm{Vol}(M_p)\sqrt{-1}$$

up to several digits. Here

$$V(z, w) = -\,\mathrm{Li}_2(zw) + \mathrm{Li}_2(z/w) + \frac{p}{4}(\log z)^2 - \log z \log w$$

and

$$\begin{aligned}
(\zeta_5, \omega_5) &= (0.1979823656 - 0.4438341209\sqrt{-1},\ 0.007552359501 - 0.5131157955\sqrt{-1}),\\
(\zeta_6, \omega_6) &= (0.3679390314 - 0.4972675889\sqrt{-1},\ 0.1027847152 \quad - 0.6654569513\sqrt{-1}),\\
(\zeta_7, \omega_7) &= (0.4855046904 - 0.5042960525\sqrt{-1},\ 0.1761405059 \quad - 0.7455559248\sqrt{-1}),\\
(\zeta_8, \omega_8) &= (0.5730134132 - 0.4940983127\sqrt{-1},\ 0.2327856161 \quad - 0.7925519927\sqrt{-1}),\\
(\zeta_9, \omega_9) &= (0.6404276706 - 0.4765868179\sqrt{-1},\ 0.2769632324 \quad - 0.8216401587\sqrt{-1}),\\
(\zeta_{10}, \omega_{10}) &= (0.6935298015 - 0.4561607978\sqrt{-1},\ 0.3118108269 \quad - 0.8402398912\sqrt{-1}).
\end{aligned}$$

Remark 4.3. For $p = 4n + 2$ with $n = 1, 2, \ldots, 100$, we can observe the same result. See Figures 1 and 2 for (ζ_p, ω_p).

4.3. $1, 2, 3$**-surgeries along the figure-eight knot.** Next we consider the case where $p = 1, 2$ and 3. Note that the manifold M_p for $p = 1, 2, 3$ is a Seifert fibered space. See [4, p. 95] for details, which was informed by K. Ichihara.

In this case, the (simplicial) volume of M_p is zero but SnapPea tells us that it has the non-trivial Chern–Simons invariant. MAPLE V shows that the same observation as Observation 4.2 holds with

$$\begin{aligned}
(\zeta_1, \omega_1) &= (0.3738178762, 0.8019377355),\\
(\zeta_2, \omega_2) &= (0.346014339\ , 0.6180339884),\\
(\zeta_3, \omega_3) &= (0.2819716801, 0.4142135623).
\end{aligned}$$

4.4. 0**-surgery of the figure-eight knot.** In this case M_0 is a torus bundle over a circle. For a detail see [4, p. 95] again. Both the volume and the Chern–Simons invariant vanish in this case and Observation 4.2 holds putting $(\zeta_0, \omega_0) = (0.381966011, 1)$.

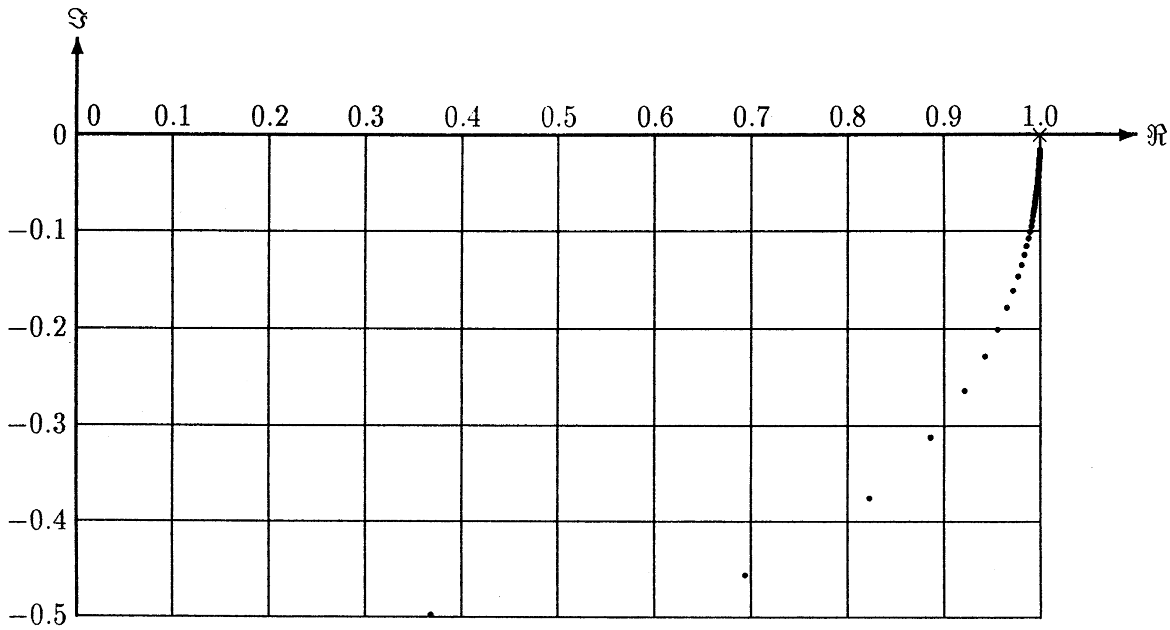

FIGURE 1. ζ_p is indicated by a dot for $p = 4n + 2$ ($n = 1, 2, \ldots, 100$). The dots approach $\zeta_\infty = 1$ indicated by $\times$ from the left when n increases.

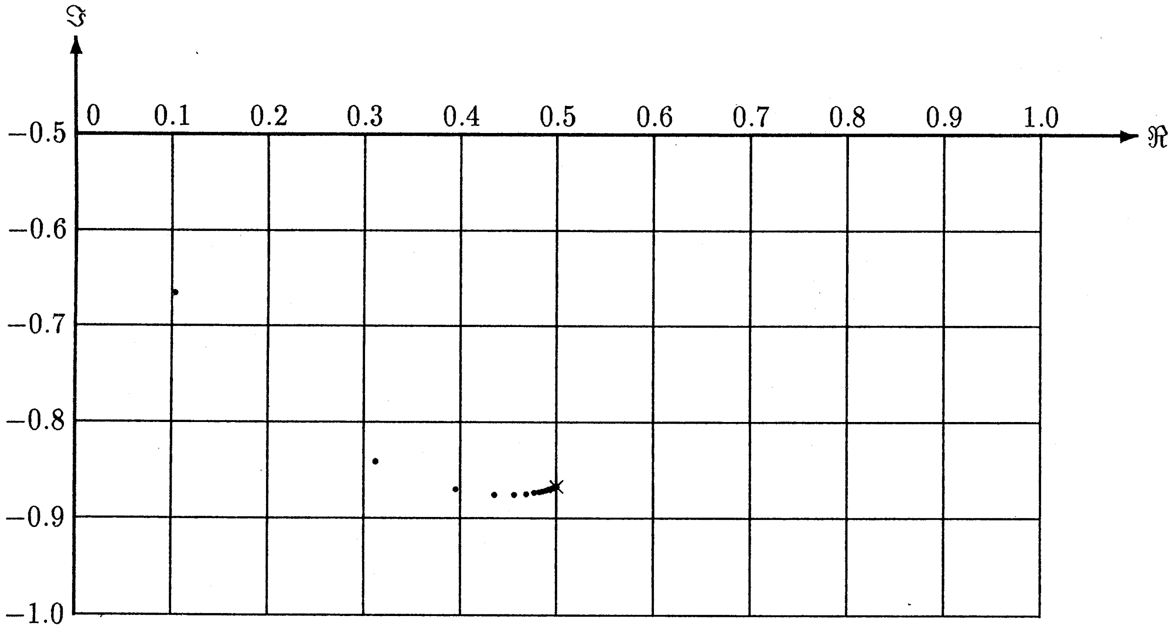

FIGURE 2. ω_p is indicated by a dot for $p = 4n + 2$ ($n = 1, 2, \ldots, 100$). The dots approach $\omega_\infty = \exp\left(-\frac{\pi\sqrt{-1}}{3}\right)$ indicated by $\times$ from the left when n increases.

4.5. **4-surgery of the figure-eight knot.** It is known that M_4 is toroidal and can be obtained by gluing the twisted I-bundle over the Klein bottle and the complement of the trefoil, which I learned from K. Motegi and M. Teragaito. The gluing map can be found in [4, p. 95] again. Therefore $\mathrm{Vol}(M_4) = 0$. (Here I use Vol for v_3 times the simplicial volume.) Computation by using MAPLE V told us that if we put $(\zeta_4, \omega_4) = (-1, -.381966011)$ then $V(\zeta_4, \omega_4) = 1.973920880 + 0.1 \times 10^{-8}\sqrt{-1}$. Note that $1.973920880 = 2\pi^2 \times 0.09999999995$. I guess this is the sum of the Chern–Simons invariants of the two pieces (with suitably chosen metrics), which one might regard as the Chern–Simons invariant of the toroidal manifold M_4.

4.6. **∞-surgery of the figure-eight knot.** Put $\zeta_\infty := \exp\left(-\frac{2\pi\sqrt{-1}}{p}\right)$ and $\omega_\infty := \exp\left(-\frac{\pi\sqrt{-1}}{3}\right)$. Then the left hand side minus the right hand side of the first equation in (4.1) is $(\zeta_\infty - 1)(\omega_\infty + 1)$ since ${\zeta_\infty}^{p/2} = -1$. That of the second equation is $\omega_\infty(\zeta_\infty - 1)^2$ since ${\omega_\infty}^2 = \omega_\infty - 1$. Noting that $\zeta_\infty \to 1$ if $p \to \infty$, $(\zeta_\infty, \omega_\infty)$ can be regarded as a solution to (4.1) for a large p.

Now we calculate $V(\zeta_\infty, \omega_\infty)$. Since $\frac{\partial \tilde{V}}{\partial z}(\zeta_\infty, \omega_\infty) \to 0$ and $\frac{\partial \tilde{V}}{\partial w}(\zeta_\infty, \omega_\infty) \to 0$ if $p \to \infty$,

$$
\begin{aligned}
V(\zeta_\infty, \omega_\infty) &= -\operatorname{Li}_2(\zeta_\infty\omega_\infty) + \operatorname{Li}_2\left(\frac{\zeta_\infty}{\omega_\infty}\right) + \frac{p}{4}(\log\zeta_\infty)^2 - \log\zeta_\infty \log\omega_\infty \\
&\xrightarrow[N\to\infty]{} -\operatorname{Li}_2\left(\exp\left(-\frac{\pi\sqrt{-1}}{3}\right)\right) + \operatorname{Li}_2\left(\exp\left(\frac{\pi\sqrt{-1}}{3}\right)\right) \\
&= \pi^2\left\{\overline{B}_2\left(\frac{1}{6}\right) - \overline{B}_2\left(-\frac{1}{6}\right)\right\} + \sqrt{-1}\left\{Л\left(\frac{\pi}{3}\right) - Л\left(-\frac{\pi}{3}\right)\right\} \\
&= \sqrt{-1} \times 2\,Л\left(\frac{\pi}{3}\right).
\end{aligned}
$$

Here I used the fact that

$$
\operatorname{Li}_2(\exp(\theta\sqrt{-1})) = \pi^2\overline{B}_2\left(\frac{\theta}{2\pi}\right) + \sqrt{-1}\,Л(\theta)
$$

with Л the Lobachevskij function and $\overline{B}_2$ the second modified Bernoulli polynomial [6, Proposition B]:

$$
\overline{B}_2(x) := -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n\pi x)}{n^2}.
$$

Note that Л is an odd function and $\overline{B}_2$ is an even function.

Thus we can write

$$
V(\zeta_\infty, \omega_\infty) = \operatorname{CS}(4_1) + \sqrt{-1}\operatorname{Vol}(4_1).
$$

This suggests that the series of optimistic limits $\left\{\underset{N\to\infty}{\text{o-lim}} \frac{2\pi\sqrt{-1}\log\tau_N(M_p)}{N}\right\}_{p=0,1,\dots,\infty}$ goes to $\lim_{N\to\infty}\frac{2\pi\sqrt{-1}\log J_N(4_1)}{N}$, agreeing with the facts that $\lim_{p\to\infty}\operatorname{Vol}(M_p) = \operatorname{Vol}(4_1)$ and that $\lim_{p\to\infty}\operatorname{CS}(M_p) = \operatorname{CS}(4_1)$. Note that $\operatorname{CS}(4_1) = 0$ since 4_1 is amphicheiral.

Remark 4.4. It was suggested by A. Kricker to observe the $p \to \infty$ limit.

5. Volume Conjecture for closed three-manifolds

Now I propose a very ambiguous conjecture.

Conjecture 5.1 (Volume conjecture for closed three-manifolds). *For any closed three-manifold M*

$$
\underset{N\to\infty}{\text{o-lim}}\frac{2\pi\sqrt{-1}\log\tau_N(M)}{N} = \operatorname{CS}(M) + \sqrt{-1}\operatorname{Vol}(M),
$$

where $\operatorname{Vol}(M) := v_3\|M\|$ *with* $\|M\|$ *the simplicial volume of* M.

A weaker but more precise conjecture is

Conjecture 5.2. *For an integer p put*

$$V_p(z,w) := -\operatorname{Li}_2(zw) + \operatorname{Li}_2\left(\frac{z}{w}\right) + \frac{p}{4}(\log z)^2 - \log z \log w.$$

Then there exists (ζ_p, ω_p) such that

(1) $\dfrac{\partial V_p}{\partial z}(\zeta_p, \omega_p) = \dfrac{\partial V_p}{\partial w}(\zeta_p, \omega_p) = 0$, *and*

(2) $V_p(\zeta_p, \omega_p) = \mathrm{CS}(M_p) + \sqrt{-1}\,\mathrm{Vol}(M_p)$.

Here M_p is the closed three-manifold obtained from the three-sphere by p-surgery along the figure-eight knot.

Remark 5.3. Conjecture 5.2 is *numerically* true (up to 8 digits or so) for $p = 0, \pm1, \pm2, \pm3, \pm5, \pm6, \dots, \pm100$.

Remark 5.4. Note that $V_p(1,w) = -\operatorname{Li}_2(w) + \operatorname{Li}_2\left(\dfrac{1}{w}\right)$ and this appears in calculations about the figure-eight knot [2, 3.15]. More precisely $\omega := \exp\left(-\dfrac{\pi\sqrt{-1}}{3}\right)$ satisfies

(1) $\dfrac{\partial V_p}{\partial w}(1,\omega) = 0$, and

(2) $V_p(1,\omega) = \mathrm{CS}(4_1) + \sqrt{-1}\,\mathrm{Vol}(4_1)$.

Finally I raise some natural problems.

Problem 5.5. Can one generalize Conjecture 5.2 to rational surgery? Compare with [15].

Problem 5.6. For any knot K (or more generally link) and any *rational* number p, does there exist a function V_p as above?

Remark 5.7. Y. Yokota told me that Conjecture 5.2 and Problem 5.5 can be solved by considering deformations of tetrahedron decomposition described in [13].

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