GL$_l$ のスーパーカスピダル表現の指標公式 (CHARACTER FORMULA FOR THE SUPERCUSPIDAL REPRESENTATIONS OF GL$_l$)

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Introduction

Let $l$ be a prime, $A$ a central simple algebra of dimension $l^2$ over a non-archimedean local field $F$ and $E/F$ an extension of degree $l$ in $A$. As is well-known ([15], [5]), any irreducible supercuspidal representation of $A^\times$ is obtained from a quasi-character of $E^\times$. The aim of this paper is to get a character formula for the irreducible supercuspidal representations of $A^\times$ on the set of elliptic regular elements. To calculate the character on the split torus is another problem. See Murnaghan’s papers ([16], [17] and [18]) for this topic. We only remark that the character value on the elliptic regular conjugacy classes determines uniquely the supercuspidal representation.

When $E/F$ is unramified, a character formula was obtained in [20]. Therefore we treat the case $E/F$ is ramified. When the residual characteristic $p$ of $F$ equals to $l$, the ramification is wild. This case is very hard to treat (see e.g. [21]). Thus we assume $p \neq l$.

Since the case $l = 2$ is already solved in [13], we also assume $l$ is odd. We note that $A$ is isomorphic to a division algebra $D = D_l$ of dimension $l^2$ over $F$ or the algebra $M_l(F)$ of $l \times l$ matrices over $F$.

Let $D_n$ be a division algebra of dimension $n^2$ over $F$. Deligne-Kazhdan-Vignéras [8] and Rogawski [19] proved an abstract matching theorem: there is a bijection between the set of equivalence classes of irreducible representations of $D_n^\times$ and that of essentially square-integrable representations of $GL_n(F)$ which preserves the characters up to $(-1)^{n-1}$. In the same case, i.e., when $n$ is prime to the residual characteristic of $F$, Moy [15] proved that there is a bijection between the same sets as above using the concrete construction of the representations given by Howe [12]. Hennirat ([10]) has shown that two correspondences coincide when $n = l \neq p$. Thus we only treat the GL$_l$ case. Our main result is Theorem 3.12. As in the unramified case, the analogue of Weyl’s character formula holds for our character formula. This does not hold when $l = p$ (cf. [22]).

Section 1 is devoted to the review of the construction of an irreducible supercuspidal representation $\pi_\theta$ (resp. $\pi_\theta'$) of GL$_l(F)$ (resp. $D_l^\times$) from a generic quasi-character $\theta$ of $E^\times$ and the known results about the representation. We note that $\pi_\theta$ is not always monomial, i.e., induced from a one-dimensional representation, but it can be written as a $Q$-linear combination of monomial representations. In fact $\pi_\theta$ is written as $Q$-linear combination of the forms ind$_H^{GL_l(F)} \rho_\theta$ where $H$ is a compact mod center subgroup of GL$_l(F)$ and $\rho_\theta$ is quasi-characters of $H$.

In section 2, we compute the character of $\pi_\theta$ up to some root numbers. Let $G = GL_l(F)$, $B$ the normalizer of an Iwahori subgroup of $G$ containing $H$ and $\eta_\theta = \text{ind}_H^B \rho_\theta$. Since we treat only elliptic regular conjugacy classes, we consider the character $\chi_{\pi_\theta}$ on $L^\times$ where $L/F$ are extensions of fields of degree $l$. Moreover the case $L = E$ is essential. By the
Frobenius formula and the result of Kutzko (\cite{14}), we have only to calculate the sum
\[ \chi_{\eta \theta}(x) = \sum_{a \in H ackslash B} \rho_{\theta}(axa^{-1}) \]
for \( x \in E \) in order to get the character formula of \( \pi_{\theta} \). Therefore it is essential to know when \( axa^{-1} \in H \), which is determined in Lemma 2.1. From this, we get the character formula of \( \eta_{\theta} \) except near the conductor (Proposition 2.2). But this formula contains the Gauss sum part \( G(y, j) \), which is calculated later. The exceptional part can be calculated directly by taking the explicit form of \( E^\times \) (Lemma 2.4). Except this lemma, there is no new result in this section. But since the proofs are short and simple and we use the property “intertwining implies conjugacy” of \( E/F \)-minimal element (very cuspidal in the terminology of Carayol \cite{4}) as the key tool, the result may be extended to \( \text{GL}_n \), at least when \( n \) is prime to \( p \). Section 3 is devoted to the calculation of the Gauss sum part \( G(y, j) \). It appears in the character formula on \( E^\times \). For this purpose, it is the point that we have only to treat the character of \( \pi_{\theta} \) on \( U_{1}^* = F^\times (1 + P) - F^\times (1 + P^2) \). For this calculation, we use the \( E^\times \)-structure of various objects. We first assume \( E/F \) is a Galois extension since \( E^\times \)-module structure can be described easily for this case. This part is analogous to section 1 of \cite{20}, but everything becomes easier since we have only to treat \( U_{1}^* \). When \( E/F \) is non-Galois, we use the base change lift. Let \( L/F \) be an unramified extension of degree \( l - 1 \). In \( L \), there exists a \( l \)-th root of unity and \( EL/L \) is Galois. Therefore we can use the tools in Galois case for \( \text{GL}_d(L) \). Let \( \text{Gal}(L/F) = \langle \tau \rangle \). By the result of Bushnell-Henniart \cite{3}, there is a base change lift \( \eta_L \) of \( \eta_{\theta} \) to \( H^*_L \) such that the twisted trace of \( \eta_L \) by \( \tau \) gives the trace of \( \eta_{\theta} \). (See Proposition 3.7 and Lemma 3.8.) We remark that we need not assume the characteristic of \( F \) is 0 since we do not use the Arthur-Clozel base change lift \cite{1}. The method to calculate the twisted trace of \( \eta_L \) is similar to that of Galois case. The complete character formula is stated as Theorem 3.12.

Closing this introduction, we compare our formula with the known results. The same type of character formula for the division algebra case was given by Corwin, Moy and Sally, Jrin \cite{6} and for \( \text{GL}_l \) case by Debacker in \cite{7}. Their formulas agree with the result given in section 2. It contains some root numbers associated with a quadratic form. In this paper, we have determined it completely in section 3. Moreover we find the Kloosterman sum appears in the character formula. These are new results of this paper. In \cite{22}, the author gave the character formula of \( \pi_{\theta} \) for \( \text{GL}_3 \) by using the decomposition of \( \pi_{\theta} \) as \( E^\times \)-module. But this need the explicit matrix form of an inverse matrix which is hard to treat for large \( l \). We can simplify the proof of the main theorem, although we treat a general prime \( l \).

**Notation**

Let \( F \) be a non-archimedean local field. We denote by \( \mathcal{O}_F, P_F, \varpi_F, k_F \) and \( v_F \) the maximal order of \( F \), the maximal ideal of \( \mathcal{O}_F \), a prime element of \( P_F \), the residue field of \( F \) and the valuation of \( F \) normalized by \( v_F(\varpi_F) = 1 \). We set \( q \) to be the number of elements in \( k_F \). Hereafter we fix an additive character \( \psi \) of \( F \) whose conductor is \( P_F \), i.e., \( \psi \) is trivial on \( P_F \) and not trivial on \( \mathcal{O}_F \). For an extension \( E \) over \( F \), we denote by \( \text{tr}_{E}, \pi_E \) the trace and norm to \( F \) respectively. We set \( \psi_E = \psi \circ \text{tr}_{E} \). The trace of matrix is denoted by \( \text{Tr} \). For an irreducible admissible representation \( \pi \) of \( \text{GL}_d(F) \), the conductoral exponent of \( \pi \) is defined to be the integer \( f(\pi) \) such that the local constant \( \varepsilon(s, \pi, \psi) \) of Godement-Jacquet \cite{9} is the form \( aq^{-s(f(\pi)-1)} \).
We call \( \pi \) minimal if
\[
 f(\pi) = \min_\eta f(\pi \otimes (\eta \circ \text{Nr})),
\]
where \( \eta \) runs through the quasi-characters of \( F^\times \). Let \( G \) be a totally disconnected, locally compact group. We denote by \( \hat{G} \) the set of (equivalence classes of) irreducible admissible representations of \( G \). For a closed subgroup \( H \) of \( G \) and a representation \( \rho \) of \( H \), we denote by \( \text{Ind}_H^G \rho \) (resp. \( \text{ind}_H^G \rho \)) the induced representation (resp. compactly induced representation) of \( \rho \) to \( G \). For a representation \( \pi \) of \( G \), we denote by \( \pi|_H \) the restriction of \( \pi \) to \( H \).

1. Construction of the representation

Let \( E \) be a ramified extension of \( F \) of degree \( l \). Then \( E \) can be embedded into \( M_l(F) \) and, up to conjugacy, the embedding is unique. Let \( G = \text{GL}_l(F) \). In this section, we review the construction of supercuspidal representations of \( G \) which are parameterized by the quasi-characters of \( E^\times \). Of course, this construction is well-known([4], [15]).

**Definition 1.1.** Let \( \theta \) be a quasi-character of \( E^\times \) and \( f(\theta) = \min\{n| \text{Ker} \theta \subset 1 + P_E^n\} \). Then \( \theta \) is called generic if \( f(\theta) \neq 1 \) mod \( l \). For a generic character \( \theta \) of \( E^\times \), \( \gamma_\theta \in P_E^{1-f(\theta)} - P_E^{2-f(\theta)} \) is defined by
\[
(1.1) \quad \theta(1+x) = \psi_E(\gamma_\theta x) \quad \text{for} \quad x \in P_E^{[f(\theta)+1]/2}.
\]
Then \( F(\gamma_\theta) = E \). We denote by \( \hat{E}_\text{gen}^\times \) the set of generic quasi-characters of \( E^\times \).

We construct an irreducible supercuspidal representation of \( G = \text{GL}_l(F) \) from \( \theta \in \hat{E}_\text{gen}^\times \). For simplicity, we set \( \gamma = \gamma_\theta \). Since \( E/F \) is tamely ramified, there exists a prime element \( \varpi_E \) of \( \mathcal{O}_E \) satisfying \( \varpi_E^l \in F \). Put \( \varpi_F = \varpi_E^l \). We identifies \( M_l(F) \) with \( \text{End}_F E \) and \( G \) with \( \text{Aut}_F E \) by the \( F \)-basis \( \{\varpi_E^{l-1}, \varpi_E^{l-2}, \ldots, \varpi_E, 1\} \) of \( E \), which is also an \( \mathcal{O}_F \)-basis of \( \mathcal{O}_E \). By the lattice flag \( \{P_E^i\}_{i \in \mathbb{Z}} \), we construct a maximal compact modulo center subgroup. The construction of the representation is well-known. For details, see [15].

**Definition 1.2.** For \( i \in \mathbb{Z} \), set
\[
 A^i = \{f \in M_l(F) | f(P_E^j) \subset P_E^{j+i} \quad \text{for all} \quad j \in \mathbb{Z}\}.
\]
Put \( K = (A^0)^\times, B = E^\times K \) and \( K^i = 1 + A^i \) for \( i \geq 1 \).

Then \( K \) is an Iwahori subgroup of \( G \) and \( B \) is a normalizer of \( K \). At first we construct an irreducible representation of \( B \) from a generic quasi-character of \( E^\times \).

Let \( \theta \) be a generic quasi-character of \( E^\times \), i.e., \( f(\theta) = n \neq 1 \) mod \( l \). There exists an element \( \gamma \in P_E^{1-n} \) such that \( \theta(1+x) = \psi_E(\gamma x) \) for \( x \in P_E^m \) where \( m = [(n+1)/2] \). Define \( \psi_\gamma \) on \( K^m \) by \( \psi_\gamma(1+x) = \psi(\text{Tr}(\gamma x)) \) for \( x \in A^m \). Then \( \psi_\gamma \) is a quasi-character of \( K^m \). Put \( H = E^\times K^m \) and define a quasi-character \( \rho_\theta \) of \( H \) by
\[
(1.2) \quad \rho_\theta(h \cdot g) = \theta(h)\psi_\gamma(g) \quad \text{for} \quad h \in E^\times, \ g \in K^m.
\]

Let \( J \) be the normalizer of \( \psi_\gamma \) in \( B \), i.e.,
\[
 J = \{a \in B \mid \psi_\gamma^a = \psi_\gamma\},
\]
where \( \psi_\gamma^a(x) = \psi_\gamma(a^{-1}xa) \) for \( x \in K^m \). Then \( J = E^\times K^{m'} \) where \( m' = [n/2] \). Put \( \eta_\theta = \text{Ind}_H^B \rho_\theta \).
When $n$ is even, i.e., $n=2m$, then $J=H=E^\times K^m$. By the Clifford theory, $\eta_\theta$ is an irreducible representation of $B$. We put

$$\kappa_\theta = \eta_\theta.$$  \hfill (1.3)

When $n$ is odd, i.e., $n=2m-1$, then $J=E^\times K^{m-1}$. Thus $\eta_\theta$ is not irreducible. In this case, we put

$$\kappa_\theta = \frac{1 - \left(\frac{q}{l}\right) q^{(l-1)/2}}{l q^{(l-1)/2}} \sum_{\chi \in \left(E^\times/F^\times(1+P_E)\right)^\ast} \eta_{\theta \otimes \chi} + \left(\frac{q}{l}\right) \eta_\theta,$$  \hfill (1.4)

where $\left(\frac{q}{l}\right)$ is the Legendre symbol. The following result is well-known (see [15]).

**Theorem 1.3.** Let the notation be as above. Then $\kappa_\theta$ is an irreducible representation of $B$. Put $\pi_\theta = \text{ind}^G_B \kappa_\theta$. Then $\pi_\theta$ is an irreducible supercuspidal representation of $G$ such that

1. the $L$-function of $\pi_\theta$ is 1;
2. $\varepsilon(\pi_\theta, \psi) = \varepsilon(\theta, \psi_E)$; in particular $f(\pi_\theta) = f(\theta) + l$;
3. $\bigcup_{E} \{\pi_\theta | \theta \in \hat{E}_{\text{gen}}^\ast\} = \{\pi \in A_0(G) | f_{\text{min}}(\pi) \equiv 0 \text{ mod } l\}$, where $E$ runs through isomorphism classes of ramified extensions of degree $l$ over $F$ and $A_0(G)$ be the set of equivalent classes of the supercuspidal representations of $G$.

**Remark.** If $\pi \in A_0(G)$ and $f_{\text{min}}(\pi) \equiv 0 \text{ mod } l$, $\pi$ can be constructed from a regular quasi-characters of $L^\times$, where $L$ is an unramified extension of $F$ of degree $l$. The character formula for such a representation was given in [20].

Next we construct an irreducible representation of $D^\times = D_l^\times$ from $\theta \in E_{\text{gen}}^\ast$. Let $f(\theta) = n$. We recall $n \equiv 1 \text{ mod } l$. We define a function $\psi_\gamma$ on $1 + P_D^m$ by $\psi_\gamma(1 + x) = \psi(\text{Tr}(\gamma x))$ for $x \in P_D$. Then $\psi_\gamma$ is a quasi-character of $1 + P_D$. $H' = E^\times (1 + P_D) \subset D^\times$ and define a quasi-character $\rho_\theta'$ of $H'$ by

$$\rho_\theta'(h \cdot g) = \theta(h)\psi_\gamma(g) \quad \text{for} \quad h \in E^\times, \quad g \in 1 + P_D.$$ \hfill (1.5)

When $n$ is even, i.e., $n=2m$, we set

$$\pi_\theta' = \text{Ind}^{D^\times}_{H'} \rho_\theta.'$$ \hfill (1.6)

When $n$ is odd, i.e., $n=2m-1$, we set

$$\pi_\theta' = \frac{1 - \left(\frac{q}{l}\right) q^{(l-1)/2}}{l q^{(l-1)/2}} \sum_{\chi \in \left(E^\times/F^\times(1+P_E)\right)^\ast} \text{Ind}^{D^\times}_{H'} \rho_\theta' \otimes \chi + \left(\frac{q}{l}\right) \text{Ind}^{D^\times}_{H'} \rho_\theta',$$ \hfill (1.7)

where $\left(\frac{q}{l}\right)$ is the Legendre symbol. The following result is essentially well-known. (See [2], [15]).

**Theorem 1.4.** Let the notation be as above. Then $\pi_\theta'$ is an irreducible minimal representation of $D^\times$ such that

1. the degree of $\pi_\theta'$ is $q^{(n-2)(l-1)/2} (q - 1) / q - 1$;
2. $\varepsilon(\pi_\theta', \psi) = \varepsilon(\theta, \psi_E)$; in particular $f(\pi_\theta') = f(\theta) + l$;
3. \( \bigcup \{ \pi'_\theta | \theta \in \widehat{E}_{\text{reg}} \} = \{ \pi' \in \widehat{D}^\times | f_{\min}(\pi') \not\equiv 0 \pmod{l} \} \), where \( E \) runs through the isomorphism classes of ramified extensions of degree \( l \) over \( F \).

4. The correspondence \( \pi'_\theta \leftrightarrow \pi_\theta \) by way of generic quasi-characters of \( E^\times \) is a bijection and preserves \( \varepsilon \)-factors and conductoral exponents. (This correspondence is a special case of Howe’s bijection (see [15]).)

On the other hand, there exists an abstract matching theorem, which is called the Deligne-Kazhdan correspondence ([8], [19]).

**Theorem 1.5.** There is a bijection between the set of irreducible representations of \( D^\times \) and the set of essentially square-integrable representations of \( G \) which preserves the characters on elliptic regular elements. In particular, it preserves \( \varepsilon \)-factors and conductoral exponents.

By the result of Henniart ([10] Theorem 8.1), these two correspondences coincide.

**Theorem 1.6.** If \( l \neq p \) is a prime, Howe’s bijection (1.4) coincides with Deligne-Kazhdan correspondence (1.5) between the set of essentially square-integrable representations of \( GL_1 \) and the set of irreducible representations of \( D^\times_l \).

At the end of this section, we quote the result of Kutzko([14]) in the form that the character formula of \( \pi_\theta \) on elliptic regular elements is essentially given by the one of \( \kappa_\theta \).

**Theorem 1.7.** Let \( x \) be an elliptic regular element of \( G \).

1. If \( F(x)/F \) is ramified and \( x \not\in F^\times(1 + P_{F(x)}) \),
   \[
   \chi_{\pi_\theta}(x) = \chi_{\kappa_\theta}(x).
   \]

2. If \( F(x)/F \) is unramified and \( x \not\in F^\times(1 + P_{F(x)}^{[n/l]+1}) \),
   \[
   \chi_{\pi_\theta}(x) = 0.
   \]

**Proof.** These are obtained by applying Proposition 5.5 in [14] to our case.

**Remark.** Since

\[
(1.8) \quad \chi_{\kappa_\theta}(x) = \begin{cases} 
\left( \frac{q}{l} \right)_1 \chi_{\eta_\theta}(x) & x \in E^\times - F^\times(1 + P_E), \\
q(l-1)/2 \chi_{\eta_\theta}(x) & x \in F^\times(1 + P_E),
\end{cases}
\]

we have only to calculate \( \chi_{\eta_\theta} \).

2. Calculation of the character

Now we begin to calculate the characters of the representations constructed in the previous section. In this section, we shall get a character formula up to some root numbers. These root numbers are calculated explicitly in the next section.

Hereafter we fix a generic character \( \theta \) and put \( \rho = \rho_\theta, \eta = \eta_\theta \) and so on. Since \( E/F \) is a totally tamely ramified extension, there exists a prime element \( \varpi_E \) of \( O_E \) such that \( \varpi_E^l \in P_E - P_E^2 \). Put \( \varpi_E = \varpi_F \). As in the previous section, we identify \( M_i(F) \) with \( \text{End}_F(E) \) by the \( F \)-basis \( \{ \varpi_E^{l-1}, \varpi_E^{l-2}, \ldots, \varpi_E, 1 \} \), which is an \( O_F \)-basis of \( O_E \). Thus we get the explicit matrix forms of various objects:
If $q \equiv 1 \mod l$, $F$ has a primitive $l$-th root of unity $\zeta$ and $E/F$ is a Galois extension. Let $\sigma$ be a generator of $\text{Gal}(E/F)$ determined by $\sigma \varpi_E = \varpi_E \zeta$. We denote the diagonal matrix $\text{diag}(1, \zeta^{i-1}, \zeta^{i-2}, \ldots, \zeta)$ by $\xi$. Then $\xi$ satisfies $\xi^l = 1$ and $\xi x \xi^{-1} = \sigma x$ for $x \in E$.

Define a natural ring morphism $R$ from $A^0$ to $k_F^l$ by the identification of $A^0/A^1$ with $k_F^l$. We note that if $R(a) = (\alpha_0, \alpha_1, \cdots, \alpha_{l-1})$, $R(\varpi_E a \varpi_E^{-1}) = (\alpha_1, \alpha_2, \cdots, \alpha_0)$.

The next lemma is the key tool for the character calculation.

**Lemma 2.1.** Let $x \in P_E^i - (F + P_E^{i+1})$, $g \in B$ and $j$ a positive integer. If $g x g^{-1} \in E^\times (1 + A^j)$, then

$$g \in \begin{cases} E^\times (1 + A^j) & \text{if } q \equiv 1 \mod l, \\ \bigcup_{k=0}^{j-1} E^\times (1 + A^j) \xi^k & \text{if } q \not\equiv 1 \mod l. \end{cases}$$

**Proof.** We may assume $g \in A_0$ by replacing $g$ by $\varpi_E^{-k} g$ if $g \in A_k$. Let $x = \varpi_E x_0$ for $x_0 \in \mathcal{O}_E^\times$ and $R(g) = (\alpha_0, \alpha_1, \cdots, \alpha_{l-1})$. Then

$$R(g x g^{-1} x^{-1}) = (\alpha_0 \alpha_i^{-1}, \alpha_0 \alpha_{i+1}^{-1}, \cdots, \alpha_0 \alpha_{l-1}^{-1}),$$

where $\alpha_s = \alpha_{s \mod l}$ for $s \in \mathbb{Z}$. Since $x \not\in F + P_E^{i+1}$, $i \not\equiv 0 \mod l$. Therefore $g x g^{-1} x^{-1} \in E^\times$ implies

$$\begin{cases} \alpha_0 = \alpha_1 = \cdots = \alpha_{l-1} & \text{if } q \equiv 1 \mod l, \\ \alpha_k = \zeta^j \alpha_0 (0 \leq k \leq l-1) & \text{otherwise,} \end{cases}$$

for some integer $j$. Since $\zeta \varpi_E \xi^{-1} = \zeta \varpi_E$, we get:

$$g \in \begin{cases} E^\times (1 + A^j) & \text{if } q \equiv 1 \mod l, \\ \bigcup_{k=0}^{j-1} E^\times (1 + A^j) \xi^k & \text{otherwise.} \end{cases}$$

(2.1) \[ \varpi_E = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \varpi_F & 0 & \cdots & \cdots & 0 \end{pmatrix} \]

(2.2) \[ K = \left\{ \begin{array}{c|c} a_{11} & \cdots & a_{1l} \\ \hline a_{21} & \cdots & a_{2l} \\ \vdots & \ddots & \vdots \\ a_{l1} & \cdots & a_{ll} \end{array} \right\} \begin{array}{c} a_{ij} \in \mathcal{O}_F \text{ if } i < j \\ a_{ii} \in \mathcal{O}_F \text{ if } i = j \\ a_{ij} \in \mathcal{O}_F \text{ if } i > j \end{array} \]

(2.3) \[ A^0 = \left\{ \begin{array}{c|c} a_{11} & \cdots & a_{1l} \\ \hline a_{21} & \cdots & a_{2l} \\ \vdots & \ddots & \vdots \\ a_{l1} & \cdots & a_{ll} \end{array} \right\} \begin{array}{c} a_{ij} \in \mathcal{O}_F \text{ if } i < j \\ a_{ii} \in \mathcal{O}_F \text{ if } i = j \\ a_{ij} \in \mathcal{O}_F \text{ if } i > j \end{array} \]

(2.4) \[ A^1 = \left\{ \begin{array}{c|c} a_{11} & \cdots & a_{1l} \\ \hline a_{21} & \cdots & a_{2l} \\ \vdots & \ddots & \vdots \\ a_{l1} & \cdots & a_{ll} \end{array} \right\} \begin{array}{c} a_{ij} \in \mathcal{O}_F \text{ if } i < j \\ a_{ii} \in \mathcal{O}_F \text{ if } i = j \\ a_{ij} \in \mathcal{O}_F \text{ if } i > j \end{array} \]
Thus we may assume $g - 1 \in A^k - (P_E^{k+1} + A^{k+1})$ for $k \geq 1$. Put $g - 1 = \varpi_E^k g_0$ and $R(g_0) = (\beta_0, \beta_1, \ldots, \beta_{l-1})$. Since

$$gxg^{-1}x^{-1} \equiv 1 + (g - 1) - x(g - 1)x^{-1} \mod A^{k+1}$$

$$\equiv 1 + \varpi_E^k (g_0 - xg_0x^{-1}) \mod A^{k+1},$$

$R(g_0 - xg_0x^{-1}) = (\beta_0 - \beta_k, \beta_1 - \beta_{1+k}, \ldots, \beta_{l-1} - \beta_{l-1+k})$. Therefore $gxg^{-1}x^{-1} \in E^\times K^{k+1}$ contradicts $g - 1 \in A^k - (P_E^{k+1} + A^{k+1})$. It implies that if $gxg^{-1}x^{-1} \in E^\times K^j$,

$$R(g_0x_0x^{-1}) = (\beta_0 - \beta]\_k, \beta_1 - \beta_{1+k}, \ldots, \beta_{l-1} - \beta_{l-1+k})$$

Therefore $gxg^{-1}x^{-1} \in E^\times K^j$ contradicts $g - 1 \in A^{k+1}$.}

Put $U_{-1} = E^\times, U_0 = F^\times \mathcal{O}_E^\times, U_i = F^\times (1 + P_E^i)$ for $i \geq 1$ and $U_i^* = U_i - U_{i+1}$ for $j \geq -1$. The previous lemma gives the character of $\eta_0$ on $E^\times - U_{n-1}$. We remark $\text{Aut}_F E = \{1\}$ if $q \not\equiv 1 \mod l$.

**Proposition 2.2.** Let $x \in U_i^*$ for $-1 \leq i < n - 1$. If $i \not\equiv 0 \mod l$, $x$ is written in the form $x = c(1 + y)$ for $c \in F$ and $y = \varpi_E^iy_0 \in \varpi_E^i \mathcal{O}_E^\times$. For $u \in k_F^\times$ and $j \in (\mathbb{Z}/l\mathbb{Z})^\times$, we define the Gauss sum part $G(u, j)$ by

$$G(u, j) = \sum_{(\alpha_0, \ldots, \alpha_{l-1}) \in k_F^l/\Delta} \psi \left( \sum_{k=0}^{l-1} u(\alpha_{k+1} - \alpha_k)\alpha_{j+k} \right),$$

where $\Delta = \{(\alpha, \ldots, \alpha) | \alpha \in k_F\}$ is the image of the diagonal embedding of $k_F$ into $k_F^l$. Then $\chi_{\eta_0}$ on $U_i^*$ is given as follows:

$$\chi_{\eta_0}(x) = \left\{ \begin{array}{ll}
\sum_{\sigma \in \text{Aut}_F E} \theta(\sigma x) & i = -1, \\
q^{[(i+1)/2(l-1)]} \sum_{\sigma \in \text{Aut}_F E} \theta(\sigma x) & i > 0 \text{ and } n - i \text{ even,} \\
q^{[i/2(l-1)]} \sum_{\sigma \in \text{Aut}_F E} \theta(\sigma x) & i > 0 \text{ and } n - i \text{ odd,}
\end{array} \right.$$

where $c = i^{-1}(n + i - 1)/2 \in (\mathbb{Z}/l\mathbb{Z})^\times$.

**Proof.** Put $\sigma x = axa^{-1}$ for $a, x \in G$. At first we treat the case $x \in U_{-1}^* = E^\times - F^\times \mathcal{O}_E^\times$. Since

$$\chi_{\eta_0}(x) = \sum_{a \in H \setminus B} \rho_\theta(ax),$$

we have only to show that if $\sigma x \in H$ for $a \in B$, then

$$a \in \left\{ \begin{array}{ll}
H & \text{if } q \not\equiv 1 \mod l, \\
\bigcup_{k=0}^{l-1} H\xi_k & \text{if } q \equiv 1 \mod l.
\end{array} \right.$$
Now we treat the case $x = c(1 + y)$ for $c \in F$ and $y \in P_{E}^{1} - (F + P_{E}^{i+1})$. We may assume $c = 1$ since $F^{\times}$ is the center of $B$. For $1 + k \in K^{[n-i+1)/2}$ and $a \in B$, we have

$$\chi_{\eta}(1 + y) = \sum_{a \in H \backslash B} \rho_{\theta}(a(1 + y))$$

$$= C \sum_{1 + k \in K^{n-i-1}/K^{[n-i+1)/2}} \sum_{a \in H \backslash B} \rho_{\theta}(a(1+k)(1 + y)),$$

where $C = \frac{1}{K^{n-i-1}/K^{[n-i+1)/2}}$. In the above expression,

$$\rho_{\theta}(a(1+k)(1 + y)) = \rho_{\theta}(1 + a y + a(ky - yk))$$
$$= \rho_{\theta}(1 + a y)\rho_{\theta}(1 + a((1 + y)^{-1}(ky - yk)))$$
$$= \rho_{\theta}(1 + a y)\psi(Tr \gamma^{a}(1 + y)^{-1}(ky - yk))$$
$$= \rho_{\theta}(1 + a y)\psi(Tr(y^{a^{-1}}\gamma - a^{-1}\gamma y)(1 + y)^{-1}k)$$

since $yk^{2} \in A^{n}$ and $a(1 + y)^{-1}(ky - yk)a^{-1} \in A^{m}$. If $y^{a^{-1}}\gamma - a^{-1}\gamma y \not\in A^{1-([n-i+1)/2]}$, the map $k \mapsto \psi(Tr(y^{a^{-1}}\gamma - a^{-1}\gamma y)(1 + y)^{-1}k))$ is a non-trivial character of $A^{n-i-1}/A^{[n-i+1)/2}$; thus

$$\sum_{k \in A^{n-i-1}/A^{[n-i+1)/2}} \psi(Tr(y^{a^{-1}}\gamma - a^{-1}\gamma y)(1 + y)^{-1}k) = 0.$$

By Lemma 3.3 in [4], $y^{a^{-1}}\gamma - a^{-1}\gamma y \in A^{1-([n-i+1)/2]}$ is equivalent to $a^{-1}\gamma \in E^{\times}K^{n-i-1}/E^{\times}K^{m}$. Thus it follows from Lemma 2.1 that

$$\chi_{\eta}(1 + y) = \sum_{\sigma \in Aut_{F} E} \sum_{1 + a \in H \backslash E \times K^{[n-i)/2}} \rho_{\theta}(1 + (1 + a)\sigma y(1 + a)^{-1}).$$

By virtue of $(1 + y)^{-1}(1 + (1+a)y) \in K^{m}$ and $(1 + y)^{-1}(1 + (1+a)y) \equiv 1 + (1 + y)^{-1}(ay - ya) + (ya - ay)a) \mod K^{n},$

$$\rho_{\theta}(1 + (1+a)y) = \theta(1 + y)\psi_{\gamma}((1 + y)^{-1}(ay - ya))\psi_{\gamma}((1 + y)^{-1}(ya - ay)a).$$

Since

$$\psi_{\gamma}((1 + y)^{-1}(ay - ya)) = \psi(Tr(y^{1} \gamma(1 + y)^{-1} - \gamma(1 + y)^{-1}y)a) = 1,$$

$$\psi_{\gamma}((1 + y)^{-1}(ya - ay)a) = \psi_{\gamma}((ya - ay)a)$$

and $|E^{\times}K^{j}/E^{\times}K^{m}| = q^{(l-1)(m-j)}$, we obtain

$$\chi_{\eta}(1 + y) = \begin{cases} q^{m-(n-i)/2} \sum_{\sigma \in Aut_{F} E} \theta(1 + a y) & n - i \text{ even}, \\ q^{m-(n-i+1)/2} \sum_{\sigma \in Aut_{F} E} \theta(1 + a y)S(n - i, \sigma) & n - i \text{ odd}, \end{cases}$$

where

$$S(n - i, \sigma) = \sum_{a \in A^{(n-i+1)/2} + E \cap A^{(n-i-1)/2} \backslash A^{(n-i-1)/2}} \psi_{\gamma}(\sigma y a - a^{\sigma} y a).$$

Now we may assume $n - i \text{ odd}$ and $\sigma = 1$. Put $y = \varpi_{E}^{i}y_{0}$, $a = \varpi_{E}^{(n-i-1)/2}a_{0}$ and $S = S(n - i, 1)$. Since

$$(ya - ay)a = \varpi_{E}^{n-1}(y_{0}\varpi_{E}^{-(n-i-1)/2}a_{0}\varpi_{E}^{(n-i-1)/2}$$

$$- \varpi_{E}^{-(n+i-1)/2}a_{0}\varpi_{E}^{(n+i-1)/2}y_{0})a_{0},$$
**Definition 2.3.** For \( a \in k_F^\times \), we define the Kloosterman sum \( \text{Kl}(a) \) by
\[
(2.6) \quad \text{Kl}(a) = \sum_{\gamma \in k_F/(\gamma \equiv 1 \mod P_F)} \psi(\gamma a) \prod_{j=0}^{l-1} k_j.
\]

**Lemma 2.4.** Let \( x = 1 + \varpi^{-1}x_0 \) for \( x_0 = \text{diag}(k_0, \ldots, k_{l-1}) \), \( (k_i \in \mathcal{O}_E^\times) \). Then
\[
\chi_{\eta_{\theta}}(x) = q^{(n-2)(l-1)/2} \frac{q^l - 1}{q - 1}.
\]

**Remark.** In [6] and [7], it is proved that the Gauss sum \( G(u, j) \) is a fourth root of unity when \( l \neq 2 \).

Next we calculate the character on \( K^{n-1} - K^n \). We state the character formula including the case \( x \notin E \). On \( K^{n-1} - K^n \), the Kloosterman sum appears in the formula.

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\[
\chi_{\eta_{\theta}}(x) = q^{(n-2)(l-1)/2} \frac{q^l - 1}{q - 1}.
\]
Proof. We use the Deligne-Kazhdan correspondence (Theorem 1.5). Since the correspondence preserves the conductoral exponents, there exists a generic character $\theta'$ such that $f(\theta') = n$ and $\chi_{\pi_{\theta}'} = \chi_{\pi_{\theta}}$. Since $\pi_{\theta}'$ is trivial on $1 + P_D^n$ and its degree is $q^{(n-2)(l-1)/2(q^l - 1)}$, $\chi_{\pi_{\theta}'}(x) = q^{(n-2)(l-1)/2(q^l - 1)}$ for $x \in 1 + P_D^n$. Consequently we have $\chi_{\pi_{\theta}}(x) = q^{n-1}(q^2 + q + 1)$ if $x \in K^n$ is elliptic regular.

The character formula on elliptic regular conjugacy classes outside $E^\times$ can be obtained easily.

Lemma 2.6. Let $x$ be an elliptic regular element of $B$. If $x$ satisfies the condition that $F(x) \not\in E$ and $x$ is not conjugate to an element of $F^\times K^n$, then $\chi_{\pi}(x) = 0$.

Proof. See Lemma 3.3 in [14].

3. Calculation of Gauss Sums

In this section, we determine the Gauss sum part $G(y, n-i)$ explicitly. Since $G(y, n-i)$ depends only on $n - i \mod l$ and $y \mod P_E$, we have only to treat the character of $\eta_{\theta}$ on $U_1^*$ by replacing $n$ big enough.

Lemma 3.1. Assume $n = 2m$. Then for $x \in U_1^*$,

\[
\chi_{\eta_{\theta}}(x) = \sum_{\sigma \in \text{Aut}_F E} \sum_{a \in H \backslash E^\times K^{m-1}} \rho_{\theta}(a^2 xa^{-1}).
\]

Proof. It follows from Lemma 2.1 that $axa^{-1} \in H$ implies $a \in E^\times K^{m-1}$. Hence our lemma.

In the calculation of the sum in the above lemma, we use the $E^\times$-module structure of various objects. When $E/F$ is a Galois extension, it is easy to treat. Thus we first assume $E/F$ is Galois, i.e., $q \equiv 1 \mod l$. We recall $\xi$ is the diagonal matrix diag$(1, \zeta^{l-1}, \zeta^{l-2}, \ldots, \zeta)$ where $\zeta$ is an $l$-th root of unity in $F$ and $\xi$ satisfies $\xi^l = 1$ and $\varpi_1 \xi^{-1} = \varpi_0$ for $x \in E$ where $\sigma$ is the generator of $\text{Gal}(E/F)$ determined by $\varpi_E = \varpi_F \zeta$. By the explicit matrix form of $E$ and $A_i$, we obtain:

\[
\begin{align*}
M_i(F) &= E \oplus E\xi \oplus \cdots \oplus E\xi^{l-1} \\
A_0 &= \mathcal{O}_E \oplus \mathcal{O}_E \xi \oplus \cdots \oplus \mathcal{O}_E \xi^{l-1} \\
A_1 &= P_E \oplus P_E \xi \oplus \cdots \oplus P_E \xi^{l-1} \\
&\quad \cdots \\
A_i &= P_i \oplus P_i \xi \oplus \cdots \oplus P_i \xi^{l-1}. \\
\end{align*}
\]

Lemma 3.2. A complete system of representatives of $H \backslash E^\times K^{m-1}$ is given by

\[
\{1 + \varpi^{l-1} \alpha_1 \xi + \cdots + \varpi^{l-1} \alpha_i \xi^{l-1} \mid \alpha_i \in k_F\}.
\]

Proof. It is obvious from (3.2).

For $a = 1 + \alpha_1 \xi + \cdots + \alpha_i \xi^{l-1} \in A^{m-1}$, $\rho(axa^{-1})$ for $x \in U_1^*$ can be expressed explicitly in terms of $\alpha_1, \ldots, \alpha_i$. At first, we determine the coefficients of $a^{-1}$ with respect to the $F$-basis $\{1, \xi, \ldots, \xi^{l-1}\}$.
Lemma 3.3. For $a = \sum_{j=0}^{l-1} \alpha_j \xi^j \ (\alpha_j \in E)$, put
\[
\Lambda(a) = \begin{pmatrix}
\sigma^0 \alpha_0 & \cdots & \sigma^{l-1} \alpha_0 \\
\sigma^1 \alpha_1 & \cdots & \sigma^{l-1} \alpha_1 \\
\vdots & \ddots & \vdots \\
\sigma^{l-j-1} \alpha_0 & \cdots & \sigma^{l-1} \alpha_{l-j-1}
\end{pmatrix} \in M_l(E)
\]
and let $\Lambda_k(a)$ be the $(1, k+1)$-cofactor of $\Lambda(a)$. Then
\[
a^{-1} = \sum_{j=0}^{l-1} \frac{\Lambda_j(a)}{|\Lambda(a)|} \xi^j,
\]
where $|\Lambda(a)|$ is the determinant of $\Lambda(a)$.

Proof. By the map $\Lambda : M_l(F) \to M_l(E)$, we can embed $M_l(F)$ into $M_l(E)$. Then our lemma follows from Cramer’s formula.

Lemma 3.4. Assume $n = 2m$ and $3(m - 1) \geq 2m$. Let $c \in F^x, y \in P_{E}^{m-1}$ and $a = 1 + \sum_{j=1}^{l-1} \alpha_j \xi^j \in K^{m-1}$. Then
\[
\rho_{\theta}(ac(1+y)a^{-1}) = \theta(c(1+y))\psi_{\gamma}
\left(\sum_{j=1}^{l-1} (\sigma^j gg^{-1} - 1)\alpha_j \xi^j \right)
\]
Proof. It is obvious that we may assume $c = 1$. Since
\[
g^{-1}aga^{-1} = 1 + (g^{-1}(a-1)g - (a-1))a^{-1}
\]
\[
= 1 + \left(\sum_{j=1}^{l-1} (\sigma^j gg^{-1} - 1)\alpha_j \xi^j \right) a^{-1},
\]
\[
\sum_{j=1}^{l-1} (\sigma^j gg^{-1} - 1)\alpha_j \xi^j \in A^m \text{ and } \text{Tr}(\gamma x \xi^j) = 0 \text{ for all } x \in E, \text{ we have:}
\]
\[
\rho_{\theta}(g^{-1}aga^{-1}) = \psi_{\gamma}
\left(\sum_{j=1}^{l-1} (\sigma^j gg^{-1} - 1)\alpha_j \xi^j \right)
\]
\[
= \psi_{\gamma}
\left(\sum_{j=1}^{l-1} (\sigma^j gg^{-1} - 1)\alpha_j \sigma^j(f_{l-j}(a)) \right)
\]
where $f_j(a) \in E$ is defined by $a^{-1} = \sum_{j=0}^{l-1} f_j(a) \xi^j$. Put $g = 1 + y$. In the last equation, $\gamma \in P_{E}^{1-n}, f_{l-j} \in P_{E}^{m-1}$ and $\sigma^j gg^{-1} - 1 \equiv \sigma^j y - y \mod P_{E}^{2m-2}$. Thus we get
\[
\rho_{\theta}(g^{-1}aga^{-1}) = \psi_{\gamma}
\left(\sum_{j=1}^{l-1} (\sigma^j \gamma f_{l-j}(a) \sigma^j - \gamma \sigma^j(f_{l-j}(a)) \alpha_j) \right)
\]
by virtue of $\text{tr}_E u \sigma^j = \text{tr}_E \sigma^j uv$ for any $u, v \in E$. By Lemma 3.3,
\[
f_{l-j}(a) = \frac{\Lambda_j(a)}{|\Lambda(a)|} \equiv \alpha_{l-j} \mod P_{E}^{2m-2}.
\]
By the assumption $3m - 3 \geq 2m$, we obtain the desired formula.
Proposition 3.5. Assume $q \equiv 1 \mod l$, $n = 2m$ and $m \geq 3$.

1. For $x \in U_1^*$,
\[
\chi_{\eta}(x) = q^{(l-1)/2} \sum_{j=0}^{l-1} \theta((\sigma^j x)).
\]

2. For an even integer $n$ and $y \in \mathcal{O}_F$, $G(y, n-1) = q^{(l-1)/2}$. In particular, $G(y, n-1)$ depends neither on $n$ nor on $y$.

Proof. By Lemmas 3.1, 3.2 and 3.4, we have for $c \in F^\times$ and $y \in 1+P_E$
\[
\chi_{\eta}(c(1+y)) = \sum_{(\alpha_1, \ldots, \alpha_{l-1}) \in (P_{E}^{m-1}/P_{E}m)} f(\alpha_1, \ldots, \alpha_{l-1}; y).
\]
where
\[
f(\alpha_1, \ldots, \alpha_{l-1}; y) = \psi_E \left( \sum_{j=1}^{l-1} (\gamma \alpha_{1-j}(a) \sigma^j - \gamma^j \alpha_{1-j}(a) \sigma^{-j} \alpha_{j}) \right)^{\sigma^j}.
\]
Put $S_j = \{(\alpha_1, \ldots, \alpha_{l-1}) \in (P_{E}^{m-1}/P_{E}m)^{l-1} | \alpha_k = 0 \text{ for } k < j, \alpha_j \neq 0 \}$ and $I_j(y) = \sum_{(\alpha_1, \ldots, \alpha_{l-1}) \in S_j} f(\alpha_1, \ldots, \alpha_{l-1}; y)$. Then
\[
\chi_{\eta}(c(1+y)) = \sum_{i=0}^{l-1} \theta((c(1+\sigma^i y))) \sum_{j=1}^{l-1} I_j(\sigma^i y).
\]
If $\alpha_1 = \cdots = \alpha_{(l-1)/2} = 0$, $f(\alpha_1, \ldots, \alpha_{l-1}; y) = 0$. Thus we have
\[
\sum_{j=(l+1)/2}^{l-1} I_j(\sigma^i y).
\]
For $1 \leq j \leq (l-1)/2$, $I_j(y)$ is proportional to
\[
\sum_{\alpha_{1-j} \in P_{E}^{m-1}/P_{E}m+1} \psi_E \left( (\gamma \alpha_j \sigma^j\alpha_{1-j} - \sigma^{-j} \gamma \alpha_{1-j} \sigma^{-j} \alpha_j) \right)^{\sigma^j}.
\]
Since $\alpha_j \neq 0$, the map
\[
\alpha_{1-j} \mapsto \gamma \alpha_j \sigma^j\alpha_{1-j} - \sigma^{-j} \gamma \alpha_{1-j} \sigma^{-j} \alpha_j
\]
is a bijection from $P_{E}^{m-1}/P_{E}m$ to $k_F$. Therefore $I_j(y) = 0$. Consequently we get the first part of our lemma. $G(y, n-1) = q^{(l-1)/2}$ follows from Proposition 2.2 and the first part. \[\square\]

Next we assume $q - 1 \not\equiv 0 \mod l$. In this situation, it is rather difficult to describe $E$-module structure of various objects since $F$ has no $l$-th primitive root of unity and $E/F$ is not Galois. In order to apply the result of Galois case, we use the base change lift of simple characters by Bushnell-Henniart([3]). Let $\zeta$ be an $l$-th root of unity and $L = F(\zeta)$. Then $L/F$ is an unramified extension of degree $l - 1$ and the generator $\tau$ of $\mathrm{Gal}(L/F)$ is determined by $\zeta^k = \zeta^k$ where $k$ is a generator of $(\mathbb{Z}/l\mathbb{Z})^\times$. We add the subscript $L$ to the base changed objects. Then $M_L(L) = M_L(F) \otimes_F L$ and $E_L = E \otimes_F L \simeq EL$. $E_L$ is a ramified Galois extension over $L$ of degree $l$, an unramified extension over $E$ of degree $l - 1$ with $\mathrm{Gal}(E_L/E) = \mathrm{Gal}(L/F) = \langle \tau \rangle$ and a non-Abelian Galois extension over $F$ of degree $l(l-1)$. (We embed $E$ into $E_L$ by the map: $x \mapsto x \otimes 1$).
As in the previous section, we identifies \( M_l(L) \) with \( \text{End}_L E_L \) and \( G_L = \text{GL}_l(L) \) with \( \text{Aut}_L E_L \) by the \( L \)-basis \( \{ \varpi_{E_{L}}^{-i}, \ldots, \varpi_{E}^{-1}, 1 \} \) of \( E_L \), which is also an \( \mathcal{O}_L \)-basis of \( \mathcal{O}_{E_L} \). By the lattice flag \( \{ P_{E_L}^j \}_{j \in \mathbb{Z}} \), we define

\[
A^j_L = \{ f \in M_l(L) \mid f(P_{E_L}^j) \subset P_{E_L}^{j+i} \quad \text{for all} \quad j \in \mathbb{Z} \}.
\]

Put \( K_L = (A^0_L)^{\times}, B_L = E_{L}^\times K_L \) and \( K^i_L = 1 + A^i_L \) for \( i \geq 1 \). For a subgroup \( M_L \subset B_L \) (resp. \( M \subset B \)), we set \( M_L^i = M \cap L^\times K^i_L \) (resp. \( M^1 = M \cap F^\times K \)). By the result of Kutzko (Theorem 1.7), it suffices to calculate the character of \( \kappa = \kappa_{\theta} \) instead of \( \pi_{\theta} \). In fact, we have only to get the character of \( \eta_{\theta}|_{B^1} \). Therefore we have only to treat the base change of \( \eta_{\theta}|_{B^1} \) to \( B^1_L \) where \( B^1_L = L^\times K^1_L \).

**Definition 3.6.** Let \( \theta \) be a generic character of \( E_{L}^{\times} \) with \( f(\theta) = n \) and \( \theta(1 + x) = \psi(\operatorname{tr}_E(\gamma x)) \) for \( x \in P_{E_L}^{n} \). We define a base change lift \( \theta_L \) of \( \theta \) to \( L^{\times} \) by \( \theta_L = \theta \circ n_{E_L/E} \). Then \( \theta_L(1 + x) = \psi_L(\operatorname{tr}_{E_L/L} \gamma x) \) for \( x \in P_{E_L}^{n} \). (Recall \( m = [(n + 1)/2] \).) The base change lift \( \rho_L \) of \( \rho_{l^1} \) to \( H^1_L = L^\times (1 + P_{E_L})K^m_L \) is defined by

\[
\rho_L(h \cdot g) = \theta_L(h) \psi_L(\operatorname{Tr} \gamma(g - 1)) \quad \text{for} \quad h \in L^\times(1 + P_{E_L}), \ g \in K^m_L.
\]

We define the base change \( \eta_L \) of \( \eta_{\theta}|_{B^1} \) to \( B^1_L \) by

\[
\eta_L = \text{Ind}_{H^1_L}^{B^1_L} \rho_L.
\]

By virtue of \( \theta_L \circ \tau = \theta_L \), we have \( \rho_L \circ \tau = \rho_L \). Thus we can define an extension \( \tilde{\rho}_L \) of \( \rho_L \) to \( H^1_{L} \times \langle \tau \rangle \) by

\[
\tilde{\rho}_L(x \times \tau) = \rho_L(x) \quad \text{for} \quad x \in H^1_{L}.
\]

Now we apply the result of Bushnell-Henniart ([3]) to our case and get the character relation between \( \eta_L \) and \( \tilde{\eta}_L \). Put \( U_{E,L,i} = L^\times (1 + P_{E_L}^i) \) for \( i > 0 \) and \( U_{E,L,i}^* = U_{E,L,i} - U_{E,L,i+1} \). By (12.19) Corollary in [3] and the fact \( \langle \tau \rangle \)-fixed space \( (L^\times K^1_L)^{\langle \tau \rangle} \) is equal to \( F^\times K^1 \), the following result follows.

**Proposition 3.7.** Let \( x \in U_{E,L,1} \). Between the set

\[
\{ g \in H^1 \setminus (E_{L}^\times K^{m-1})^1 \mid gn_{E_L/E}(x)g^{-1} \in H^1 \}
\]

and the set

\[
\{ h \in H^1_L \setminus (E_{L}^\times K^{m-1})^1 \mid hx^\tau h^{-1} \in H^1_L \},
\]

there is a bijection \( \psi \) with the property

\[
\rho_L(\psi(g)x^\tau(\psi(g))^{-1}) = \rho(gn_{E_L/E}(x)g^{-1}).
\]

Combining this with Lemma 3.1, we have:

**Lemma 3.8.**

\[
(3.3) \quad \chi_{\eta_{\theta}}(n_{E_L/E}(x)) = \sum_{a \in H^1_L \setminus (E_{L}^\times K^{m-1})^1, ax^\tau a^{-1} \in H^1_L} \rho_L(ax^\tau a^{-1}).
\]

Since \( n_{E_L/E}(L^\times (1 + P_{E_L}^i)) = F^\times (1 + P_{E}^i) \), it suffices to calculate the right hand side of (3.3) for \( x \in U_{L,1}^* \).

As in the Galois case, set \( \xi = \text{diag}(1, \zeta^{l-1}, \zeta^{l-2}, \ldots, \zeta) \in M_l(L) \). Then \( \xi \) satisfies \( \xi^l = 1 \), \( \zeta^l = \zeta^k \) and

\[
\xi x^\xi^{-1} = \sigma x \quad \text{for any} \quad x \in E_L,
\]
where $\sigma$ is the generator of $\text{Gal}(E_L/L)$ determined by $^{\sigma}w_E = w_E\zeta$. Moreover we have $\tau\sigma\tau^{-1} = \sigma^k$ and

\[
\begin{align*}
M_i(L) &= E_L \oplus E_L\xi \oplus \cdots \oplus E_L\xi^{i-1} \\
A_{L}^0 &= \mathcal{O}_{E_L} \oplus \mathcal{O}_{E_L}\xi \oplus \cdots \oplus \mathcal{O}_{E_L}\xi^{l-1} \\
A_{L}^1 &= P_{E_L} \oplus P_{E_L}\xi \oplus \cdots \oplus P_{E_L}\xi^{l-1} \\
A_{L}^{l-1} &= P_{E_L}^{l-1} \oplus P_{E_L}^{l-1}\xi \oplus \cdots \oplus P_{E_L}^{l-1}\xi^{l-1}. 
\end{align*}
\]

We note that any element of $K_1^1$ can be written in the form $(1+\alpha_1\xi+\alpha_2\xi^2+\cdots+\alpha_{l-1}\xi^{l-1})$ for $\alpha_i \in P_{E_L}$.

**Lemma 3.9.** Let $i < m$ and $a = 1+\alpha_1\xi+\alpha_2\xi^2+\cdots+\alpha_{l-1}\xi^{l-1}$ for $\alpha_j \in \mathcal{O}_E$ and $x \in U_{E_L}^*$. Then $ax\tau a^{-1} \in H_L$ is equivalent to $\alpha_j \in P_{m-i}$ and $\alpha_{jk} = \tau^{i-j}\alpha_k$ for all $j$. (The suffix of $\alpha_j$ is extended to $\mathbb{Z}$ by $\alpha_j = \alpha_{j \text{mod} \cdot l}$.)

**Proof.** It follows from Lemma 3.2 that if $ax\tau a^{-1} \in H_L$, there exist $\gamma_0 \in \mathcal{O}_E^\times$ and $\gamma_j \in P_{E_L}^m$ for $1 \leq j \leq l - 1$ such that

\[
(1 + \alpha_1\xi + \cdots + \alpha_{l-1}\xi^{l-1})x = \gamma_0(1 + \gamma_1\xi + \gamma_2\xi^2 + \cdots + \gamma_{l-1}\xi^{l-1})
\]

\[
(1 + \tau\alpha_1\xi^k + \tau\alpha_2\xi^{2k} + \cdots + \tau\alpha_{l-1}\xi^{(l-1)k}).
\]

It implies

\[
\begin{align*}
x &= \gamma_0(1 + \gamma_{l-k}\sigma^{l-k}\alpha_1 + \gamma_{l-2k}\sigma^{l-2k}\alpha_2 + \cdots + \gamma_k\sigma^k\alpha_{l-1}) \\
\alpha_k^{\sigma^k} &= \gamma_0(\gamma_k + \tau\alpha_1 + \gamma_{l-k}\sigma^{l-k}\alpha_2 + \cdots + \gamma_{2k}\sigma^{2k}\alpha_{l-1}) \\
&\quad \cdots \\
\alpha_{l-k}\sigma^{l-k} &= \gamma_0(\gamma_{l-k} + \gamma_{l-2k}\sigma^{l-2k}\alpha_1 + \cdots + \tau\alpha_{l-1}).
\end{align*}
\]

Thus we have

\[
\alpha_{jk}^{\sigma^k}x = x\tau a_j \bmod P_{E_L}^m \quad (j \in (\mathbb{Z}/l\mathbb{Z})^\times).
\]

By eliminating $\alpha_{k^2}, \alpha_{k^3}, \ldots, \alpha_{k^{l-1}}$, we get

\[
\alpha_k = n_{E_E/E}(x)^{\sigma^k}n_{E_E/E}(x)^{-1}\alpha_k \bmod P_{E_L}^m.
\]

Since $n_{E_E/E}(x)^{\sigma^k}n_{E_E/E}(x)^{-1} \in 1 + P_{E_L}^i - P_{E_L}^{i+1}$, $\alpha_k \in P_{E_L}^{m-i}$. By $x^{\sigma^k}x^{-1} \in 1 + P_{E_L}^i$, we obtain $\alpha_{kj} \in P_{E_L}^{m-i}$ and $\alpha_{kj} = \tau^{(j-1)}\alpha_k \bmod P_{E_L}^m$ for $j = 1, \ldots, l - 1$.

**Lemma 3.10.** Let $x \in 1 + P_{E_L} - P_{E_L}^2$ and $a = 1 + \sum_{j=1}^{l-1} \tau^{(j-1)}\alpha_j \xi^j$ for $\alpha \in P_{E_L}^{m-1}$. Then

\[
\rho_L(ax\tau a^{-1} x^{-1}) = \psi_E(\text{tr}_{E_L/E}(\alpha^{\sigma^{k(j-1)}}) \text{tr}_{E_L/E}(x - 1)).
\]

**Proof.** By Lemma 3.9, $ax\tau a^{-1} \in H_L$. Since $\rho_L(ax\tau a^{-1} g^{-1}) = \rho_L(ax\tau a^{-1} g^{-1})$. Thus we can prove the lemma by the same way as Lemma 3.4.

It is time to get the character value of $\chi_\eta$ on $U_1^*$. 

**Proposition 3.11.** Let $x \in 1 + P_{E_L} - P_{E_L}^2$ and $n = 2m > 6$. Then

\[
\chi_\eta(n_{E_E/E}(x)) = -q^{(l-1)/2} \bmod (n_{E_E/E}(x))
\]

and

\[
G(y, j) = -q^{(l-1)/2}
\]

for all $y \in k_E$ and $j$ odd.
Proof. By Proposition 3.7, Lemmas 3.8, 3.9, 3.10 and 3.11, we have:
\[
\chi_{\eta}(n_{E_{L}/E}(x)) = \theta_{L}(x) \sum_{\alpha \in P_{E_{L}}^{m-1}/P_{E_{L}}^{m}} \psi_{E}(\text{tr}_{E_{L}/E}(\alpha^{\sigma(l-1)/2} \alpha) \text{tr}_{E_{L}/E}(x-1)).
\]
Let \( E' \) be the \((\sigma^{(l-1)/2})\)-fixed field. Then \( E_{L}/E' \) is a quadratic unramified extension.
Since \( \alpha^{\sigma(l-1)/2} \alpha = n_{E_{L}/E'}(\alpha) \) and \( n_{E_{L}/E'} \) induces a surjection from \( \omega_{E}^{m-1} O_{E_{L}}/1 + P_{E_{L}} \) to \( \omega_{E}^{2m-2} O_{E'}/1 + P_{E'} \) and each fiber of the map has \( q^{(l-1)/2} + 1 \) elements. Thus we get
\[
\chi_{\eta}(n_{E_{L}/E}(x)) = (1 - (q^{(l-1)/2} + 1))\theta_{L}(x).
\]
and it follows from Proposition 2.2 that \( G(y, j) = -q^{(l-1)/2} \) for all \( y \in k_{F} \) and \( j \) odd. \( \square \)

Putting all these together, we can state the character formula.

**Theorem 3.12.** Let \( E \) be a ramified extension of \( F \) with degree \( l \), \( \theta \) a generic quasi-character of \( E^{\times} \) with \( f(\theta) = n \) and \( \pi = \pi_{\theta} \) the irreducible supercuspidal representation of \( GL_{l}(F) \) defined in section 1. Put \( U_{0} = F^{\times} O_{E}^{\times} \), \( U_{j} = F^{\times}(1 + P_{E}^{j}) \) and \( U_{j}^{*} = U_{j} - U_{j+1} \) for \( j \geq 1 \). Let \( x \) be an elliptic regular element of \( GL_{l}(F) \) and \( \text{Aut}_{F}E \) the group of automorphism of \( E \) over \( F \).

1. If \( F(x)/F \) is unramified, then
\[
\chi_{\pi}(x) = \begin{cases} 
0 & \text{if } x \notin F^{\times}(1 + P_{F(x)}^{n-1}), \\
q^{(n-2)(l-1)/2} \left( q^{1} - 1 \right) \theta(c) & \text{if } c = c(1+y) \text{ for } c \in F^{\times}, y \in P_{F(x)}^{n}.
\end{cases}
\]

2. If \( F(x)/F \) is ramified and \( F(x) \not\cong E \), then
\[
\chi_{\pi}(x) = \begin{cases} 
0 & \text{if } x \notin F^{\times}(1 + P_{F(x)}^{n-1}), \\
q^{(n-2)(l-1)/2} \theta(c) \text{Kl}((\gamma\omega_{E}^{n-1})^{l} \prod_{j=0}^{l-1} k_{j}) & \text{if } x = c(1 + \omega_{E}^{n-1} \text{diag}(k_{0}, \ldots, k_{l-1}) + z) \text{ for } c \in F^{\times}, k_{i} \in k_{F}^{\times}, z \in P_{E}^{n}, \\
q^{(n-2)(l-1)/2} \left( q^{1} - 1 \right) \theta(c) & \text{if } x = c(1+y) \text{ for } c \in F^{\times}, y \in P_{F(x)}^{n}.
\end{cases}
\]

3. When \( x \in E \), then
\[
\chi_{\pi}(x) = \begin{cases} 
\left( \frac{q}{l} \right)^{n} \sum_{\sigma \in \text{Aut}_{F}E} \theta(\omega_{x}) & \text{if } x \in E^{\times} - U_{0}, \\
(-1)^{n-j}q^{(l-1)/2} \sum_{\sigma \in \text{Aut}_{F}E} \theta(\omega_{x}) & \text{if } x \in U_{j}^{*} \text{ for } 1 \leq j \leq n-1, \\
q^{(n-2)(l-1)/2} \theta(c) \text{Kl}((\gamma\omega_{E}^{n-1}x_{0})^{j}) & \text{if } x = c(1 + \omega_{E}^{n-1}x_{0}) \text{ for } c \in F^{\times}, x_{0} \in O_{E}^{\times}, \\
q^{(n-2)(l-1)/2} \theta(c) & \text{if } x = c(1+y) \text{ for } c \in F^{\times}, y \in P_{F(x)}^{n}.
\end{cases}
\]
(See (2.6) for the definition of the Kloosterman sum \( \text{Kl}(a) \).)
Proof. It follows from (1.8), Theorem 1.7, Lemmas 2.4, 2.5, 2.6, Propositions 2.2, 3.5 and 3.11.

Remark. By Theorem 1.6, the character formula of the representation $\pi'_0$ of $D^\times$ is given by the same formula for $\pi_0$.

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